Convergence and Stability

Partial Differential Equations

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Objectives

In this lesson we will:

- develop formulas for and properties of eigenvalues and eigenvectors of tridiagonal matrices, and
- explore the stability properties of iterative methods for solving the heat, wave, and Poisson's equations.

Eigensystems of Tridiagonal Matrices

Lemma

Suppose A is an $(n-1)\times(n-1)$ tridiagonal matrix of the form

$$A = \begin{bmatrix} a & b & 0 & \cdots & 0 & 0 & 0 \\ c & a & b & \cdots & 0 & 0 & 0 \\ \vdots & \vdots & \vdots & & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & c & a & b \\ 0 & 0 & 0 & \cdots & 0 & c & a \end{bmatrix}$$

with real or complex entries. If $b\,c \neq 0$ then the eigenvalues of A are

$$\lambda_j = a + 2b\sqrt{\frac{c}{b}}\cos\frac{j\pi}{n},$$

with corresponding eigenvectors

$$\mathbf{v}_j = \left(\left(\frac{c}{b}\right)^{1/2} \sin\frac{j\pi}{n}, \ldots, \left(\frac{c}{b}\right)^{(n-1)/2} \sin\frac{(n-1)j\pi}{n} \right)^T,$$

for $j = 1, 2, \ldots, n-1$.

Eigensystems of Block-Structured Matrices

Theorem

Let matrix A be an $NM \times NM$ matrix written in block form as

$$A = \begin{bmatrix} A_{1,1} & A_{1,2} & \cdots & A_{1,M} \\ A_{2,1} & A_{2,2} & \cdots & A_{2,M} \\ \vdots & \vdots & & \vdots \\ A_{M,1} & A_{M,2} & \cdots & A_{M,M} \end{bmatrix}.$$

Suppose each block $A_{i,j}$ is an $N \times N$ matrix and all the matrices $A_{i,j}$ have a set of N linearly independent eigenvectors $\{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_N\}$ in common. Then the eigenvalues of matrix A are the eigenvalues of the matrices

$$\Lambda_k = \begin{bmatrix} \lambda_{1,1}^{(k)} & \lambda_{1,2}^{(k)} & \cdots & \lambda_{1,M}^{(k)} \\ \lambda_{2,1}^{(k)} & \lambda_{2,2}^{(k)} & \cdots & \lambda_{2,M}^{(k)} \\ \vdots & \vdots & & \vdots \\ \lambda_{M,1}^{(k)} & \lambda_{M,2}^{(k)} & \cdots & \lambda_{M,M}^{(k)} \end{bmatrix} \quad \text{for } k = 1, 2, \ldots, N,$$

where $\lambda_{i,j}^{(k)}$ is the eigenvalue of $A_{i,j}$ corresponding to the common eigenvector $\mathbf{v}_k$.

Stability

- An algorithm (such as a finite difference scheme) is **stable** if small changes in the input data result in proportionately small changes in the output data.
- If an algorithm is not stable, then it is labeled **unstable**.
- If e_0 is the error present in the data and e_n is the error after n subsequent calculations, then the growth rate of the error is **linear** if $e_n \propto n\, e_0$ and the growth rate of the error is **exponential** if $e_n \propto \gamma^n e_0$ for some $\gamma > 1$.

Sources of Error

There are three primary types of error: truncation error, measurement error, and rounding error.

- Truncation error is due to the use of a finite number of terms taken from a Taylor series to develop the approximations to various derivatives and derivative operators used in the partial differential equations.
- Measurement error comes from approximations used to set the initial and boundary conditions of an initial boundary value problem.
- Round-off errors result from the machine arithmetic used by computing devices.

Even if measurement errors are eliminated, truncation error and rounding error will still be present.

Heat Equation

Recall the explicit scheme for approximating the solution to the heat/diffusion equation.

$$\mathbf{u}^{(j+1)} = A(r)\,\mathbf{u}^{(j)}$$

$$\begin{bmatrix} u_1^{j+1} \\ u_2^{j+1} \\ u_3^{j+1} \\ \vdots \\ u_{N-2}^{j+1} \\ u_{N-1}^{j+1} \end{bmatrix} = \begin{bmatrix} 1-2r & r & 0 & \cdots & 0 & 0 \\ r & 1-2r & r & \cdots & 0 & 0 \\ 0 & r & 1-2r & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & 1-2r & r \\ 0 & 0 & 0 & \cdots & r & 1-2r \end{bmatrix} \begin{bmatrix} u_1^{j} \\ u_2^{j} \\ u_3^{j} \\ \vdots \\ u_{N-2}^{j} \\ u_{N-1}^{j} \end{bmatrix}.$$

Matrix $A(r)$ is real, tridiagonal, and symmetric, thus the eigenvalues are all distinct and the eigenvectors form a basis for the vector space $\mathbb{R}^{N-1}$.

Heat Equation

Suppose after the completion of the calculation of $\mathbf{u}^{(j)}$, the vector can be expressed as $\mathbf{u}^{(j)} = \mathbf{w}^{(j)} + \mathbf{e}^{(0)}$ where $\mathbf{w}^{(j)}$ is the exact solution of the finite difference equations. The error in the calculation of $\mathbf{u}^{(j)}$ is therefore $\mathbf{e}^{(0)}$.

After n additional time steps forward

$$\begin{aligned} \mathbf{u}^{(j+n)} &= (A(r))^n \mathbf{u}^{(j)} \\ \mathbf{w}^{(j+n)} + \mathbf{e}^{(n)} &= (A(r))^n (\mathbf{w}^{(j)} + \mathbf{e}^{(0)}) \\ \mathbf{e}^{(n)} &= (A(r))^n \mathbf{e}^{(0)}. \end{aligned}$$

Since the eigenvectors of $A(r)$ form a basis for $\mathbb{R}^{N-1}$ then there exist constants $c_1, c_2, \ldots, c_{N-1}$ such that $\mathbf{e}^{(0)} = c_1\mathbf{v}_1 + c_2\mathbf{v}_2 + \cdots + c_{N-1}\mathbf{v}_{N-1}$.

$$\mathbf{e}^{(n)} = (A(r))^n \sum_{k=1}^{N-1} c_k \mathbf{v}_k = \sum_{k=1}^{N-1} c_k \lambda_k^n \mathbf{v}_k.$$

Heat Equation

The explicit finite difference scheme for the heat equation is stable if $|\lambda_k| \leq 1$ for $k = 1, 2, \ldots, N-1$.

$$\lambda_k = 1 - 2r + 2r\cos\frac{k\pi}{N} = 1 - 4r\sin^2\frac{k\pi}{2N}$$

and $|\lambda_k| \leq 1$ if $r \leq 1/2$. Thus the explicit finite difference scheme given in is stable if $r \leq 1/2$ and unstable for $r > 1/2$.