Laplace's and Poisson's Equations

Partial Differential Equations

J Robert Buchanan

Department of Mathematics

Fall 2025

Objectives

In this lesson we will:

- develop a 5-point approximation to the Laplacian differential operator,
- explore the structure of the system of algebraic equations induces by the 5-point approximation,
- develop a 9-point approximation to the Laplacian differential operator, and
- explore the structure of the system of algebraic equations induces by the 9-point approximation.

Numerical Approximation

- Suppose the plane has been gridded with a spacing of $h > 0$ in both the x and y directions.
- Let $u_i^j = u(i\,h, j\,h)$ for some integers i and j, then

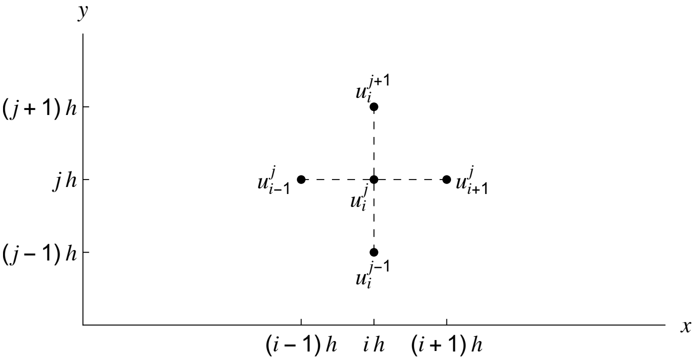
$$u_{xx} + u_{yy} = \frac{u(x+h,y) - 2u(x,y) + u(x-h,y)}{h^2} + \frac{u(x,y+h) - 2u(x,y) + u(x,y-h)}{h^2} + O(h^2).$$

- Poisson's equation can be approximated by the finite difference formula

$$\frac{1}{h^2}(u_{i+1}^j + u_{i-1}^j - 4u_i^j + u_i^{j+1} + u_i^{j-1}) = f(i\,h, j\,k),$$

  which has $O(h^2)$ truncation error.
- The five-point differencing approximation to the Laplacian will be denoted as $\Delta_5 u_i^j$.

Stencil



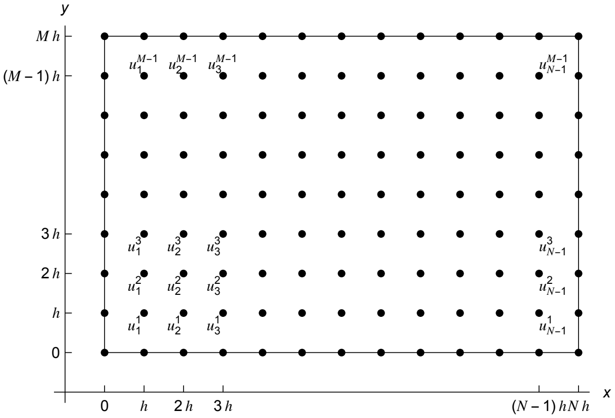
Linear System

If the domain of Poisson's equation is the rectangle $R = \{(x, y) \mid 0 < x < N\,h,\, 0 < y < M\,h\}$ and the boundary conditions are Dirichlet type with $u(x, y) = g(x, y)$ on ∂R then the finite difference approximation can be written in the form of a linear system $A\,\mathbf{u} = \mathbf{b}$.

The vector of unknowns $\mathbf{u}$ contains $(M-1)(N-1)$ elements arranged in what is called the **natural ordering** of the interior nodes,

$$\mathbf{u} = \left(u_1^1, \ldots, u_{N-1}^1, u_1^2, \ldots, u_{N-1}^2, \ldots, u_1^{M-1}, \ldots, u_{N-1}^{M-1}\right)^T$$

Natural Ordering of Nodes



Assuming Dirichlet boundary conditions, the boundary points in the discretization have values given by

$$u_i^0 = g(i\,h, 0), \quad u_i^M = g(i\,h, M\,h), \quad u_0^j = g(0, j\,h), \text{ and } u_N^j = g(N\,h, j)$$

for $i = 1, 2, \ldots, N-1$ and $j = 1, 2, \ldots, M-1$.

Matrix Form

Matrix A is a square matrix with $(M-1)^2(N-1)^2$ entries. Let $\hat{A}$ be the $(N-1) \times (N-1)$ tridiagonal matrix:

$$\hat{A} = \begin{bmatrix} 4 & -1 & 0 & \cdots & 0 & 0 & 0 \\ -1 & 4 & -1 & \cdots & 0 & 0 & 0 \\ \vdots & \vdots & \vdots & & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & -1 & 4 & -1 \\ 0 & 0 & 0 & \cdots & 0 & -1 & 4 \end{bmatrix},$$

let I be the $(N-1) \times (N-1)$ identity matrix, and let 0 represent the $(N-1) \times (N-1)$ matrix of all zero entries. Matrix A has the block structure,

$$A = \begin{bmatrix} \hat{A} & -I & 0 & \cdots & 0 & 0 & 0 \\ -I & \hat{A} & -I & \cdots & 0 & 0 & 0 \\ \vdots & \vdots & \vdots & & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & -I & \hat{A} & -I \\ 0 & 0 & 0 & \cdots & 0 & -I & \hat{A} \end{bmatrix}.$$

There are $M-1$ blocks in each row or column. Note that A is strictly diagonally dominant, thus ensuring there is a unique solution to the linear system $A\mathbf{u} = \mathbf{b}$.

Vector $\mathbf{b}$

Vector $\mathbf{b}$ also has $(M-1)(N-1)$ elements. Using the natural ordering of the interior points

$$\mathbf{b} = -h^2\left(f_1^1, f_2^1, \dots, f_{N-1}^1, f_1^2, f_2^2, \dots, f_{N-1}^2, \dots, f_1^{M-1}, f_2^{M-1}, \dots, f_{N-1}^{M-1}\right)^T$$

$$+\left(g_1^0, g_2^0, \dots, g_{N-1}^0, \underbrace{0, \dots, 0}_{(N-1)(M-3)}, g_1^M, g_2^M, \dots, g_{N-1}^M\right)^T$$

$$+\left(g_0^1, \underbrace{0, \dots, 0}_{N-3}, g_N^1, g_0^2, \underbrace{0, \dots, 0}_{N-3}, g_N^2, \dots, g_0^{M-1}, \underbrace{0, \dots, 0}_{N-3}, g_N^{M-1}\right)^T$$

where to save space $f_i^j = f(i\,h, j\,h)$ and $g_i^j = g(i\,h, j\,h)$.

Example

Derive the linear system of equations for approximating the solution to the following boundary value problem using the 5-point approximation:

$$\Delta u = e^{x-y} \text{ on } R$$
$$u(x, y) = x\,y \text{ on } \partial R.$$

Let the domain of the boundary value problem be $R = \{(x, y) \,|\, 0 < x < 4,\ 0 < y < 3\}$. Use $h = 1$ for simplicity.

Solution

$$
\begin{aligned}
4u_1^1 - u_2^1 - u_1^2 &= -1 \\
-u_1^1 + 4u_2^1 - u_3^1 - u_2^2 &= -e \\
-u_2^1 + 4u_3^1 - u_3^2 &= 4 - e^2 \\
-u_1^1 + 4u_1^2 - u_2^2 &= 3 - e^{-1} \\
-u_2^1 - u_1^2 + 4u_2^2 - u_3^2 &= 5 \\
-u_3^1 - u_2^2 + 4u_3^2 &= 17 - e.
\end{aligned}
$$

9-Point Approximation

A nine-point $O(h^4)$ finite difference representation of Δu is

$$6h^2\Delta_9 u_i^j = 4\left(u_{i-1}^j + u_{i+1}^j + u_i^{j-1} + u_i^{j+1}\right) - 20u_i^j + \left(u_{i-1}^{j-1} + u_{i+1}^{j-1} + u_{i-1}^{j+1} + u_{i+1}^{j+1}\right).$$

There exists a 9-point approximation to Poisson's equation.

$$\Delta_9 u_i^j = f(i\,h, j\,h) + \frac{h^2}{12}\Delta f(i\,h, j\,h).$$

This is a finite difference approximation to Poisson's equation with a truncation error of $O(h^4)$.

9-Point Approximation

Practical use of the 9-point approximation requires that function f be known and its Laplacian can be calculated. If f is only represented by a table of values, the three-point differencing formula yields the $O(h^4)$ discrete approximation to Poisson's equation:

$$\Delta_9 u_i^j = \frac{1}{12}\left(f_{i-1}^j + f_{i+1}^j + 8f_i^j + f_i^{j-1} + f_i^{j+1}\right),$$

where $f_i^j = f(i\,h, j\,h)$.
Multiplying both sides by $-6h^2$ enables the linear system to be written as

$$\begin{aligned}20u_i^j - 4(u_{i-1}^j + u_{i+1}^j + u_i^{j-1} + u_i^{j+1}) - (u_{i-1}^{j-1} + u_{i+1}^{j-1} + u_{i-1}^{j+1} + u_{i+1}^{j+1})\\ = -\frac{h^2}{2}(f_{i-1}^j + f_{i+1}^j + 8f_i^j + f_i^{j-1} + f_i^{j+1})\end{aligned}$$

for $i = 1, 2, \ldots, N-1$ and $j = 1, 2, \ldots, M-1$.

Matrix/Vector Form

If the boundary conditions are of Dirichlet type, the linear system of equations can be written in matrix form as

$$A\mathbf{u} = -(h^2/2)B\mathbf{f} + \mathbf{b}$$

where A is an $(M-1)(N-1) \times (M-1)(N-1)$ block banded matrix,

$$A = \begin{bmatrix} \hat{A} & -\hat{B} & 0 & \cdots & 0 & 0 & 0 \\ -\hat{B} & \hat{A} & -\hat{B} & \cdots & 0 & 0 & 0 \\ \vdots & \vdots & \vdots & & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & -\hat{B} & \hat{A} & -\hat{B} \\ 0 & 0 & 0 & \cdots & 0 & -\hat{B} & \hat{A} \end{bmatrix}.$$

Each block denoted by $\hat{A}$, $\hat{B}$, or 0 is an $(N-1) \times (N-1)$ matrix.

Blocks

As before 0 denotes the matrix of all zero entries. Matrices $\hat{A}$ and $\hat{B}$ are tridiagonal matrices with the following structures,

$$\hat{A} = \begin{bmatrix} 20 & -4 & 0 & \cdots & 0 & 0 & 0 \\ -4 & 20 & -4 & \cdots & 0 & 0 & 0 \\ \vdots & \vdots & \vdots & & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & -4 & 20 & -4 \\ 0 & 0 & 0 & \cdots & 0 & -4 & 20 \end{bmatrix}$$

and

$$\hat{B} = \begin{bmatrix} 4 & 1 & 0 & \cdots & 0 & 0 & 0 \\ 1 & 4 & 1 & \cdots & 0 & 0 & 0 \\ \vdots & \vdots & \vdots & & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & 1 & 4 & 1 \\ 0 & 0 & 0 & \cdots & 0 & 1 & 4 \end{bmatrix}.$$

Blocks

Matrix B is also $(M-1)(N-1) \times (M-1)(N-1)$ with a block banded structure,

$$B = \begin{bmatrix} \hat{C} & I & 0 & \cdots & 0 & 0 & 0 \\ I & \hat{C} & I & \cdots & 0 & 0 & 0 \\ \vdots & \vdots & \vdots & & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & I & \hat{C} & I \\ 0 & 0 & 0 & \cdots & 0 & I & \hat{C} \end{bmatrix}.$$

Each block denoted by $\hat{C}$, I, or 0 is an $(N-1) \times (N-1)$ matrix. Matrix I is an identity matrix. Matrix $\hat{C}$ is a tridiagonal matrix with the following structure:

$$\hat{C} = \begin{bmatrix} 8 & 1 & 0 & \cdots & 0 & 0 & 0 \\ 1 & 8 & 1 & \cdots & 0 & 0 & 0 \\ \vdots & \vdots & \vdots & & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & 1 & 8 & 1 \\ 0 & 0 & 0 & \cdots & 0 & 1 & 8 \end{bmatrix}.$$

Vector $\mathbf{b}$

Vector $\mathbf{b}$ contains $(M-1)(N-1)$ elements and will be written as the sum of three vectors, $\mathbf{b} = \mathbf{f}_0 + \mathbf{g}_4 + \mathbf{g}$ where

$$\mathbf{f}_0 = -\frac{h^2}{2}\left(f_0^1, \underbrace{0,\ldots,0}_{N-3}, f_N^1, f_0^2, \underbrace{0,\ldots,0}_{N-3}, f_N^2, \ldots, f_0^{M-1}, \underbrace{0,\ldots,0}_{N-3}, f_N^{M-1}\right)^T$$
$$-\frac{h^2}{2}\left(f_1^0, f_2^0, \ldots, f_{N-1}^0, \underbrace{0,\ldots,0}_{(N-1)(M-3)}, f_1^M, f_2^M, \ldots, f_{N-1}^M\right)^T$$

has elements which depend on the right-hand side of Poisson's equation.

$$\mathbf{g}_4 = 4\left(g_1^0, g_2^0, \ldots, g_{N-1}^0, \underbrace{0,\ldots,0}_{(N-1)(M-3)}, g_1^M, g_2^M, \ldots, g_{N-1}^M\right)^T$$
$$+4\left(g_0^1, \underbrace{0,\ldots,0}_{N-3}, g_N^1, g_0^2, \underbrace{0,\ldots,0}_{N-3}, g_N^2, \ldots, g_0^{M-1}, \underbrace{0,\ldots,0}_{N-3}, g_N^{M-1}\right)^T$$

Vector **b**

$$\mathbf{g} = \left(g_0^0, g_1^0, \ldots, g_{N-2}^0, \underbrace{0, \ldots, 0}_{(N-1)(M-3)}, g_0^M, g_1^M, \ldots, g_{N-2}^M \right)^T$$

$$+ \left(g_2^0, g_3^0, \ldots, g_N^0, \underbrace{0, \ldots, 0}_{(N-1)(M-3)}, g_2^M, g_3^M, \ldots, g_N^M \right)^T$$

$$+ \left(g_0^2, \underbrace{0, \ldots, 0}_{N-3}, g_N^2, g_0^3, \underbrace{0, \ldots, 0}_{N-3}, g_N^3, \ldots, g_0^{M-1}, \underbrace{0, \ldots, 0}_{N-3}, g_N^{M-1}, \underbrace{0, \ldots, 0}_{N-1} \right)^T$$

$$+ \left(\underbrace{0, \ldots, 0}_{N-1}, g_0^1, \underbrace{0, \ldots, 0}_{N-3}, g_N^1, g_0^2, \underbrace{0, \ldots, 0}_{N-3}, g_N^2, \ldots, g_0^{M-2}, \underbrace{0, \ldots, 0}_{N-3}, g_N^{M-2}, \right)$$

Example

Derive the linear system of equations for approximating the solution to the following boundary value problem using the 9-point approximation:

$$\Delta u = e^{x-y} \text{ on } R$$
$$u(x, y) = x\,y \text{ on } \partial R.$$

Let the domain of the boundary value problem be $R = \{(x, y) \,|\, 0 < x < 4,\ 0 < y < 3\}$. Use $h = 1$ for simplicity.

Solution

$$
\begin{aligned}
-20u_1^1 + u_2^1 + u_1^2 + 4u_2^2 &= 4 + e^{-1} + e \\
u_1^1 - 20u_2^1 + u_3^1 + 4u_1^2 + u_2^2 + 4u_3^2 &= 1 + 4e + e^2 \\
u_2^1 - 20u_3^1 + 4u_2^2 + u_3^2 &= -36 + e + 4e^2 + e^3 \\
u_1^1 + 4u_2^1 - 20u_1^2 + u_2^2 &= -26 + e^{-2} + e^{-1} \\
4u_1^1 + u_2^1 + 4u_3^1 + u_1^2 - 20u_2^2 + u_3^2 &= -50 + e^{-1} + e \\
4u_2^1 + u_3^1 + u_2^2 - 20u_3^2 &= -104 + 4e + e^2.
\end{aligned}
$$