Wave Equation

Partial Differential Equations

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Objectives

In this lesson we will:

- develop explicit methods for numerically approximating the solution to the wave equation, and
- develop implicit methods for numerically approximating the solution to the wave equation.

Wave Equation

Consider the initial value problem for the wave equation,

$$\begin{aligned} u_{tt} &= u_{xx} \text{ for } 0 < x < 1 \text{ and } t > 0 \\ u(x,0) &= f(x) \text{ for } 0 < x < 1 \\ u_t(x,0) &= g(x) \text{ for } 0 < x < 1. \end{aligned}$$

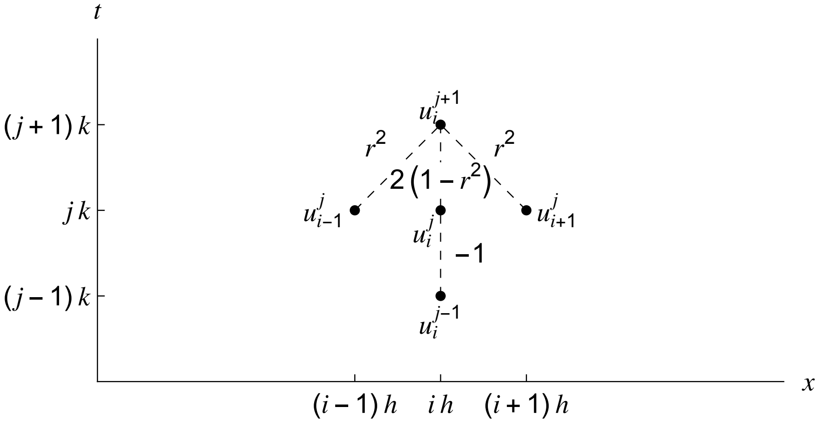
The differential equation may be approximated as

$$\frac{u_i^{j+1} - 2u_i^j + u_i^{j-1}}{k^2} = \frac{u_{i+1}^j - 2u_i^j + u_{i-1}^j}{h^2}.$$

The truncation error of this approximation is $O(k^2) + O(h^2)$. The time step size is $\Delta t = k$ and the spatial step size is $\Delta x = h = 1/N$. Let $r = k/h$ and solve for u_i^{j+1}.

$$u_i^{j+1} = r^2 u_{i+1}^j + 2(1 - r^2)u_i^j + r^2 u_{i-1}^j - u_i^{j-1}.$$

Stencil



Initial Conditions

If the initial conditions for the equation are $u(x, 0) = f(x)$ and $u_t(x, 0) = g(x)$ for $0 < x < 1$ then

$$u_i^0 = f(i\,h) \text{ for } i = 1, 2, \ldots, N-1.$$

Using the centered difference formula for the first derivative with respect to t yields

$$\frac{u_i^1 - u_i^{-1}}{2k} = g(i\,h) \iff u_i^{-1} = u_i^1 - 2kg(i\,h)$$

for $i = 1, 2, \ldots, N-1$. Set $j = 0$ and use the expression just derived for u_i^{-1}.

$$u_i^1 = \frac{r^2}{2} f((i+1)h) + (1 - r^2) f(i\,h) + \frac{r^2}{2} f((i-1)h) + kg(i\,h)$$

for $i = 1, 2, \ldots, N-1$.

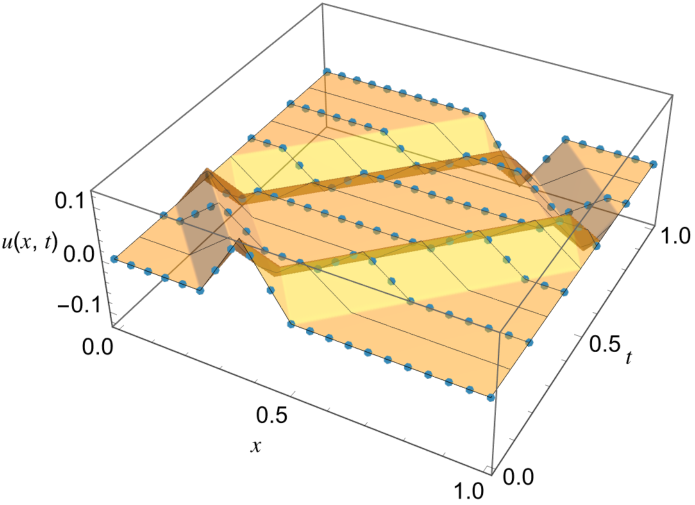
Example

Use the explicit finite difference formulas to approximate the solution to the initial boundary value problem

$$u_{tt} = u_{xx} \text{ for } 0 < x < 1 \text{ and } t > 0$$
$$u(0,t) = u(1,t) = 0 \text{ for } t > 0$$
$$u_t(x,0) = 0 \text{ for } 0 < x < 1$$
$$u(x,0) = \begin{cases} 0 & \text{if } 0 \leq x \leq 1/4, \\ x - 1/4 & \text{if } 1/4 \leq x \leq 3/8, \\ 1/2 - x & \text{if } 3/8 \leq x \leq 1/2, \\ 0 & \text{if } 1/2 \leq x \leq 1. \end{cases}$$

Use $h = 1/20$ and $k = 1/20$. Compare the finite difference approximation to the infinite series solution at $t = 0,\ 1/5,\ 2/5,\ 3/5,\ 4/5,\ 1$.

Solution



Implicit Scheme

For a twice continuously differentiable function f,

$$f(x) = \frac{f(x+h) + f(x-h)}{2} + O(h^2)$$

which is equivalent to the following equation,

$$f(x) = \frac{f(x+h) + 2f(x) + f(x-h)}{4} + O(h^2).$$

If $f = u_{xx}(x,t)$ then

$$\begin{aligned} u_{xx}(x,t) &= \frac{u_{xx}(x,t-k) + 2u_{xx}(x,t) + u_{xx}(x,t+k)}{4} \\ &= \frac{u(x+h,t-k) - 2u(x,t-k) + u(x-h,t-k)}{4h^2} \\ &\quad + \frac{u(x+h,t) - 2u(x,t) + u(x-h,t)}{2h^2} \\ &\quad + \frac{u(x+h,t+k) - 2u(x,t+k) + u(x-h,t+k)}{4h^2} \\ &\quad + O(h^2) + O(k^2). \end{aligned}$$

Implicit Scheme

Therefore the wave equation can be approximated by the differencing scheme,

$$\frac{u_i^{j+1} - 2u_i^j + u_i^{j-1}}{k^2} = \frac{u_{i+1}^{j+1} - 2u_i^{j+1} + u_{i-1}^{j+1}}{4h^2} + \frac{u_{i+1}^{j} - 2u_i^{j} + u_{i-1}^{j}}{2h^2} + \frac{u_{i+1}^{j-1} - 2u_i^{j-1} + u_{i-1}^{j-1}}{4h^2}.$$

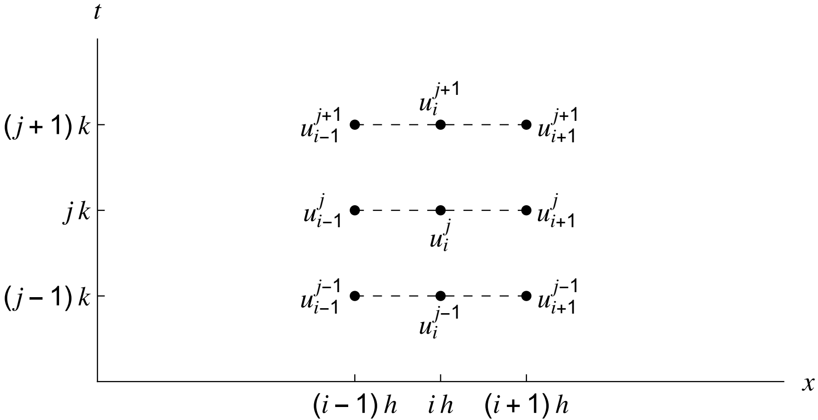
The truncation error in this finite difference approximation is $O(k^2) + O(h^2)$.

The approximation can be written as

$$-r^2 u_{i+1}^{j+1} + 2(2 + r^2)u_i^{j+1} - r^2 u_{i-1}^{j+1} = 2r^2 u_{i+1}^j + 4(2 - r^2)u_i^j + 2r^2 u_{i-1}^j + r^2 u_{i+1}^{j-1} - 2(2 + r^2)u_i^{j-1} + r^2 u_{i-1}^{j-1}$$

for $i = 1, 2, \ldots, N - 1$ where $r = k/h$.

Stencil



t
x
$(j+1)\,k$
$j\,k$
$(j-1)\,k$
$(i-1)\,h$
$i\,h$
$(i+1)\,h$
u_{i-1}^{j+1}
u_i^{j+1}
u_{i+1}^{j+1}
u_{i-1}^{j}
u_i^{j}
u_{i+1}^{j}
u_{i-1}^{j-1}
u_i^{j-1}
u_{i+1}^{j-1}

Matrix/Vector Form

Suppose the boundary conditions are of the homogeneous Dirichlet type. Define $A(z)$ as an $(N-1)\times(N-1)$ matrix:

$$A(z) = \begin{bmatrix} 2(2+z) & -z & 0 & \cdots & 0 & 0 & 0 \\ -z & 2(2+z) & -z & \cdots & 0 & 0 & 0 \\ \vdots & \vdots & \vdots & & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & -z & 2(2+z) & -z \\ 0 & 0 & 0 & \cdots & 0 & -z & 2(2+z) \end{bmatrix}.$$

The implicit finite difference scheme can be written in matrix/vector form as

$$A(r^2)\mathbf{u}^{j+1} = 2A(-r^2)\mathbf{u}^j - A(r^2)\mathbf{u}^{j-1}$$

When $j = 0$

$$A(r^2)\mathbf{u}^1 = 2A(-r^2)\mathbf{u}^0 - A(r^2)\mathbf{u}^{-1} = A(-r^2)\mathbf{u}^0 + kA(r^2)\begin{bmatrix} u_t(h,0) \\ u_t(2h,0) \\ \vdots \\ u_t((N-1)h,0) \end{bmatrix}.$$

Example

Use the implicit finite difference scheme to approximate the solution to the following initial boundary value problem.

$$u_{tt} = u_{xx} \text{ for } 0 < x < 1 \text{ and } t > 0$$
$$u(0,t) = u(1,t) = 0 \text{ for } t > 0$$
$$u(x,0) = x(1-x) \text{ for } 0 < x < 1$$
$$u_t(x,0) = \sin(5\pi x) \text{ for } 0 < x < 1$$

Use $N = 10$, $h = 1/10$, and $k = 1/20$ and iterate the solution until $t = 1/2$.

Solution (1 of 3)

Given the values of h and k the parameter $r = 1/2$ and thus

$$A(\frac{1}{4}) = \begin{bmatrix} \frac{9}{2} & -\frac{1}{4} & 0 & \cdots & 0 & 0 & 0 \\ -\frac{1}{4} & \frac{9}{2} & -\frac{1}{4} & \cdots & 0 & 0 & 0 \\ \vdots & \vdots & \vdots & & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & -\frac{1}{4} & \frac{9}{2} & -\frac{1}{4} \\ 0 & 0 & 0 & \cdots & 0 & -\frac{1}{4} & \frac{9}{2} \end{bmatrix}$$

and

$$2A(-\frac{1}{4}) = \begin{bmatrix} 7 & \frac{1}{2} & 0 & \cdots & 0 & 0 & 0 \\ \frac{1}{2} & 7 & \frac{1}{2} & \cdots & 0 & 0 & 0 \\ \vdots & \vdots & \vdots & & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & \frac{1}{2} & 7 & \frac{1}{2} \\ 0 & 0 & 0 & \cdots & 0 & \frac{1}{2} & 7 \end{bmatrix}$$

For the sake of brevity denote $A(1/4)$ as simply A and $A(-1/4)$ as simply B.

Solution (2 of 3)

Setting $u_i^0 = (i\,h)(1 - i\,h)$ for $i = 1, 2, \ldots, N - 1$ and solving the tridiagonal system:

$$A \begin{bmatrix} u_1^1 \\ u_2^1 \\ \vdots \\ u_{N-1}^1 \end{bmatrix} = B \begin{bmatrix} h(1-h) \\ 2h(1-2h) \\ \vdots \\ (N-1)h(1-(N-1)h) \end{bmatrix} + kA \begin{bmatrix} \sin(5\pi h) \\ \sin(10\pi h) \\ \vdots \\ \sin(5(N-1)\pi h) \end{bmatrix}$$

produce $\mathbf{u}^1$. Iterate to move forward in t.

Solution (3 of 3)