Students' Solution Manual

to accompany

A First Course in Partial Differential Equations

This document contains a subset of the solutions and supporting derivations to the exercises in *A First Course in Partial Differential Equations*, 2nd edition, by Dr. J Robert Buchanan and Dr. Zhoude Shao. If any typographical or mathematical errors are found herein, please contact one of the authors with the specifics of the error and it will be corrected. Email addresses for the authors can be found at `www.millersville.edu/math/`. Where possible the exercises have been checked with the *Wolfram* (formerly *Mathematica*) application. The textbook was prepared using LaTeX and specifically the xsim (eXercise Sheets IMproved) package. The solution numbers correspond to the exercise numbers found in the textbook. The exercises themselves are not repeated in this solution manual.

1.1.1 For $i = 1, 2, \ldots, n$ let $u_i(x,t)$ be a solution to

$$Au_{tt} + Bu_{tx} + Cu_{xx} + Du_t + Eu_x + Fu = G_i$$

valid on $\Omega \subset \mathbb{R}^2$ and let $c_1, c_2, \ldots, c_n$ be constants. Define $v(x,t) = \sum_{i=1}^n c_i u_i(x,t)$, then

$$\begin{aligned}
&Av_{tt} + Bv_{tx} + Cv_{xx} + Dv_t + Ev_x + Fv \\
&= A\left(\sum_{i=1}^n c_i u_i(x,t)\right)_{tt} + B\left(\sum_{i=1}^n c_i u_i(x,t)\right)_{tx} + C\left(\sum_{i=1}^n c_i u_i(x,t)\right)_{xx} + D\left(\sum_{i=1}^n c_i u_i(x,t)\right)_t \\
&\quad + E\left(\sum_{i=1}^n c_i u_i(x,t)\right)_x + F\sum_{i=1}^n c_i u_i(x,t) \\
&= \left(\sum_{i=1}^n c_i A u_i(x,t)\right)_{tt} + \left(\sum_{i=1}^n c_i B u_i(x,t)\right)_{tx} + \left(\sum_{i=1}^n c_i C u_i(x,t)\right)_{xx} + \left(\sum_{i=1}^n c_i D u_i(x,t)\right)_t \\
&\quad + \left(\sum_{i=1}^n c_i E u_i(x,t)\right)_x + \sum_{i=1}^n c_i F u_i(x,t) \\
&= \sum_{i=1}^n c_i \left(Au_{i,tt} + Bu_{i,tx} + Cu_{i,xx} + Du_{i,t} + Eu_{i,x} + Fu_i\right) \\
&= \sum_{i=1}^n c_i G_i.
\end{aligned}$$

If $G_i = 0$ then $v(x,t)$ solves Eq. (1.2).

1.1.3

(a) If $u(x,y) = e^{ax}\sin ay$ then

$$\begin{aligned}
u_{xx}(x,y) &= a^2 e^{ax}\sin ay \\
u_{yy}(x,y) &= -a^2 e^{ax}\sin ay
\end{aligned}$$

and

$$u_{xx} + u_{yy} = a^2 e^{ax}\sin ay - a^2 e^{ax}\sin ay = 0.$$

Thus $u(x,y)$ is a solution.

(b) If $u(x,y,z) = (x^2+y^2+z^2)^{-1/2}$ then

$$\begin{aligned}
u_{xx}(x,y,z) &= (2x^2 - y^2 - z^2)(x^2+y^2+z^2)^{-5/2} \\
u_{yy}(x,y,z) &= (-x^2 + 2y^2 - z^2)(x^2+y^2+z^2)^{-5/2} \\
u_{zz}(x,y,z) &= (-x^2 - y^2 + 2z^2)(x^2+y^2+z^2)^{-5/2}
\end{aligned}$$

and

$$u_{xx} + u_{yy} + u_{zz} = \left[(2x^2 - y^2 - z^2) + (-x^2 + 2y^2 - z^2) + (-x^2 - y^2 + 2z^2)\right](x^2 + y^2 + z^2)^{-5/2}$$
$$= (0)(x^2 + y^2 + z^2)^{-5/2}$$
$$= 0.$$

Thus $u(x, y, z)$ is a solution.

(c) If $u(x,t) = e^{a(t-x)}$ then

$$u_{tt} = a^2 e^{a(t-x)}$$
$$u_{xx} = a^2 e^{a(t-x)}$$

and

$$u_{tt} = a^2 e^{a(t-x)} = u_{xx}.$$

Thus $u(x, t)$ is a solution.

1.1.5 Let $u(x, t) = e^{at} \sin(bx)$ be a solution to the PDE.

$$0 = ae^{at} \sin(bx) - (-b^2 e^{at} \sin(bx))0 = (a + b^2) \sin(bx)$$

Thus for all $b \in \mathbb{R}$ and $a = -b^2$ the function $u(x, t)$ solves the partial differential equation.

1.1.7

(a) Let $u_c(x, y) = c\,x^2/2 + (1 - c)y^2/2$ then

$$u_{xx}(x, y) = c$$
$$u_{yy}(x, y) = 1 - c$$

and

$$u_{xx} + u_{yy} = c + 1 - c = 1.$$

Thus $u(x, y)$ is a solution.

(b) The Laplacian operator is linear, thus the Laplacian of $v(x, y)$ is the Laplacian of $u_c(x, y)$ (which is 1) plus the Laplacian of $u(x, y)$ (which is 0).

(c) Let $a = 2$ and $c = 3$ then $v(x, y) = e^{2x} \sin 2y + u_3(x, y)$ is the desired solution.

1.2.2

(a) $u_{xx} + u_{tx} + u_{tt} = 0$

$$4AC - B^2 = 4(1)(1) - (1)^2 = 3 > 0 \implies \text{elliptic}$$

(b) $5u_{xx} + 3u_{tx} + 2u_{tt} = 0$

$$4AC - B^2 = 4(2)(5) - (3)^2 = 32 \implies \text{elliptic}$$

(c) $t^2 u_{tt} - 2x\,t\,u_{tx} + x^2 u_{xx} = 0$

$$4AC - B^2 = 4(t^2)(x^2) - (-2x\,t)^2 = 0 \implies \text{parabolic}$$

(d) $x^2u_{tt} + 2x\,t\,u_{tx} + t^2u_{xx} = 0$

$$4AC - B^2 = 4(x^2)(t^2) - (2x\,t)^2 = 0 \implies \text{parabolic}$$

1.2.4

(a) $(1+t)u_{tt} - 2x\,t\,u_{tx} - x^2u_{xx} = 0$

$$4AC - B^2 = 4(1+t)(-x^2) - (-2x\,t)^2 = -4x^2(1+t+t^2)$$

The PDE is hyperbolic on the set $\{(x,t)\,|\,x \neq 0\}$ and parabolic on the line where $x = 0$.

(b) $16u_{tt} + 8u_{tx} + 9u_{xx} + u_x = 0$

$$4AC - B^2 = 4(16)(9) - (8)^2 = 512 > 0$$

The PDE is elliptic on the entire (x,t)-plane.

(c) $u_{tt} + 4t\,u_{tx} + 4x\,u_{xx} = 0$

$$4AC - B^2 = 4(1)(4x) - (4t)^2 = 16(x - t^2)$$

The PDE is elliptic on the set $\{(x,t)\,|\,x > t^2\}$, hyperbolic on the set $\{(x,t)\,|\,x < t^2\}$, and parabolic on the curve where $x = t^2$.

1.2.6 Using the chain rule for derivatives,

$$\begin{aligned} v_x &= \gamma v_z \\ v_{xx} &= \gamma^2 v_{zz}. \end{aligned}$$

Substituting these derivatives into the result of Exercise 1.2.5 produces,

$$\begin{aligned} 0 &= v_{tt} + 4\gamma^2 v_{zz} - \frac{21}{4}v \\ &= v_{tt} + v_{zz} - \frac{21}{4}v \end{aligned}$$

if $\gamma = 1/2$.

1.2.8 Differentiating and substituting into the partial differential equation yields

$$\begin{aligned} e^{\alpha x+\beta t}(\beta v + v_t) &= ae^{\alpha x+\beta t}(\alpha^2 v + 2\alpha v_x + v_{xx}) + be^{\alpha x+\beta t}(\alpha v + v_x) + ce^{\alpha x+\beta y}v \\ \beta v + v_t &= a(\alpha^2 v + 2\alpha v_x + v_{xx}) + b(\alpha v + v_x) + cv \\ v_t &= av_{xx} + (2a\alpha + b)v_x + (a\alpha^2 + b\alpha + c - \beta)v \\ v_t &= av_{xx} \end{aligned}$$

when $\alpha = -b/(2a)$ and $\beta = c - b^2/(4a)$.

1.3.2

(a)

$$\begin{aligned} u_t &= 0.86u_{xx} \text{ for } 0 < x < L \text{ and } t > 0 \\ u(0,t) &= u(L,t) = 10 \text{ for } t > 0 \\ u(x,0) &= 100 \text{ for } 0 < x < L \end{aligned}$$

(b)

$$\begin{aligned} u_t &= 0.86 u_{xx} \text{ for } 0 < x < L \text{ and } t > 0 \\ u(0,t) &= u_x(L,t) = 0 \text{ for } t > 0 \\ u(x,0) &= 100 \text{ for } 0 < x < L \end{aligned}$$

(c)

$$\begin{aligned} u_t &= 0.86 u_{xx} \text{ for } 0 < x < L \text{ and } t > 0 \\ u(0,t) &= 0 \text{ and } u(L,t) = 100 \text{ for } t > 0 \\ u(x,0) &= 100 \text{ for } 0 < x < L \end{aligned}$$

(d) The temperature distributions will be as follows.

- In the first case $\lim_{t\to\infty} u(x,t) = 10$ for all $0 \le x \le L$.
- In the second case $\lim_{t\to\infty} u(x,t) = 0$ for all $0 \le x \le L$.
- In the third case $\lim_{t\to\infty} u(x,t) = 100x/L$.

1.3.4 Differentiate the solution and substitute into Eq. (1.26).

$$\begin{aligned} \frac{\partial}{\partial t}\left[1 - x^2 - 2\kappa\, t\right] &= \kappa \frac{\partial^2}{\partial x^2}\left[1 - x^2 - 2\kappa\, t\right] \\ -2\kappa &= -2\kappa \end{aligned}$$

Hence $u(x,t)$ solves Eq. (1.26). When $x = 0$ then $u(0,t) = 1 - 2\kappa\, t$ for $t \ge 0$. When $x = L$ then $u(0,t) = 1 - L^2 - 2\kappa\, t$ for $t \ge 0$. When $t = 0$ then $u(0,t) = 1 - x^2$ for $0 < x < L$.

1.3.6

(a) $u(x,t) = e^{-\kappa\, t} \sin x$; $u(x,0) = \sin x,\ u(0,t) = u(\pi,t) = 0$

$$\begin{aligned} u_t &= -kappa e^{-\kappa\, t} \sin x \\ u_{xx} &= -e^{-\kappa\, t} \sin x \end{aligned}$$

Substituting these partial derivatives into the heat equation produces,

$$-\kappa e^{-\kappa\, t} \sin x = \kappa(-e^{-\kappa\, t} \sin x)$$

which is satisfied for all t and $0 < x < \pi$. The initial and boundary conditions are satisfied as well.

$$\begin{aligned} u(x,0) &= e^{-\kappa(0)} \sin x = \sin x \\ u(0,t) &= e^{-\kappa t} \sin(0) = 0 \\ u(\pi,t) &= e^{-\kappa t} \sin(\pi) = 0 \end{aligned}$$

(b) $u(x,t) = e^{-\kappa\, t} \cos x$; $u(x,0) = \cos x,\ u_x(0,t) = u_x(\pi,t) = 0$

$$\begin{aligned} u_t &= -\kappa e^{-\kappa\, t} \cos x \\ u_x &= -e^{-\kappa\, t} \sin x \\ u_{xx} &= -e^{-\kappa\, t} \cos x \end{aligned}$$

Substituting these partial derivatives into the heat equation produces,

$$-\kappa e^{-\kappa t}\sin x = \kappa(-e^{-\kappa t}\sin x)$$

which is satisfied for all t and $0 < x < \pi$. The initial and boundary conditions are satisfied as well.

$$\begin{aligned} u(x,0) &= e^{-\kappa(0)}\cos x = \cos x \\ u_x(0,t) &= -e^{-\kappa t}\sin(0) = 0 \\ u_x(\pi,t) &= -e^{-\kappa t}\sin(\pi) = 0 \end{aligned}$$

(c) $u(x,t) = 1/2 + (1/2)e^{-4\kappa t}\cos(2x)$; $u(x,0) = \cos^2 x$, $u_x(0,t) = u_x(\pi,t) = 0$

$$\begin{aligned} u_t &= -2\kappa e^{-4\kappa t}\cos(2x) \\ u_x &= -e^{-4\kappa t}\sin(2x) \\ u_{xx} &= -2e^{-4\kappa t}\cos(2x) \end{aligned}$$

Substituting these partial derivatives into the heat equation produces,

$$-2\kappa e^{-4\kappa t}\cos(2x) = \kappa(-2e^{-4\kappa t}\cos(2x))$$

which is satisfied for all t and $0 < x < \pi$. The initial and boundary conditions are satisfied as well.

$$\begin{aligned} u(x,0) &= 1/2 + (1/2)e^{-4\kappa(0)}\cos(2x) = 1/2 + (1/2)\cos(2x) = \cos^2 x \\ u_x(0,t) &= -e^{-4\kappa t}\sin(2(0)) = 0 \\ u_x(\pi,t) &= -e^{-4\kappa t}\sin(2\pi) = 0 \end{aligned}$$

1.4.1 The equilibrium position of the string satisfies the boundary value problem

$$\begin{aligned} c^2U''(x) - g &= 0 \\ U(0) = U(L) &= 0. \end{aligned}$$

Integrating twice produces the general solution

$$U(x) = \frac{gx^2}{2c^2} + Ax + B.$$

Since $U(0) = 0$ then $B = 0$. Since $U(L) = 0$ then $A = -gL/(2c^2)$ and therefore the equilibrium position of the string is given by

$$U(x) = \frac{gx^2}{2c^2} - \frac{gLx}{2c^2}.$$

1.4.3 Differentiate the function and substitute into the two-dimensional wave equation.

$$\begin{aligned} u_{tt} &= -kc^2(m^2+n^2)\pi^2\sin(n\pi x)\sin(m\pi y)\sin(\sqrt{m^2+n^2}c\pi t) \\ u_{xx} &= -kn^2\pi^2\sin(n\pi x)\sin(m\pi y)\sin(\sqrt{m^2+n^2}c\pi t) \\ u_{yy} &= -km^2\pi^2\sin(n\pi x)\sin(m\pi y)\sin(\sqrt{m^2+n^2}c\pi t) \end{aligned}$$

Upon substitution,

$$\begin{aligned} -kc^2(m^2+n^2)\pi^2\sin(n\pi x)\sin(m\pi y)\sin(\sqrt{m^2+n^2}c\pi t) &= -c^2kn^2\pi^2\sin(n\pi x)\sin(m\pi y)\sin(\sqrt{m^2+n^2}c\pi t) \\ &\quad - c^2km^2\pi^2\sin(n\pi x)\sin(m\pi y)\sin(\sqrt{m^2+n^2}c\pi t) \\ &= -c^2k(m^2+n^2)\pi^2\sin(n\pi x)\sin(m\pi y)\sin(\sqrt{m^2+n^2}c\pi t) \end{aligned}$$

1.4.5 Since $u_{tt} = c^2 u_{xx}$ by assumption, then Differentiating both sides with respect to x yields

$$u_{ttx} = c^2 u_{xxx}$$
$$(u_x)_{tt} = c^2 (u_x)_{xx}.$$

Hence $u_x(x,t)$ is also a solution to the wave equation.
1.5.1

$$u_x = \frac{2(x-x_0)}{(x-x_0)^2+(y-y_0)^2}$$
$$u_{xx} = \frac{-2(x-x_0)^2+2(y-y_0)^2}{\left[(x-x_0)^2+(y-y_0)^2\right]^2}$$
$$u_y = \frac{2(y-y_0)}{(x-x_0)^2+(y-y_0)^2}$$
$$u_{yy} = \frac{2(x-x_0)^2-2(y-y_0)^2}{\left[(x-x_0)^2+(y-y_0)^2\right]^2}$$
$$u_{xx}+y_{yy} = \frac{-2(x-x_0)^2+2(y-y_0)^2+2(x-x_0)^2-2(y-y_0)^2}{\left[(x-x_0)^2+(y-y_0)^2\right]^2} = 0$$

1.5.3

$$u_{xx}+u_{yy} = \sinh x \cos y + (-\sinh x \cos y) = 0$$

1.5.5 Begin by finding the partial derivatives of $u(x,y)$.

$$u_x = \frac{-y}{x^2+y^2}$$
$$u_{xx} = \frac{2x\,y}{(x^2+y^2)^2}$$
$$u_y = \frac{x}{x^2+y^2}$$
$$u_{yy} = \frac{-2x\,y}{(x^2+y^2)^2}$$

Now it is seen that

$$u_{xx}+u_{yy} = \frac{2x\,y}{(x^2+y^2)^2} + \frac{-2x\,y}{(x^2+y^2)^2} = 0.$$

1.5.7 Begin by taking partial derivatives.

$$u_r = \frac{2(-2r+(1+r^2)\cos\theta)}{(1+r^2-2r\cos\theta)^2}$$
$$u_{rr} = \frac{4(3r^2-r(r^2+3)\cos\theta+\cos(2\theta))}{(1+r^2-2r\cos\theta)^3}$$
$$u_\theta = \frac{2r(r^2-1)\sin\theta}{(1+r^2-2r\cos\theta)^2}$$
$$u_{\theta\theta} = \frac{2r(r^2-1)((1+r^2)\cos\theta+r(-3+\cos(2\theta)))}{(1+r^2-2r\cos\theta)^3}$$

Substitute the partial derivatives into the left-hand side of Laplace's equation in polar coordinates.

$$
\begin{aligned}
u_{rr} + \frac{1}{r}u_r + \frac{1}{r^2}u_{\theta\theta} &= \frac{4(3r^2 - r(r^2+3)\cos\theta + \cos(2\theta))}{(1+r^2-2r\cos\theta)^3} + \frac{2(-2r + (1+r^2)\cos\theta)}{r(1+r^2-2r\cos\theta)^2} \\
&\quad + \frac{2r(r^2-1)((1+r^2)\cos\theta + r(-3+\cos(2\theta)))}{r^2(1+r^2-2r\cos\theta)^3} \\
&= \frac{4r^2(3r^2 - r(r^2+3)\cos\theta + \cos(2\theta)) + 2r^2(-2r+(1+r^2)\cos\theta)(1+r^2-2r\cos\theta)}{r^2(1+r^2-2r\cos\theta)^3} \\
&\quad + \frac{2r^3(r^2-1)((1+r^2)\cos\theta + r(-3+\cos(2\theta)))}{r^2(1+r^2-2r\cos\theta)^3} \\
&= \frac{8(-1+r^2+\cos^2\theta - r^2\cos^2\theta + \sin^2\theta - r^2\sin^2\theta)}{(1+r^2-2r\cos\theta)^3} \\
&= 0
\end{aligned}
$$

1.6.2

(a) This is a mathematical model of a one-dimensional rod of unit length whose ends are kept at constant temperature of 0. The initial temperature distribution along the length of the rod is given by $\sin(\pi x)$.

(b) Since the ends of the rod are kept at temperature 0 and are not insulated, in the long term all heat energy will flow out of the rod into the surrounding environment. Thus $\lim_{t\to\infty} u(x,t) = 0$.

(c) The fundamental solution expressed in Eq. (1.58) solves the initial boundary value problem when $n=1$, $L=1$, and $B_1 = 1$. Thus

$$u(x,t) = e^{-\pi^2\kappa t}\sin(\pi x)$$

and

$$\lim_{t\to\infty} u(x,t) = \lim_{t\to\infty} e^{-\pi^2\kappa t}\sin(\pi x) = 0.$$

1.6.4 The finite sum solution expressed in Eq. (1.59) solves the initial boundary value problem when $\kappa = 4$, $L = 2$, and $N = 6$. In this case let $B_1 = 1$, $B_2 = -1/2$, $B_3 = B_4 = B_5 = 0$, and $B_6 = 3$ and thus

$$u(x,t) = e^{-\pi^2 t}\sin\left(\frac{\pi x}{2}\right) - \frac{1}{2}e^{-4\pi^2 t}\sin(\pi x) + 3e^{-36\pi^2 t}\sin(3\pi x).$$

1.6.6 Assume there is a product solution of the form $u(x,t) = X(x)T(t)$ then

$$
\begin{aligned}
X(x)T'(t) &= \kappa X''(x)T(t) \\
\frac{X(x)T'(t)}{\kappa X(x)T(t)} &= \frac{\kappa X''(x)T(t)}{\kappa X(x)T(t)} \\
\frac{T'(t)}{\kappa T(t)} &= \frac{X''(x)}{X(x)} = -c
\end{aligned}
$$

where c is a constant. The boundary conditions can be written as

$$X'(0)T(t) = X'(L)T(t) = 0 \implies X'(0) = X'(L) = 0.$$

The implied boundary value problem is

$$
\begin{aligned}
X''(x) + cX(x) &= 0 \\
X'(0) = X'(L) &= 0.
\end{aligned}
$$

There are three cases to consider.

Case $c = 0$**:** which implies $X(x) = Bx + A$. The boundary conditions require $A = 0$ and thus the nontrivial solutions may be expressed as $X_0(x) = A_0$ where A_0 is a nonzero constant.

Case $c > 0$**:** in which case $c = \lambda^2$ for some $\lambda > 0$. The general solution to the ordinary differential equation is $X(x) = A_n \cos(\lambda x) + B_n \sin(\lambda x)$. The boundary conditions require $B_n = 0$. If $A_n \neq 0$ then $\lambda = \lambda_n = n\pi/L$ with $n \in \mathbb{N}$. Thus $X_n(x) = A_n \cos(n\pi x/L)$ is a nontrivial solution corresponding to $c = n^2\pi^2/L^2$.

Case $c < 0$**:** in which case $c = -\lambda^2$ for some $\lambda > 0$. The general solution to the ordinary differential equation is $X(x) = A_n \cosh(\lambda x) + B_n \sinh(\lambda x)$. The boundary conditions require $A_n = B_n = 0$. There are no nontrivial solutions in this case.

The eigenfunctions are members of the set $\{\cos(n\pi x/L)\}_{n=0}^{\infty}$ with corresponding eigenvalues $\{n^2\pi^2/L^2\}_{n=0}^{\infty}$.

1.6.8 Assume there is a product solution of the form $u(x,t) = X(x)T(t)$ then

$$\begin{aligned} X(x)T'(t) &= \kappa X''(x)T(t) \\ \frac{X(x)T'(t)}{\kappa X(x)T(t)} &= \frac{\kappa X''(x)T(t)}{\kappa X(x)T(t)} \\ \frac{T'(t)}{\kappa T(t)} &= \frac{X''(x)}{X(x)} = -c \end{aligned}$$

where c is a constant. The boundary conditions can be written as

$$X(0)T(t) = X'(L)T(t) = 0 \implies X(0) = X'(L) = 0.$$

The implied boundary value problem is

$$\begin{aligned} X''(x) + cX(x) &= 0 \\ X(0) = X'(L) &= 0. \end{aligned}$$

There are three cases to consider.

Case $c = 0$**:** which implies $X(x) = Bx + A$. The boundary conditions require $A = B = 0$ and thus there are no nontrivial solutions in this case.

Case $c > 0$**:** in which case $c = \lambda^2$ for some $\lambda > 0$. The general solution to the ordinary differential equation is $X(x) = A_n \cos(\lambda x) + B_n \sin(\lambda x)$. The boundary condition $X(0) = 0$ requires $A_n = 0$. If $B_n \neq 0$ then $\lambda = \lambda_n = (2n-1)\pi/(2L)$ with $n \in \mathbb{N}$. Thus $X_n(x) = B_n \sin((2n-1)\pi x/(2L))$ is a nontrivial solution corresponding to $c = (2n-1)^2\pi^2/(2L)^2$.

Case $c < 0$**:** in which case $c = -\lambda^2$ for some $\lambda > 0$. The general solution to the ordinary differential equation is $X(x) = A_n \cosh(\lambda x) + B_n \sinh(\lambda x)$. The boundary conditions require $A_n = B_n = 0$. There are no nontrivial solutions in this case.

The eigenfunctions are members of the set $\left\{\sin \frac{(2n-1)\pi x}{2L}\right\}_{n=1}^{\infty}$ with corresponding eigenvalues $\left\{\frac{(2n-1)^2\pi^2}{(2L)^2}\right\}_{n=1}^{\infty}$.

2.1.2 Both Alice and Bob are correct. Differentiating Alice's solution produces

$$\begin{aligned} u_x &= -\frac{c}{a}e^{-cx/a}f(bx-ay) + be^{-cx/a}f'(bx-ay) \\ u_y &= -ae^{-cx/a}f'(bx-ay), \end{aligned}$$

and thus

$$\begin{aligned} au_x + bu_y + cu &= -ce^{-cx/a}f(bx-ay) + abe^{-cx/a}f'(bx-ay) - abe^{-cx/a}f'(bx-ay) + ce^{-cx/a}f(bx-ay) \\ &= 0. \end{aligned}$$

Differentiating Bob's solution produces

$$u_x = be^{-cy/b}g(bx - ay)$$
$$u_y = -\frac{c}{b}e^{-cy/b}g(bx - ay) - ae^{-cy/b}g'(bx - ay),$$

and thus

$$\begin{aligned} au_x + bu_y + cu &= abe^{-cy/b}g(bx - ay) - ce^{-cy/b}g(bx - ay) - abe^{-cy/b}g'(bx - ay) + ce^{-cy/b}g(bx - ay) \\ &= 0. \end{aligned}$$

2.1.4

(a)

$$\frac{dx}{dy} = x \implies x = ke^y$$

and assuming $u(x, y) = u(x(y), y)$ then

$$\frac{du}{dy} = u_x ke^y + u_y = y \implies u(x, y) = \frac{1}{2}y^2 + f(k) = \frac{1}{2}y^2 + f(xe^{-y}).$$

Using the boundary condition,

$$u(x, 0) = f(x) = x^2$$

and thus

$$u(x, y) = \frac{1}{2}y^2 + x^2e^{-2y}.$$

(b)

$$\frac{dy}{dx} = \frac{1}{x} \implies y = k + \ln x$$

and assuming $u(x, y) = u(x, y(x))$ then

$$\frac{du}{dx} = u_x + \frac{1}{x}u_y = \frac{y}{x} = \frac{k + \ln x}{x} \implies u(x, y) = k\ln x + \frac{1}{2}(\ln x)^2 + f(k) = y\ln x - \frac{1}{2}(\ln x)^2 + f(y - \ln x)$$

Using the boundary condition,

$$u(x, 0) = -\frac{1}{2}(\ln x)^2 + f(-\ln x) = x^2$$

and thus $f(z) = e^{-2z} + z^2/2$ and consequently

$$u(x, y) = \frac{1}{2}y^2 + x^2e^{-2y}.$$

2.1.6 According to Exercise 2.1.2 the general solution to this partial differential equation can be expressed as

$$u(x, y) = e^{-4x}f(3x - 2y).$$

Attempting to impose the side condition results in

$$e^x = u(x, (3x - 1)/2) = e^{-4x}f(3x - (3x - 1)) \implies f(1) = e^{5x}.$$

This is impossible since $f(1)$ is a constant. Thus there is no particular solution in this case.

2.1.8 The system of characteristic ordinary differential equations can be written as

$$\begin{aligned}
\frac{dx}{dt} &= 1 \implies x = t + A \\
\frac{dy}{dt} &= e^x \implies y = e^{t+A} + B = e^x + B \\
\frac{dz}{dt} &= e^z \implies z = -\ln(C - A - t) = -\ln(C - x)
\end{aligned}$$

where A, B, and C are constants. Assuming that $u(x, y, z) \equiv u(x, y(x), z(x))$ then

$$\frac{du}{dx} = (2x - e^x)e^u \implies e^{-u} = e^x - x^2 + f(B, C) = e^x - x^2 + f(y - e^x, x + e^{-z})$$

where f is an arbitrary differentiable function.

2.2.1

(a)
$$(ku_1)(ku_{1,x}) + ku_{1,y} = k^2(u_1 u_{1,x}) + ku_{1,y} = 0 \iff k = 0 \text{ or } k = 1.$$

(b) Define $v(x, y) = u_1(x, y) + u_2(x, y)$ then

$$\begin{aligned}
v\,v_x + v_y &= (u_1 + u_2)(u_{1,x} + u_{2,x}) + (u_{1,y} + u_{2,y}) \\
&= u_1 u_{1,x} + u_{1,y} + u_2 u_{2,x} + u_{2,y} + u_1 u_{2,x} + u_2 u_{1,x} \\
&= u_1 u_{2,x} + u_2 u_{1,x} \neq 0
\end{aligned}$$

in general.

(c) The partial differential equation is not linear in the dependent variable.

2.2.3

(a) The system of characteristic equations can be expressed as

$$\begin{aligned}
\frac{dx}{dt} &= 1, \quad x(0) = s \\
\frac{dy}{dt} &= -1, \quad y(0) = 2s - 1 \\
\frac{du}{dt} &= u^2, \quad u(0) = 4.
\end{aligned}$$

The solutions to the characteristic system are

$$x = t + s, \quad y = -t + 2s - 1, \quad u = \frac{-4}{4t - 1}.$$

Solving for s and t in terms of x and y and substituting into the expression for u yields the solution

$$u(x, y) = \frac{12}{3 - 4(2x - y - 1)}.$$

(b) The system of characteristic equations can be expressed as

$$\begin{aligned}
\frac{dx}{dt} &= x, \quad x(0) = s \\
\frac{dy}{dt} &= y, \quad y(0) = s^3 \\
\frac{du}{dt} &= \sec u, \quad u(0) = 0.
\end{aligned}$$

The solutions to the characteristic system are

$$x = se^t, \quad y = s^3 e^t, \quad \sin u = t.$$

Solving for s and t in terms of x and y and substituting into the expression for u yields the solution

$$u(x, y) = \sin^{-1}\left(\frac{1}{2}\ln\left[\frac{x^3}{y}\right]\right).$$

(c) The system of characteristic equations can be expressed as

$$\begin{aligned} \frac{dx}{dt} &= 1, \quad x(0) = s^2 \\ \frac{dy}{dt} &= y^2, \quad y(0) = s \\ \frac{du}{dt} &= \cos u, \quad u(0) = 1. \end{aligned}$$

The solutions to the characteristic system are

$$x = t + s^2, \quad y = \frac{s}{1 - st}, \quad \ln|\sec u + \tan u| = t + \ln(\sec 1 + \tan 1).$$

Function u is defined implicitly.

(d) The system of characteristic equations can be expressed as

$$\begin{aligned} \frac{dx}{dt} &= y + u, \quad x(0) = s \\ \frac{dy}{dt} &= y, \quad y(0) = 1 \\ \frac{du}{dt} &= x - y, \quad u(0) = s + 1. \end{aligned}$$

The differential equation for y is readily solved to find $y(t) = e^t$. Substituting this expression in the remaining characteristic differential equations and solving results in

$$x(t) = -e^{-t} + (1 + s)e^t = (1 + s)y - 1/y \text{ and } u(t) = e^{-t} + se^t = sy + 1/y.$$

Eliminating parameter s produces

$$u(x, y) = \frac{2}{y} + x - y.$$

2.2.5 The system of differential equations for the characteristic curves can be expressed as

$$\begin{aligned} \frac{dx}{dt} &= 1, \quad x(0) = 0 \\ \frac{dy}{dt} &= u, \quad y(0) = s \\ \frac{du}{dt} &= 0, \quad u(0) = f(s). \end{aligned}$$

The solution to the first equation is $x(t) = t$. The solution to the third equation is $u(t) = f(s)$. Using this in the second equation produces $y(t) = f(s)t + s$ and thus

$$y - f(s)t = s \iff y - u(x, y)x = s \iff u(x, y) = f(y - u(x, y)x).$$

2.3.2

$$q = 10\rho = 10\left(\frac{1000}{10}\right) = 1000 \text{ vehicles/hour.}$$

From Eq. (2.42),

$$10 = u_{\max}\left(1 - \frac{1000/10}{133.333}\right) \implies u_{\max} = 40 \text{ kilometers/hour.}$$

2.3.4 The derivative $d\rho/dx$ is undefined when $f'(s) \neq 0$ and

$$1 - \frac{2u_{\max}t}{\rho_{\max}}f'(s) = 0 \iff t = \frac{\rho_{\max}}{2u_{\max}f'(s)}.$$

2.3.6 The time of the first shock will occur when $f'(s)$ is maximized.

$$f'(s) = \frac{-6\rho_{\max}s}{(3+3s^2)^2}$$
$$f''(s) = \frac{2\rho_{\max}(3s^2-1)}{3(1+s^2)^3} = 0 \implies s = \frac{-1}{\sqrt{3}}$$

Note that $f'(-1/\sqrt{3}) = \rho_{\max}\sqrt{3}/8$, therefore

$$t_0 = \frac{4}{u_{\max}\sqrt{3}}$$

is the time of the first shock. The location of the first shock is

$$x = u_{\max}\left(1 - \frac{2f(s)}{\rho_{\max}}\right)t + s = u_{\max}\left(1 - \frac{2f(-1/\sqrt{3})}{\rho_{\max}}\right)\frac{4}{u_{\max}\sqrt{3}} - \frac{1}{\sqrt{3}} = \frac{1}{\sqrt{3}}.$$

At t_0 the maximum density is located at $x = 4/(3\sqrt{3})$.
3.1.2 The least period is $\text{lcm}(T_1, T_2)$.
3.1.4

(a)
$$f'(x+T) = \lim_{z\to x+T}\frac{f(z)-f(x+T)}{z-(x+T)} = \lim_{z-T\to x}\frac{f(z-T)-f(x)}{(z-T)-x} = f'(x)$$

(b)
$$G(x+T) = \int_a^{x+T} f(t)\,dt = \int_a^{a+T} f(t)\,dt + \int_{a+T}^{x+T} f(t)\,dt = 0 + \int_a^x f(t)\,dt = G(x)$$

3.2.1 Consider the sum and difference of angles formulas for the cosine.

$$\cos(\alpha+\beta) = \cos\alpha\cos\beta - \sin\alpha\sin\beta$$
$$\cos(\alpha-\beta) = \cos\alpha\cos\beta + \sin\alpha\sin\beta$$

Add the two equations and then divide by 2 to produce Eq. (3.8). Subtract the first equation from the second and divide by 2 to produce Eq. (3.10). Now recall the sum and difference of angles formulas for the sine.

$$\sin(\alpha+\beta) = \sin\alpha\cos\beta + \cos\alpha\sin\beta$$
$$\sin(\alpha-\beta) = \sin\alpha\cos\beta - \cos\alpha\sin\beta$$

Subtract the second equation from the first and divide by 2 to produce Eq. (3.9).

3.2.3 Computing the inner product reveals,

$$\langle x, x^2 \rangle = \int_{-1}^{1} x \cdot x^2 \, dx = \left[\frac{1}{4}x^4\right]_{x=-1}^{x=1} = \frac{1}{4} - \frac{1}{4} = 0.$$

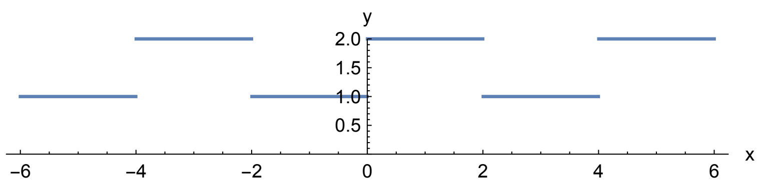
3.2.5 If $f(x) = ax^2 + bx + c$ is orthogonal to all linear functions on $[-1, 1]$ then it is orthogonal to both x and 1 on $[-1, 1]$. Consider the system of equations:

$$\begin{aligned}
\langle ax^2 + bx + c, 1 \rangle &= \int_{-1}^{1} (ax^2 + bx + c)(1) \, dx = \frac{2}{3}a + 2c = 0 \\
\langle ax^2 + bx + c, x \rangle &= \int_{-1}^{1} (ax^2 + bx + c)(x) \, dx = \int_{-1}^{1} (ax^3 + bx^2 + cx) \, dx = \frac{2}{3}b = 0.
\end{aligned}$$

Hence $b = 0$ and $a = -3c$. Thus there are infinitely many solutions. In particular if $c = 1$ then $f(x) = -3x^2 + 1$ is orthogonal to all linear functions on $[-1, 1]$.

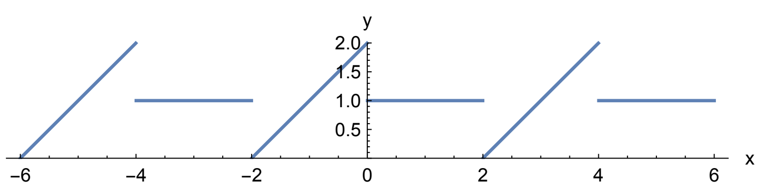
3.3.1

(a) $f(x) = \begin{cases} 1 & \text{if } -2 < x < 0 \\ 2 & \text{if } 0 < x \leq 2 \end{cases}$



$$\begin{aligned}
a_0 &= \frac{1}{2}\int_{-2}^{2} f(x) \, dx = \frac{1}{2}\int_{-2}^{0} 1 \, dx + \frac{1}{2}\int_{0}^{2} 2 \, dx = 3 \\
a_n &= \frac{1}{2}\int_{-2}^{2} f(x) \cos\frac{n\pi x}{2} \, dx = \frac{1}{2}\int_{-2}^{0} \cos\frac{n\pi x}{2} \, dx + \frac{1}{2}\int_{0}^{2} 2\cos\frac{n\pi x}{2} \, dx = 0 \\
b_n &= \frac{1}{2}\int_{-2}^{2} f(x) \sin\frac{n\pi x}{2} \, dx = \frac{1}{2}\int_{-2}^{0} \sin\frac{n\pi x}{2} \, dx + \frac{1}{2}\int_{0}^{2} 2\sin\frac{n\pi x}{2} \, dx = \frac{1 - (-1)^n}{n\pi} \\
f(x) &\sim \frac{3}{2} + \sum_{n=1}^{\infty} \frac{1 - (-1)^n}{n\pi} \sin\frac{n\pi x}{2}
\end{aligned}$$

(b) $f(x) = \begin{cases} 2 + x & \text{if } -2 < x < 0 \\ 1 & \text{if } 0 < x < 2 \end{cases}$



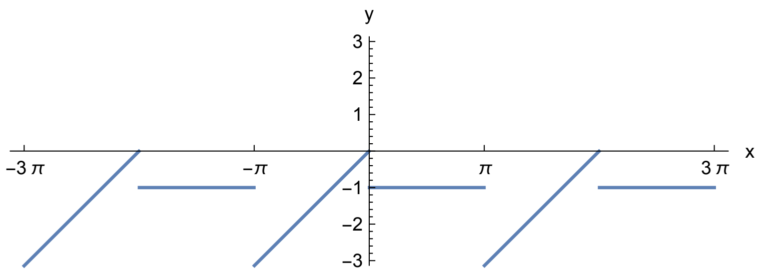
$$a_0 = \frac{1}{2}\int_{-2}^{2} f(x)\,dx = \frac{1}{2}\int_{-2}^{0}(2+x)\,dx + \frac{1}{2}\int_{0}^{2} 1\,dx = 2$$

$$a_n = \frac{1}{2}\int_{-2}^{2} f(x)\cos\frac{n\pi x}{2}\,dx = \frac{1}{2}\int_{-2}^{0}(2+x)\cos\frac{n\pi x}{2}\,dx + \frac{1}{2}\int_{0}^{2}\cos\frac{n\pi x}{2}\,dx = \frac{2}{n^2\pi^2}(1-(-1)^n)$$

$$b_n = \frac{1}{2}\int_{-2}^{2} f(x)\sin\frac{n\pi x}{2}\,dx = \frac{1}{2}\int_{-2}^{0}(2+x)\sin\frac{n\pi x}{2}\,dx + \frac{1}{2}\int_{0}^{2}\sin\frac{n\pi x}{2}\,dx = \frac{-1}{n\pi}(1+(-1)^n)$$

$$f(x) \sim 1 + \sum_{n=1}^{\infty}\left[\frac{2}{n^2\pi^2}(1-(-1)^n)\cos\frac{n\pi x}{2} - \frac{1}{n\pi}(1+(-1)^n)\sin\frac{n\pi x}{2}\right]$$

(c) $f(x) = \begin{cases} x & \text{if } -\pi < x < 0 \\ -1 & \text{if } 0 < x < \pi \end{cases}$



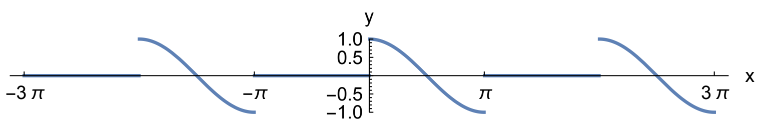
$$a_0 = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\,dx = \frac{1}{\pi}\int_{-\pi}^{0} x\,dx - \frac{1}{\pi}\int_{0}^{\pi} 1\,dx = -1 - \frac{\pi}{2}$$

$$a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos(nx)\,dx = \frac{1}{\pi}\int_{-\pi}^{0} x\cos(nx)\,dx - \frac{1}{\pi}\int_{0}^{\pi}\cos(nx)\,dx = \frac{1}{n^2\pi}(1-(-1)^n)$$

$$b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\sin(nx)\,dx = \frac{1}{\pi}\int_{-\pi}^{0} x\sin(nx)\,dx - \frac{1}{\pi}\int_{0}^{\pi}\sin(nx)\,dx = \frac{-1-(\pi-1)(-1)^n}{n\pi}$$

$$f(x) \sim -\frac{1}{2} - \frac{\pi}{4} + \sum_{n=1}^{\infty}\left[\frac{1}{n^2\pi}(1-(-1)^n)\cos(nx) - \frac{1+(\pi-1)(-1)^n}{n\pi}\sin(nx)\right]$$

(d) $f(x) = \begin{cases} 0 & \text{if } -\pi < x < 0 \\ \cos x & \text{if } 0 < x < \pi \end{cases}$



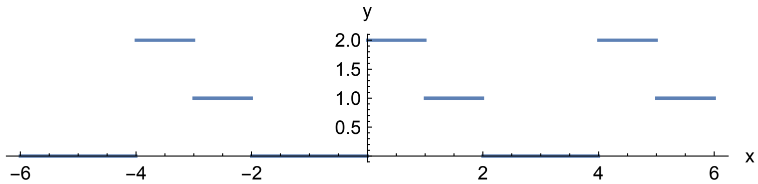
$$a_0 = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\,dx = \frac{1}{\pi}\int_{0}^{\pi}\cos x\,dx = 0$$

$$a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos(nx)\,dx = \frac{1}{\pi}\int_{0}^{\pi}\cos x\cos(nx)\,dx = \begin{cases} 1/2 & \text{if } n=1 \\ 0 & \text{otherwise} \end{cases}$$

$$b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\sin(nx)\,dx = \frac{1}{\pi}\int_{0}^{\pi}\cos x\sin(nx)\,dx = \begin{cases} 0 & \text{if } n=1 \\ \frac{n(1+(-1)^n)}{\pi(n^2-1)} & \text{otherwise} \end{cases}$$

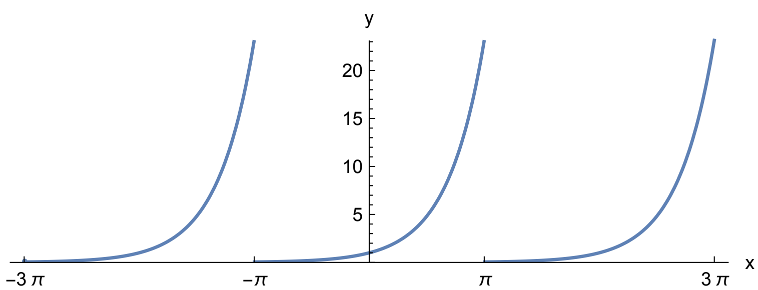
$$f(x) \sim \frac{1}{2}\cos x + \sum_{n=2}^{\infty}\frac{n(1+(-1)^n)}{\pi(n^2-1)}\sin(nx)$$

(e) $f(x) = \begin{cases} 0 & \text{if } -2 < x < 0 \\ 2 & \text{if } 0 < x < 1 \\ 1 & \text{if } 1 < x < 2 \end{cases}$



$$\begin{aligned}
a_0 &= \frac{1}{2}\int_{-2}^{2} f(x)\,dx = \frac{1}{2}\int_0^1 2\,dx + \frac{1}{2}\int_1^2 1\,dx = \frac{3}{2} \\
a_n &= \frac{1}{2}\int_{-2}^{2} f(x)\cos\frac{n\pi x}{2}\,dx = \frac{1}{2}\int_0^1 2\cos\frac{n\pi x}{2}\,dx + \frac{1}{2}\int_1^2 \cos\frac{n\pi x}{2}\,dx = \frac{\sin(n\pi/2)}{n\pi} \\
b_n &= \frac{1}{2}\int_{-2}^{2} f(x)\sin\frac{n\pi x}{2}\,dx = \frac{1}{2}\int_0^1 2\sin\frac{n\pi x}{2}\,dx + \frac{1}{2}\int_1^2 \sin\frac{n\pi x}{2}\,dx = \frac{2-(-1)^n-\cos(n\pi/2)}{n\pi} \\
f(x) &\sim \frac{3}{4} + \sum_{n=1}^{\infty}\left[\frac{\sin(n\pi/2)}{n\pi}\cos\frac{n\pi x}{2} + \frac{2-(-1)^n-\cos(n\pi/2)}{n\pi}\sin\frac{n\pi x}{2}\right]
\end{aligned}$$

(f) $f(x) = e^{a\,x}$, for $-\pi < x < \pi$, where $a \neq 0$ is a constant.



$$\begin{aligned}
a_0 &= \frac{1}{\pi}\int_0^{\pi} e^{ax}\,dx = \frac{2\sinh(a\pi)}{a\pi} \\
a_n &= \frac{1}{\pi}\int_{-\pi}^{\pi} e^{ax}\cos(nx)\,dx = \frac{2(-1)^n a\sinh(a\pi)}{(a^2+n^2)\pi} \\
b_n &= \frac{1}{\pi}\int_{-\pi}^{\pi} e^{ax}\sin(nx)\,dx = -\frac{2n(-1)^n\sinh(a\pi)}{(a^2+n^2)\pi} \\
f(x) &\sim \frac{\sinh(a\pi)}{a\pi} + \sum_{n=1}^{\infty}\left[\frac{2(-1)^n a\sinh(a\pi)}{(a^2+n^2)\pi}\cos(nx) - \frac{2n(-1)^n\sinh(a\pi)}{(a^2+n^2)\pi}\sin(nx)\right]
\end{aligned}$$

3.3.3 If $a \in \mathbb{Z}$ the Fourier series is $\cos(|a|\,x)$. If $a \notin \mathbb{Z}$ then

$$a_0 = \frac{1}{\pi}\int_{-\pi}^{\pi} \cos(a\,x)\,dx = \frac{2\sin(a\pi)}{a\pi}$$
$$a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} \cos(a\,x)\cos(n\,x)\,dx = \frac{2a(-1)^n\sin(a\pi)}{(a^2-n^2)\pi}$$
$$b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} \cos(a\,x)\sin(n\,x)\,dx = 0$$
$$f(x) \sim \frac{\sin(a\pi)}{a\pi} + \sum_{n=1}^{\infty}\left[\frac{2a(-1)^n\sin(a\pi)}{(a^2-n^2)\pi}\cos(nx).\right]$$

3.3.5 In Example 3.4 it was determined that

$$|\sin x| \sim \frac{2}{\pi} - \frac{4}{\pi}\sum_{k=1}^{\infty}\frac{\cos(2kx)}{4k^2-1}.$$

Thus if $x = 0$,

$$|\sin 0| = \frac{2}{\pi} - \frac{4}{\pi}\sum_{k=1}^{\infty}\frac{\cos(0)}{4k^2-1}$$
$$\frac{2}{\pi} = \frac{4}{\pi}\sum_{k=1}^{\infty}\frac{1}{4k^2-1}$$
$$\frac{1}{2} = \sum_{k=1}^{\infty}\frac{1}{4k^2-1}.$$

When $x = \pi/2$,

$$|\sin\frac{\pi}{2}| = \frac{2}{\pi} - \frac{4}{\pi}\sum_{k=1}^{\infty}\frac{\cos(2k\pi/2)}{4k^2-1}$$
$$1 = \frac{2}{\pi} - \frac{4}{\pi}\sum_{k=1}^{\infty}\frac{(-1)^k}{4k^2-1}$$
$$\frac{1}{2} - \frac{\pi}{4} = \sum_{k=1}^{\infty}\frac{(-1)^k}{4k^2-1}.$$

3.4.2 Take the case that $f(x)$ is even then

$$\frac{d}{dx}\left[f(-x)\right] = \frac{d}{dx}\left[f(x)\right]$$
$$-f'(-x) = f'(x)$$
$$f'(-x) = -f'(x),$$

so $f'(x)$ is an odd function. When $f(x)$ is an odd function,

$$\frac{d}{dx}\left[f(-x)\right] = \frac{d}{dx}\left[-f(x)\right]$$
$$-f'(-x) = -f'(x)$$
$$f'(-x) = f'(x),$$

so $f'(x)$ is an even function.

3.4.4

(a) Suppose f and g are even functions, then

$$(f \circ g)(-x) = f(g(-x)) = f(g(x)) = (f \circ g)(x),$$

so $f \circ g$ is an even function.

(b) Suppose f and g are odd functions, then

$$(f \circ g)(-x) = f(g(-x)) = f(-g(x)) = -f(g(x)) = -(f \circ g)(x)$$

so $f \circ g$ is an odd function.

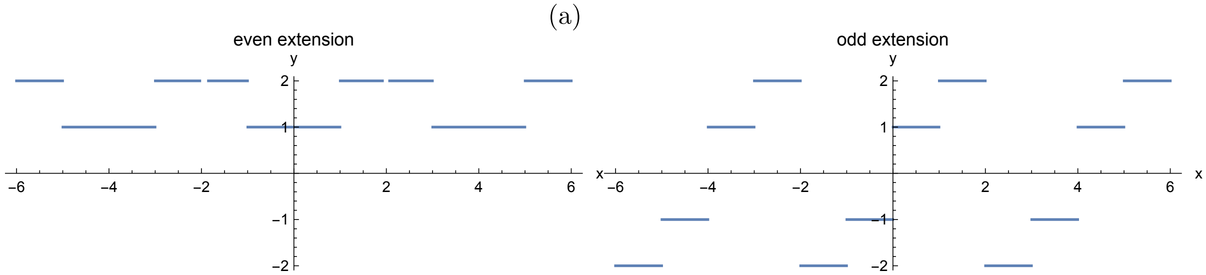
(c) Suppose f is an even function and g is an odd function, then

$$\begin{aligned} (f \circ g)(-x) &= f(g(-x)) = f(-g(x)) = f(g(x)) = (f \circ g)(x) \\ (g \circ f)(-x) &= g(f(-x)) = g(f(x)) = (g \circ f)(x) \end{aligned}$$

thus $f \circ g$ and $g \circ f$ are even functions.

3.5.1

(a)



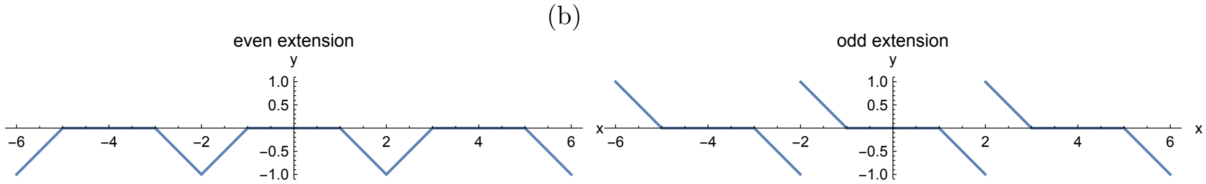
Fourier cosine series:

$$\begin{aligned} a_0 &= \frac{2}{2}\int_0^2 f(x)\,dx = \int_0^1 1\,dx + \int_1^2 2\,dx = 3 \\ a_n &= \frac{2}{2}\int_0^2 f(x)\cos\frac{n\pi x}{2}\,dx = \int_0^1 \cos\frac{n\pi x}{2}\,dx + \int_1^2 2\cos\frac{n\pi x}{2}\,dx = -\frac{2\sin(n\pi/2)}{n\pi} \\ f(x) &\sim \frac{3}{2} - \sum_{n=1}^{\infty} \frac{2\sin(n\pi/2)}{n\pi}\cos\frac{n\pi x}{2} \end{aligned}$$

Fourier sine series:

$$\begin{aligned} b_n &= \frac{2}{2}\int_0^2 f(x)\sin\frac{n\pi x}{2}\,dx = \int_0^1 \sin\frac{n\pi x}{2}\,dx + \int_1^2 2\sin\frac{n\pi x}{2}\,dx = \frac{4}{n\pi}\left(\cos^2\frac{n\pi}{4} - (-1)^n\right) \\ f(x) &\sim \sum_{n=1}^{\infty} \frac{4}{n\pi}\left(\cos^2\frac{n\pi}{4} - (-1)^n\right)\sin\frac{n\pi x}{2} \end{aligned}$$

(b)



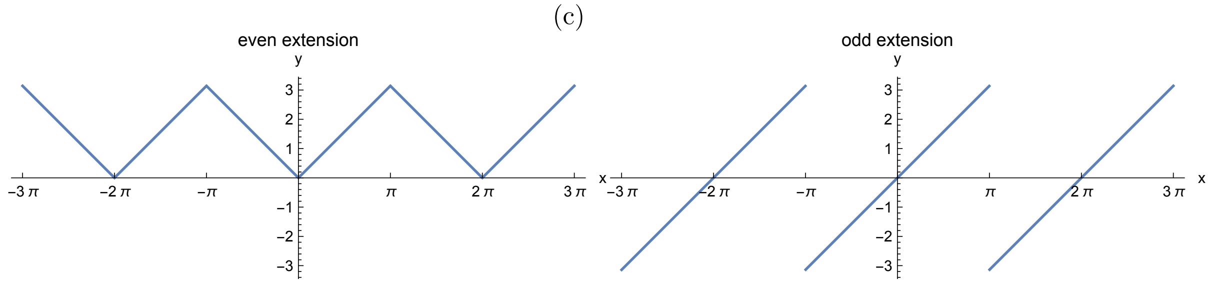
Fourier cosine series:

$$a_0 = \frac{2}{2}\int_0^2 f(x)\,dx = \int_1^2 (1-x)\,dx = -\frac{1}{2}$$
$$a_n = \frac{2}{2}\int_0^2 f(x)\cos\frac{n\pi x}{2}\,dx = \int_1^2 (1-x)\cos\frac{n\pi x}{2}\,dx = \frac{4}{n^2\pi^2}\left(\cos\frac{n\pi}{2} - (-1)^n\right)$$
$$f(x) \sim -\frac{1}{4} + \sum_{n=1}^{\infty} \frac{4}{n^2\pi^2}\left(\cos\frac{n\pi}{2} - (-1)^n\right)\cos\frac{n\pi x}{2}$$

Fourier sine series:

$$b_n = \frac{2}{2}\int_0^2 f(x)\sin\frac{n\pi x}{2}\,dx = \int_1^2 (1-x)\sin\frac{n\pi x}{2}\,dx = \frac{2}{n^2\pi^2}\left(n\pi(-1)^n + 2\sin\frac{n\pi}{2}\right)$$
$$f(x) \sim \sum_{n=1}^{\infty} \frac{2}{n^2\pi^2}\left(n\pi(-1)^n + 2\sin\frac{n\pi}{2}\right)\sin\frac{n\pi x}{2}$$

(c)



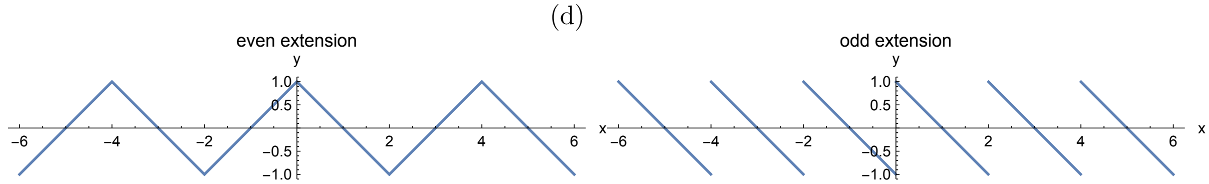
Fourier cosine series:

$$a_0 = \frac{2}{\pi}\int_0^{\pi} x\,dx = \pi$$
$$a_n = \frac{2}{\pi}\int_0^{\pi} x\cos(nx)\,dx = \frac{2}{n^2\pi}(-1+(-1)^n)$$
$$f(x) \sim \frac{\pi}{2} + \sum_{n=1}^{\infty}\frac{2}{n^2\pi}(-1+(-1)^n)\cos(nx)$$

Fourier sine series:

$$b_n = \frac{2}{\pi}\int_0^{\pi} x\sin(nx)\,dx = -\frac{2}{n}(-1)^n$$
$$f(x) \sim -\sum_{n=1}^{\infty}\frac{2(-1)^n}{n}\sin(nx)$$

(d)



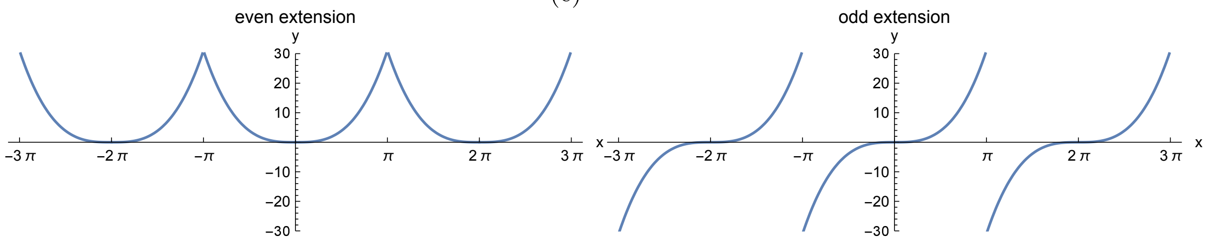
Fourier cosine series:

$$a_0 = \frac{2}{2}\int_0^2 (1-x)\,dx = 0$$
$$a_n = \frac{2}{2}\int_0^2 (1-x)\cos\frac{n\pi x}{2}\,dx = \frac{4}{n^2\pi^2}(1-(-1)^n)$$
$$f(x) \sim \sum_{n=1}^{\infty} \frac{4}{n^2\pi^2}(1-(-1)^n)\cos\frac{n\pi x}{2}$$

Fourier sine series:

$$b_n = \frac{2}{2}\int_0^2 (1-x)\sin\frac{n\pi x}{2}\,dx = \frac{2}{n\pi}(1+(-1)^n)$$
$$f(x) \sim \sum_{n=1}^{\infty} \frac{2}{n\pi}(1+(-1)^n)\sin\frac{n\pi x}{2}$$

(e)



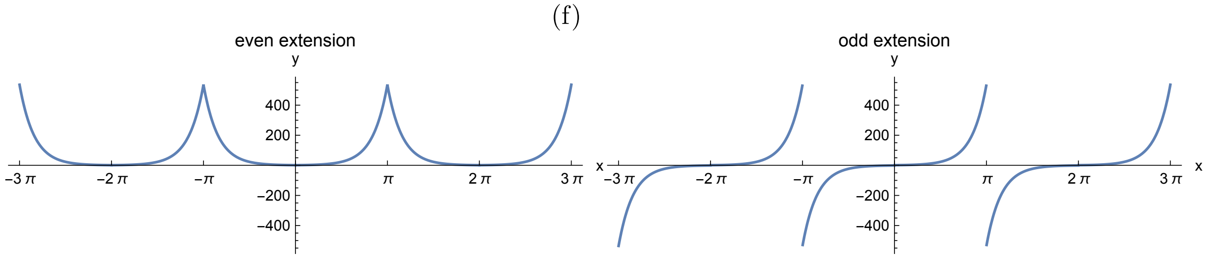
Fourier cosine series:

$$a_0 = \frac{2}{\pi}\int_0^{\pi} x^3\,dx = \frac{\pi^3}{2}$$
$$a_n = \frac{2}{\pi}\int_0^{\pi} x^3\cos(n\,x)\,dx = \frac{2}{n^4\pi}(6+3(-1)^n(n^2\pi^2-2))$$
$$f(x) \sim \frac{\pi^3}{4} + \sum_{n=1}^{\infty} \frac{2}{n^4\pi}(6+3(-1)^n(n^2\pi^2-2))\cos(n\,x)$$

Fourier sine series:

$$b_n = \frac{2}{\pi}\int_0^{\pi} x^3\sin(n\,x)\,dx = \frac{2(-1)^n}{n^3}(6-n^2\pi^2)$$
$$f(x) \sim \sum_{n=1}^{\infty} \frac{2(-1)^n}{n^3}(6-n^2\pi^2)\sin(n\,x)$$

(f)



Fourier cosine series:

$$a_0 = \frac{2}{\pi}\int_0^\pi e^{2x}\,dx = \frac{2e^\pi}{\pi}\sinh\pi$$
$$a_n = \frac{2}{\pi}\int_0^\pi e^{2x}\cos(nx)\,dx = \frac{-4}{(n^2+4)\pi}(1-(-1)^n e^{2\pi})$$
$$f(x) \sim \frac{e^\pi}{\pi}\sinh\pi - \sum_{n=1}^\infty \frac{4}{(n^2+4)\pi}(1-(-1)^n e^{2\pi})\cos(nx)$$

Fourier sine series:

$$b_n = \frac{2}{\pi}\int_0^\pi e^{2x}\sin(nx)\,dx = \frac{2n}{(n^2+4)\pi}(1-(-1)^n e^{2\pi})$$
$$f(x) \sim \sum_{n=1}^\infty \frac{2n}{(n^2+4)\pi}(1-(-1)^n e^{2\pi})\sin(nx)$$

3.5.3 Fourier cosine series:

$$a_0 = \frac{2}{\pi}\int_0^\pi e^{ax}\,dx = \frac{2}{a\pi}(e^{a\pi}-1)$$
$$a_n = \frac{2}{\pi}\int_0^\pi e^{ax}\cos(nx)\,dx = \frac{-2a}{(a^2+n^2)\pi}(1-(-1)^n e^{a\pi})$$
$$f(x) \sim \frac{1}{a\pi}(e^{a\pi}-1) - \sum_{n=1}^\infty \frac{2a}{(a^2+n^2)\pi}(1-(-1)^n e^{a\pi})\cos(nx)$$

Fourier sine series:

$$b_n = \frac{2}{\pi}\int_0^\pi e^{ax}\sin(nx)\,dx = \frac{2n}{(a^2+n^2)\pi}(1-(-1)^n e^{a\pi})$$
$$f(x) \sim \sum_{n=1}^\infty \frac{2n}{(a^2+n^2)\pi}(1-(-1)^n e^{a\pi})\sin(nx)$$

3.5.5

$$a_0 = \frac{1}{L}\int_{-L}^{L} e^x\,dx = \left[\frac{1}{L}e^x\right]_{x=-L}^{x=L} = \frac{2}{L}\sinh L$$
$$a_n = \frac{1}{L}\int_{-L}^{L} e^x\cos\frac{n\pi x}{L}\,dx = \left[\frac{e^x}{n^2\pi^2+L^2}(L\cos\frac{n\pi x}{L} + n\pi\sin\frac{n\pi x}{L})\right]_{x=-L}^{x=L}$$
$$= \frac{2L(-1)^n}{n^2\pi^2+L^2}\sinh L$$
$$b_n = \frac{1}{L}\int_{-L}^{L} e^x\sin\frac{n\pi x}{L}\,dx = \left[\frac{e^x}{n^2\pi^2+L^2}(L\sin\frac{n\pi x}{L} - n\pi\cos\frac{n\pi x}{L})\right]_{x=-L}^{x=L}$$
$$= -\frac{2n\pi(-1)^n}{n^2\pi^2+L^2}\sinh L$$
$$e^x \sim \frac{1}{L}\sinh L + \sum_{k=1}^\infty \left[\frac{2(-1)^n}{n^2\pi^2+L^2}\sinh L\left(L\cos\frac{n\pi x}{L} - n\pi\sin\frac{n\pi x}{L}\right)\right]$$

3.6.1

(a) Piecewise smooth

(b) Neither

(c) Neither

(d) Piecewise continuous

(e) Piecewise smooth

(f) Piecewise continuous

(g) Piecewise smooth

(h) Neither (undefined on $[-1, 0)$)

3.6.3 Fourier sine series:

$$\begin{aligned}
b_n &= \frac{2}{\pi}\int_0^\pi f(x)\sin(nx)\,dx \\
&= \frac{2}{\pi}\int_0^1 x\sin(nx)\,dx - \frac{2}{\pi}\int_1^2 \sin(nx)\,dx + \frac{2}{\pi}\int_2^\pi 2\sin(nx)\,dx \\
&= \frac{2}{n^2\pi}\left(-2n(-1)^n - 2n\cos n + 3n\cos(2n) + \sin n\right) \\
f(x) &\sim \sum_{n=1}^\infty \frac{2}{n^2\pi}\left(-2n(-1)^n - 2n\cos n + 3n\cos(2n) + \sin n\right)\sin(nx) \\
&= \begin{cases} 0 & \text{if } x = -\pi \\ -2 & \text{if } -\pi < x < -2 \\ -1/2 & \text{if } x = -2 \\ 1 & \text{if } -2 < x < -1 \\ 0 & \text{if } x = -1 \\ x & \text{if } -1 < x < 1 \\ 0 & \text{if } x = 1 \\ -1 & \text{if } 1 < x < 2 \\ 1/2 & \text{if } x = 2 \\ 2 & \text{if } 2 < x < \pi \\ 0 & \text{if } x = \pi \end{cases}
\end{aligned}$$

Fourier cosine series:

$$\begin{aligned}
a_0 &= \frac{2}{\pi}\int_0^\pi f(x)\,dx \\
&= \frac{2}{\pi}\int_0^1 x\,dx - \frac{2}{\pi}\int_1^2 1\,dx + \frac{2}{\pi}\int_2^\pi 2\,dx \\
&= 4 - \frac{9}{\pi} \\
a_n &= \frac{2}{\pi}\int_0^\pi f(x)\cos(nx)\,dx \\
&= \frac{2}{\pi}\int_0^1 x\cos(nx)\,dx - \frac{2}{\pi}\int_1^2 \cos(nx)\,dx + \frac{2}{\pi}\int_2^\pi 2\cos(nx)\,dx \\
&= \frac{2}{n^2\pi}\left(-1+\cos n + 2n\sin n - 3n\sin(2n)\right) \\
f(x) &\sim 2 - \frac{9}{2\pi} + \sum_{n=1}^\infty \frac{2}{n^2\pi}\left(-1+\cos n + 2n\sin n - 3n\sin(2n)\right)\cos(nx) \\
&= \begin{cases} 2 & \text{if } -\pi \le x < -2 \\ -1/2 & \text{if } x = -2 \\ -1 & \text{if } -2 < x < -1 \\ 0 & \text{if } x = -1 \\ |x| & \text{if } -1 < x < 1 \\ 0 & \text{if } x = 1 \\ -1 & \text{if } 1 < x < 2 \\ 1/2 & \text{if } x = 2 \\ 2 & \text{if } 2 < x \le \pi \end{cases}
\end{aligned}$$

3.6.5 If f is continuous at x_0 then by the Dirichlet Convergence Theorem, the Fourier series evaluated at $x = x_0$ converges to $f(x_0)$. Thus $f(x_0) = 0$.

3.6.7

$$\begin{aligned}
a_0 &= \int_{-1}^1 f(x)\,dx = \int_{-1}^0 1\,dx + \int_0^1 2\,dx = 3 \\
a_n &= \int_{-1}^1 f(x)\cos(n\pi x)\,dx = \int_{-1}^0 \cos(n\pi x)\,dx + \int_0^1 2\cos(n\pi x)\,dx = 0 \\
b_n &= \int_{-1}^1 f(x)\sin(n\pi x)\,dx = \int_{-1}^0 \sin(n\pi x)\,dx + \int_0^1 2\sin(n\pi x)\,dx = \frac{1-(-1)^n}{n\pi} \\
f(x) &\sim \frac{3}{2} + \sum_{n=1}^\infty \frac{2}{(2n-1)\pi}\sin((2n-1)\pi x)
\end{aligned}$$

Let $x = 1/2$, then

$$\begin{aligned}
f(1/2) &= \frac{3}{2} + \sum_{n=1}^\infty \frac{2}{(2n-1)\pi}\sin((2n-1)\pi/2) \\
2 &= \frac{3}{2} + \frac{2}{\pi}\sum_{n=1}^\infty \frac{(-1)^{n-1}}{2n-1} \\
\frac{\pi}{4} &= \sum_{n=1}^\infty \frac{(-1)^{n-1}}{2n-1}.
\end{aligned}$$

3.7.1 Let $x_0 \in (-1,1)$ then the sum $\sum_{k=0}^{\infty} x_0^k$ is a geometric series which converges to $1/(1-x_0)$. Since x_0 was arbitrary in $(-1,1)$ then the series converges pointwise on the interval.
3.7.3 Let $\epsilon > 0$ and assume the sequence $\{f_k\}_{k=1}^{\infty}$ converges uniformly to f on $[a,b]$. By definition there exists $N \in \mathbb{N}$ such that for all $ngeN$, then $|f_n(x) - f(x)|^2 < \epsilon^2$ for all $x \in [a,b]$. This implies

$$\left(\int_a^b |f_n(x) - f(x)|^2 \, dx \right)^{1/2} < \left(\int_a^b \epsilon^2 \, dx \right)^{1/2} = \epsilon\sqrt{b-a},$$

for all $n \geq N$. Since ϵ is arbitrary then $f_n \to f$ on $[a,b]$ in the L^2 sense.
3.7.5

(a) The even extension of $f(x)$ is continuous on $[-2,2]$, $f'(x)$ is piecewise continuous on $[-2,2]$, and $f(-2) = f(2)$. Thus the Fourier cosine series for $f(x)$ converges uniformly to $f(x)$ on $[-2,2]$.

(b) Let $M_n = \frac{1}{(2k-1)^2}$, then

$$\sum_{k=1}^{\infty} \frac{1}{(2k-1)^2} \cos \frac{(2k-1)\pi x}{2} \leq \sum_{k=1}^{\infty} \frac{1}{(2k-1)^2}.$$

Applying Parseval's identity to the Fourier series of the function $f(x)$ in Exercise 3.6.7 produces

$$\frac{1}{2} \int_{-1}^{1} (f(x))^2 \, dx = \frac{5}{2} = \frac{a_0^2}{4} + \frac{1}{2} \sum_{n=1}^{\infty} (a_n^2 + b_n^2) = \frac{9}{4} + \frac{2}{\pi^2} \sum_{n=1}^{\infty} \frac{1}{(2n-1)^2}.$$

This implies $\sum_{n=1}^{\infty} \frac{1}{(2n-1)^2} = \pi^2/8$. Hence

$$\sum_{k=1}^{\infty} \frac{1}{(2k-1)^2} \cos \frac{(2k-1)\pi x}{2} \leq \frac{\pi^2}{8}$$

and by the Weierstrass M-test the Fourier cosine series converges uniformly to $f(x)$ and $[0,2]$.
3.7.7

(a) Consider the limit of the sequence as $n \to \infty$.

$$\lim_{n\to\infty} \frac{nx + \cos(nx^2)}{n} = \lim_{n\to\infty} \left(x + \frac{\cos(nx^2)}{n} \right) = x$$

(b) Let $\epsilon > 0$ and let $N = 1 + [\![\frac{1}{\epsilon}]\!]$. For all $n \geq N$,

$$|f_n(x) - x| = \left| \frac{nx + \cos(nx^2)}{n} - x \right| = \left| \frac{\cos(nx^2)}{n} \right| \leq \frac{1}{n} < \epsilon$$

for all $x \in \mathbb{R}$. Thus the convergence is uniform.

3.8.2 The even extension of $f(x)$ is continuous on $[-\pi,\pi]$, $f'(x)$ is piecewise continuous on $[-\pi,\pi]$, and $f(-\pi) = f(\pi)$. Thus the Fourier cosine series for $f(x)$ converges uniformly to $f(x)$ on $[-\pi,\pi]$. Thus the Fourier cosine series for $f(x)$ can be differentiated term by term.
3.8.4 Let the Fourier series representation of $f(x)$ be

$$f(x) \sim \frac{a_0}{2} + \sum_{n=1}^{\infty} \left(a_n \cos \frac{n\pi x}{L} + b_n \sin \frac{n\pi x}{L} \right),$$

then by Thm. 3.5 the Fourier series representation of $f'(x)$ is

$$f'(x) \sim \frac{\pi}{L}\sum_{n=1}^{\infty}\left(nb_n \cos\frac{n\pi x}{L} - na_n \sin\frac{n\pi x}{L}\right).$$

By Bessel's inequality

$$\frac{\pi^2}{L^2}\sum_{n=1}^{\infty}\left((na_n)^2 + (nb_n)^2\right) \le \frac{1}{L}\int_{-L}^{L}(f'(x))^2\,dx < \infty,$$

since $f'(x)$ is piecewise continuous on $[-L, L]$. Thus

$$\lim_{n\to\infty}(na_n) = 0 = \lim_{n\to\infty}(nb_n).$$

3.9.1

(a)

$$\begin{aligned}\frac{1}{2\pi}\int_{-\pi}^{\pi}x^2\,dx &= \frac{1}{2}\sum_{n=1}^{\infty}\left(\frac{2(-1)^{n+1}}{n}\right)^2\\ \frac{\pi^2}{3} &= 2\sum_{n=1}^{\infty}\frac{1}{n^2}\\ \frac{\pi^2}{6} &= \sum_{n=1}^{\infty}\frac{1}{n^2}\end{aligned}$$

(b)

$$\begin{aligned}\frac{\pi^2}{6} &= \sum_{n=1}^{\infty}\frac{1}{n^2} = \sum_{n=1}^{\infty}\frac{1}{(2n-1)^2} + \sum_{n=1}^{\infty}\frac{1}{(2n)^2}\\ \frac{\pi^2}{6} - \left(\frac{1}{4}\right)\left(\frac{\pi^2}{6}\right) = \frac{\pi^2}{8} &= \sum_{n=1}^{\infty}\frac{1}{(2n-1)^2}\end{aligned}$$

3.9.3 On the interval $(-\pi, \pi)$

$$x^3 \sim \sum_{n=1}^{\infty}\frac{2(-1)^n(6 - n^2\pi^2)}{n^3}\sin(nx).$$

According to Parseval's identity

$$\begin{aligned}\frac{1}{2\pi}\int_{-\pi}^{\pi}(x^3)^2\,dx &= \frac{1}{2}\sum_{n=1}^{\infty}\left[\frac{2(-1)^n(6 - n^2\pi^2)}{n^3}\right]^2\\ \frac{\pi^6}{7} &= \sum_{n=1}^{\infty}\frac{72}{n^6} - \sum_{n=1}^{\infty}\frac{24\pi^2}{n^4} + \sum_{n=1}^{\infty}\frac{2\pi^4}{n^2}\\ \frac{\pi^6}{7} + \frac{4\pi^6}{15} - \frac{\pi^6}{3} &= 72\sum_{n=1}^{\infty}\frac{1}{n^6}\\ \frac{\pi^6}{945} &= \sum_{n=1}^{\infty}\frac{1}{n^6}.\end{aligned}$$

Taking just the odd indexed terms

$$\frac{\pi^6}{945} = \sum_{n=1}^{\infty} \frac{1}{n^6} = \sum_{n=1}^{\infty} \frac{1}{(2n-1)^6} + \sum_{n=1}^{\infty} \frac{1}{(2n)^6}$$

$$\frac{\pi^6}{945} - \left(\frac{1}{64}\right)\left(\frac{\pi^6}{945}\right) = \frac{\pi^6}{960} = \sum_{n=1}^{\infty} \frac{1}{(2n-1)^6}$$

3.9.5 We may define the odd, 2π-periodic extension of f as

$$f_o(x) = \begin{cases} -f(-x) & \text{if } -\pi \le x < 0 \\ 0 & \text{if } x = 0 \\ f(x) & \text{if } 0 < x \le \pi. \end{cases}$$

Now calculate the Fourier series coefficients for $f_o(x)$ on $[-\pi, \pi]$.

$$\begin{aligned} a_0 &= \frac{1}{\pi}\int_{-\pi}^{\pi} f_o(x)\,dx = 0 \\ a_n &= \frac{1}{\pi}\int_{-\pi}^{\pi} f_o(x)\cos(nx)\,dx = 0 \\ b_n &= \frac{1}{\pi}\int_{-\pi}^{\pi} f_o(x)\sin(nx)\,dx = \frac{2}{\pi}\int_{0}^{\pi} f(x)\sin(nx)\,dx = 0 \end{aligned}$$

By Parseval's identity,

$$\frac{1}{2\pi}\int_{-\pi}^{\pi} (f_o(x))^2\,dx = 0$$

which implies $(f_o(x))^2 = 0$ on $[-\pi, \pi]$ which in turn implies $f(x) = 0$ on $[0, \pi]$.
3.10.2 Let m, $n \in \mathbb{Z}$ with $m \neq n$, then

$$\begin{aligned} \langle e^{im\pi x/L}, e^{in\pi x/L}\rangle &= \int_{-L}^{L} e^{im\pi x/L}\overline{e^{in\pi x/L}}\,dx \\ &= \int_{-L}^{L} e^{im\pi x/L}e^{-in\pi x/L}\,dx \\ &= \int_{-L}^{L} e^{i(m-n)\pi x/L}\,dx \\ &= \left[\frac{L}{i(m-n)\pi}e^{i(m-n)\pi x/L}\right]_{x=-L}^{x=L} \\ &= \frac{L}{i(m-n)\pi}\left(e^{i(m-n)\pi} - e^{-i(m-n)\pi}\right) \\ &= \frac{L}{i(m-n)\pi}\left(\cos((m-n)\pi) - \cos(-(m-n)\pi)\right) \\ &= 0. \end{aligned}$$

When $m = n$, then

$$\begin{aligned}\langle e^{in\pi x/L}, e^{in\pi x/L}\rangle &= \int_{-L}^{L} e^{in\pi x/L}\overline{e^{in\pi x/L}}\,dx\\ &= \int_{-L}^{L} e^{in\pi x/L}e^{-in\pi x/L}\,dx\\ &= \int_{-L}^{L} 1\,dx\\ &= 2L.\end{aligned}$$

3.10.4 For $n \in \mathbb{Z}$,

$$\begin{aligned}c_n &= \frac{1}{2L}\int_{-L}^{L} e^x e^{-in\pi x/L}\,dx\\ &= \frac{1}{2L}\int_{-L}^{L} e^x e^{\left(1-\frac{in\pi}{L}\right)x}\,dx\\ &= \left[\frac{1}{2(L-in\pi)}e^{\left(1-\frac{in\pi}{L}\right)x}\right]_{x=-L}^{x=L}\\ &= \frac{1}{2(L-in\pi)}\left(e^{(L-in\pi)} - e^{-(L-in\pi)}\right)\\ &= \frac{\sinh(L-in\pi)}{L-in\pi}.\end{aligned}$$

Therefore

$$e^x = \sum_{n=-\infty}^{\infty}\frac{\sinh(L-in\pi)}{L-in\pi}e^{in\pi x/L}.$$

3.10.6 By definition,

$$\begin{aligned}\sinh z &= \frac{e^z - e^{-z}}{2} = \frac{e^{a+bi} - e^{-a-bi}}{2} = \frac{e^{a+bi} - e^{-a-bi}}{2} = \frac{e^a e^{bi} - e^{-a}e^{-bi}}{2}\\ &= \frac{e^a(\cos b + i\sin b) - e^{-a}(\cos b - i\sin b)}{2} = \frac{e^a\cos b - e^{-a}\cos b + i(e^a\sin b + e^{-a}\sin b)}{2} = \sinh a\cos b + i\cosh a\sin b\end{aligned}$$

Hence $\operatorname{Re}(\sinh z) = \sinh a\cos b$ and $\operatorname{Im}(\sinh z) = \cosh a\sin b$.

3.11.2 The limit

$$\lim_{x\to 2kL} D_N(x) = \lim_{x\to 2kL}\frac{\sin\left(\frac{(2N+1)\pi x}{2L}\right)}{2\sin\frac{\pi x}{2L}}$$

is indeterminate of the form $0/0$. Apply l'Hôpital's rule.

$$\begin{aligned}\lim_{x\to 2kL}\frac{\frac{(2N+1)\pi}{2L}\cos\left(\frac{(2N+1)\pi x}{2L}\right)}{\frac{2\pi}{2L}\cos\frac{\pi x}{2L}} &= \frac{(2N+1)\cos((2N+1)k\pi)}{2\cos(k\pi)}\\ &= \frac{2N+1}{2}\end{aligned}$$

for each integer k.

3.11.4

$$\begin{aligned}\left|a_k\cos\frac{k\pi x}{L} + b_k\sin\frac{k\pi x}{L}\right| &= \left|R_k\sin\delta\cos\frac{k\pi x}{L} + R_k\cos\delta\sin\frac{k\pi x}{L}\right|\\ &= \left|R_k\sin\left(\frac{k\pi x}{L}+\delta\right)\right| \le |R_k|\end{aligned}$$

where $a_k = R_k \sin\delta$ and $b_k = R_k \cos\delta$.

$$R_k^2 = a_k^2 + b_k^2 \implies |R_k| = \sqrt{a_k^2 + b_k^2}$$

4.1.1 Using separation of variables and the boundary conditions, the general solution of the boundary value problem is

$$u(x,t) = \sum_{n=1}^{\infty} b_n e^{-100n^2\pi^2 t} \sin(n\pi x).$$

If $t = 0$ then

$$u(x,0) = \sum_{n=1}^{\infty} b_n \sin(n\pi x) = \sin 2\pi x - \sin 5\pi x$$

and thus $b_2 = 1$, $b_5 = -1$, and $b_n = 0$ if $n \neq 2$ and $n \neq 5$. Thus the solution to the initial boundary value problem is

$$u(x,t) = e^{-400\pi^2 t} \sin(2\pi x) - e^{-2500\pi^2 t} \sin(5\pi x).$$

4.1.3 Using separation of variables and the boundary conditions, the general solution of the boundary value problem is

$$u(x,t) = \sum_{n=1}^{\infty} b_n e^{-100n^2\pi^2 t} \sin(n\pi x).$$

If $t = 0$ then

$$u(x,0) = \sum_{n=1}^{\infty} b_n \sin(n\pi x) = \begin{cases} 0 & \text{if } 0 \le x < 1/4 \\ 100 & \text{if } 1/4 \le x \le 3/4 \\ 0 & \text{if } 3/4 < x \le 1. \end{cases}$$

The coefficients

$$\begin{aligned} b_n &= 2\int_0^1 u(x,0)\sin(n\pi x)\,dx = 200\int_{1/4}^{3/4} \sin(n\pi x)\,dx = 200\left[-\frac{1}{n\pi}\cos(n\pi x)\right]_{x=1/4}^{x=3/4} \\ &= \frac{200}{n\pi}\left(\cos\frac{n\pi}{4} - \cos\frac{3n\pi}{4}\right). \end{aligned}$$

Thus the solution to the initial boundary value problem is

$$u(x,t) \sim \frac{200}{\pi}\sum_{n=1}^{\infty}\left(\cos\frac{n\pi}{4} - \cos\frac{3n\pi}{4}\right)\frac{e^{-n^2\pi^2 t}}{n}\sin(n\pi x).$$

4.1.5 The following initial boundary value problem models the metal rod.

$$\begin{aligned} u_t &= \frac{1}{4}u_{xx} \text{ for } 0 < x < 1 \text{ and } t > 0 \\ u(0,t) &= u(1,t) = 0 \text{ for } t > 0 \\ u(x,0) &= \begin{cases} 50 & \text{if } 0 < x < 1/2 \\ 0 & \text{if } 1/2 \le x \le 1 \end{cases} \end{aligned}$$

The coefficients of the Fourier sine series formal solution are

$$b_n = 2\int_0^{1/2} 50\sin(n\pi x)\,dx = \frac{200}{n\pi}\sin^2\left(\frac{n\pi}{4}\right).$$

Thus the formal solution can be expressed as

$$u(x,t) = \frac{200}{\pi}\sum_{n=1}^{\infty}\frac{e^{-n^2\pi^2 t/4}}{n}\sin^2\left(\frac{n\pi}{4}\right)\sin(n\pi x).$$

4.1.7 Assuming a product solution of the form $u(x,t) = X(x)T(t)$ then

$$\begin{aligned} X(x)T'(t) &= X''(x)T(t) \\ \frac{X(x)T'(t)}{X(x)T(t)} &= \frac{X''(x)T(t)}{X(x)T(t)} \\ \frac{T'(t)}{T(t)} &= \frac{X''(x)}{X(x)} = -c \end{aligned}$$

where c is a constant. The function $X(x)$ must satisfy the boundary value problem:

$$\begin{aligned} X''(x) + cX(x) = 0 \\ X(0) = X'(1) = 0. \end{aligned}$$

If $c \leq 0$ there are no nontrivial solutions. If $c = \lambda^2 > 0$ with $\lambda > 0$ the general solution to the ordinary differential equation is $X(x) = A\cos(\lambda x) + B\sin(\lambda x)$. Since $X(0) = A$ then $A = 0$.

$$X'(1) = B\lambda\cos(\lambda) = 0 \implies \lambda = \lambda_n = \frac{(2n-1)\pi}{2}$$

Thus the eigenvalues and associated eigenfunctions are $\lambda_n = (2n-1)\pi/2$ and $X_n(x) = \sin((2n-1)\pi x/2)$ for $n \in \mathbb{N}$. The formal solution has the form

$$u(x,t) \sim \sum_{n=1}^{\infty} b_n e^{-(2n-1)^2\pi^2 t/4} \sin\frac{(2n-1)\pi x}{2}.$$

The Fourier series coefficients are

$$b_n = 2\int_0^1 x(2-x)\sin\frac{(2n-1)\pi x}{2}\,dx = \frac{32}{(2n-1)^3\pi^3}$$

and thus the formal solution is

$$u(x,t) \sim \frac{32}{\pi^3}\sum_{n=1}^{\infty} \frac{e^{-(2n-1)^2\pi^2 t/4}}{(2n-1)^3} \sin\frac{(2n-1)\pi x}{2}.$$

4.1.9 Let $u(x,t) = X(x)T(t)$ then

$$\begin{aligned} X(x)T'(t) &= \kappa X''(x)T(t) - cX(x)T(t) \\ \frac{X(x)T'(t)}{\kappa X(x)T(t)} &= \frac{\kappa X''(x)T(t)}{\kappa X(x)T(t)} - \frac{cX(x)T(t)}{\kappa X(x)T(t)} \\ \frac{T'(t)}{\kappa T(t)} &= \frac{X''(x)}{X(x)} - \frac{c}{\kappa} = -\gamma \end{aligned}$$

where γ is a constant. Thus function $X(x)$ must satisfy the boundary value problem

$$\begin{aligned} X''(x) + \left(\gamma - \frac{c}{\kappa}\right)X(x) = 0 \\ X(0) = X(L) = 0. \end{aligned}$$

If $\gamma \leq c/\kappa$ there are no nontrivial solutions to the boundary value problem. If $\gamma > c/\kappa$ the general solution to the ordinary differential equation is

$$X(x) = A\cos\left(\sqrt{\gamma - \frac{c}{\kappa}}x\right) + B\sin\left(\sqrt{\gamma - \frac{c}{\kappa}}x\right).$$

Since $X(0) = 0$ then $A = 0$. Since $X(L) = 0$ then

$$B \sin\left(\sqrt{\gamma - \frac{c}{\kappa}} L\right) = 0 \implies \gamma = \gamma_n = \frac{c}{\kappa} + \frac{n^2\pi^2}{L^2}$$

where $n \in \mathbb{N}$. The eigenfunctions are

$$X_n(x) = \sin \frac{n\pi x}{L}.$$

Substituting the eigenvalues into the ordinary differential equation for $T(t)$ produces

$$T_n(t) = e^{-(c+\kappa n^2\pi^2/L^2)t}.$$

Thus the product solutions take the form

$$u_n(x,t) = e^{-(c+\kappa n^2\pi^2/L^2)t} \sin \frac{n\pi x}{L}.$$

The formal solution to the initial boundary value problem is

$$u(x,t) = \sum_{n=1}^{\infty} b_n e^{-(c+\kappa n^2\pi^2/L^2)t} \sin \frac{n\pi x}{L}$$

where

$$b_n = \frac{2}{L} \int_0^L f(x) \sin \frac{n\pi x}{L}\, dx.$$

4.1.11

(a) Suppose $X(x)$ is a solution of the boundary value problem

$$\begin{aligned} X''(x) + \lambda^2 X(x) &= 0 \\ X(0) = 0 \text{ and } X'(1) + X(1) &= 0. \end{aligned}$$

The general solution to the ordinary differential equation has the form

$$X(x) = A\cos(\lambda x) + B\sin(\lambda x).$$

Since $X(0) = 0$ then $A = 0$. Thus $X(x) = B\sin(\lambda x)$ and $X'(x) = \lambda B\cos(\lambda x)$. The boundary condition at $x = 1$ implies the equation.

$$\lambda\cos(\lambda) + \sin(\lambda) = 0 \implies \tan\lambda = -\lambda$$

Numerically solving the equation produces the first four positive roots

$$\lambda_1 \approx 2.02876 < \lambda_2 \approx 4.91318 < \lambda_3 \approx 7.97867 < \lambda_4 \approx 11.0855.$$

(b) Product solutions:

$$\begin{aligned} u_1(x,t) &= e^{-2\lambda_1^2 t}\sin(\lambda_1 x) \\ u_2(x,t) &= e^{-2\lambda_2^2 t}\sin(\lambda_2 x) \\ u_3(x,t) &= e^{-2\lambda_3^2 t}\sin(\lambda_3 x) \\ u_4(x,t) &= e^{-2\lambda_4^2 t}\sin(\lambda_4 x) \end{aligned}$$

4.2.2

(a) Steady-state temperature distribution:

$$U(x) = A + \frac{(B-A)x}{L}.$$

(b) Initial boundary value problem solved by $v(x,t)$.

$$\begin{aligned} v_t &= \kappa v_{xx} \text{ for } 0 < x < L,\, t > 0 \\ v(0,t) &= 0 \\ v(L,t) &= 0 \\ v(x,0) &= \sin\frac{\pi x}{L} - A - \frac{(B-A)x}{L} \end{aligned}$$

(c) Fourier series for $v(x,t)$:

$$\begin{aligned} v(x,t) &= \sum_{n=1}^{\infty} b_n e^{-\kappa n^2\pi^2 t/L^2} \sin\frac{n\pi x}{L} \\ v(x,0) &= \sum_{n=1}^{\infty} b_n \sin\frac{n\pi x}{L} = \sin\frac{\pi x}{L} - A - \frac{(B-A)x}{L} \\ b_n &= \frac{2}{L}\int_0^L \left(\sin\frac{\pi x}{L} - A - \frac{(B-A)x}{L}\right)\sin\frac{n\pi x}{L}\,dx \\ b_1 &= \frac{2}{L}\int_0^L \left(\sin\frac{\pi x}{L} - A - \frac{(B-A)x}{L}\right)\sin\frac{\pi x}{L}\,dx = 1 - \frac{2}{\pi}(A+B) \\ b_n &= \frac{2}{L}\int_0^L \left(\sin\frac{\pi x}{L} - A - \frac{(B-A)x}{L}\right)\sin\frac{n\pi x}{L}\,dx \\ &= -\frac{2}{L}\int_0^L \left(A + \frac{(B-A)x}{L}\right)\sin\frac{n\pi x}{L}\,dx = \frac{2}{n\pi}((-1)^n B - A) \text{ (for } n \geq 2) \\ v(x,t) &\sim \sin\frac{\pi x}{L} + \frac{2}{\pi}\sum_{n=1}^{\infty} \frac{((-1)^n B - A}{n} e^{-\kappa n^2\pi^2 t/L^2} \sin\frac{n\pi x}{L}. \end{aligned}$$

(d) Solution to the original initial boundary value problem:

$$u(x,t) \sim A + \frac{(B-A)x}{L} + \sin\frac{\pi x}{L} + \frac{2}{\pi}\sum_{n=1}^{\infty} \frac{((-1)^n B - A}{n} e^{-\kappa n^2\pi^2 t/L^2} \sin\frac{n\pi x}{L}$$

4.2.4

(a) Reference temperature distribution $r(x,t)$:

$$\begin{aligned} r_x(x,t) &= Bx/L \\ r(x,t) &= Bx^2/(2L). \end{aligned}$$

(b) Initial boundary value problem solved by $v(x,t)$:

For $0 < x < L$ and $t > 0$,

$$\begin{aligned} u_t &= \kappa u_{xx} \\ (v+r)_t &= \kappa (v+r)_{xx} \\ v_t + 0 &= \kappa \left(v_{xx} + \frac{B}{L} \right) \\ v_t &= \kappa v_{xx} + \frac{\kappa B}{L}. \end{aligned}$$

At $x = 0$,

$$\begin{aligned} u_x(0,t) &= 0 \\ (v+r)_x(0,t) &= 0 \\ v_x(0,t) + 0 &= 0 \\ v_x(0,t) &= 0. \end{aligned}$$

At $x = L$,

$$\begin{aligned} u_x(L,t) &= B \\ (v+r)_x(L,t) &= B \\ v_x(L,t) + B &= B \\ v_x(L,t) &= 0. \end{aligned}$$

At $t = 0$,

$$\begin{aligned} u(x,0) &= f(x) \\ v(x,0) + r(x,0) &= f(x) \\ v(x,0) &= f(x) - \frac{Bx^2}{2L}. \end{aligned}$$

Thus $v(x,t)$ must satisfy the following initial boundary value problem.

$$\begin{aligned} v_t &= \kappa v_{xx} + \frac{\kappa B}{L} \text{ for } 0 < x < L \text{ and } t > 0 \\ v_x(0,t) &= 0 \\ v_x(L,t) &= 0 \\ v(x,0) &= f(x) - \frac{Bx^2}{2L} \end{aligned}$$

(c) Fourier series for $v(x,t)$:

Suppose $v(x,t) = v_1(x,t) + v_2(x,t)$ where $v_1(x,t)$ solves the initial boundary value problem

$$\begin{aligned} (v_1)_t &= \kappa (v_1)_{xx} \text{ for } 0 < x < L \text{ and } t > 0 \\ (v_1)_x(0,t) &= 0 \\ (v_1)_x(L,t) &= 0 \\ v_1(x,0) &= f(x) - \frac{Bx^2}{2L} \end{aligned}$$

and $v_2(x,t)$ solves the initial boundary value problem

$$\begin{aligned}(v_2)_t &= \kappa(v_2)_{xx} + \frac{\kappa B}{L} \text{ for } 0 < x < L \text{ and } t > 0\\ (v_2)_x(0,t) &= 0\\ (v_2)_x(L,t) &= 0\\ v_2(x,0) &= 0.\end{aligned}$$

Then $v_1(x,t)$ can be expressed formally as

$$v_1(x,t) \sim \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n e^{-\kappa n^2\pi^2 t/L^2} \cos\left(\frac{n\pi x}{L}\right)$$

where

$$a_n = \frac{2}{L}\int_0^L \left(f(x) - \frac{Bx^2}{2L}\right)\cos\left(\frac{n\pi x}{L}\right)\, dx.$$

Assume $v_2(x,t)$ can be expressed as

$$v_2(x,t) \sim \frac{a_0(t)}{2} + \sum_{n=1}^{\infty} a_n(t) e^{-\kappa n^2\pi^2 t/L^2} \cos\left(\frac{n\pi x}{L}\right).$$

If we differentiate this solution we obtain

$$\begin{aligned}(v_2)_t &\sim \frac{a_0'(t)}{2} + \sum_{n=1}^{\infty}\left(a_n'(t) - \frac{\kappa n^2\pi^2}{L^2}a_n(t)\right) e^{-\kappa n^2\pi^2 t/L^2}\cos\left(\frac{n\pi x}{L}\right)\\ (v_2)_{xx} &\sim -\sum_{n=1}^{\infty}\frac{n^2\pi^2}{L^2}a_n(t)e^{-\kappa n^2\pi^2 t/L^2}\cos\left(\frac{n\pi x}{L}\right).\end{aligned}$$

Therefore $(v_2)_t - \kappa(v_2)_{xx} = \kappa B/L$ can be expressed as

$$\frac{\kappa B}{L} \sim \frac{a_0'(t)}{2} + \sum_{n=1}^{\infty} a_n'(t)e^{-\kappa n^2\pi^2 t/L^2}\cos\left(\frac{n\pi x}{L}\right).$$

Thus we can make $a_0'(t) = 2\kappa B/L$ and $a_n'(t) = 0$ for $n \geq 1$. Consequently $a_0(t) = 2\kappa Bt/L$ and $a_n(t) = 0$ for $n \geq 0$. This implies

$$v_2(x,t) = \frac{\kappa Bt}{L}.$$

Therefore

$$v(x,t) \sim \frac{\kappa Bt}{L} + \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n e^{-\kappa n^2\pi^2 t/L^2}\cos\left(\frac{n\pi x}{L}\right).$$

(d) Solution to the original initial boundary value problem:

$$u(x,t) \sim \frac{Bx^2}{2L} + \frac{\kappa Bt}{L} + \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n e^{-\kappa n^2\pi^2 t/L^2}\cos\left(\frac{n\pi x}{L}\right).$$

4.2.6

(a) Reference temperature distribution $r(x,t)$:

Let $r(x,t)$ be the function which interpolates between the nonhomogeneous boundary conditions.

$$r(x,t) = 1 - x + \frac{xt}{2}$$

(b) Let $u(x,t) = v(x,t) + r(x,t)$ then $u_t = v_t + \frac{x}{2}$ and $u_{xx} = v_{xx}$. Likewise $u(0,t) = v(0,t) + r(0,t) = v(0,t) + 1 = 1$ and $u(1,t) = v(1,t) + r(1,t) = v(1,t) + \frac{t}{2} = \frac{t}{2}$. When $t = 0$, $u(x,0) = v(x,0) + r(x,0) = v(x,0) + 1 - x$. Consequently $v(x,t)$ satisfies the initial boundary value problem below.

$$\begin{aligned} v_t &= v_{xx} \text{ for } 0 < x < 1 \text{ and } t > 0 \\ v(0,t) &= 0 \text{ for } t > 0 \\ v(1,t) &= 0 \text{ for } t > 0 \\ v(x,0) &= x - x^2 \text{ for } 0 < x < 1 \end{aligned}$$

(c) Fourier series for $v(x,t)$:

$$\begin{aligned} b_n &= 2\int_0^1 (x - x^2)\sin(n\pi x)\,dx \\ &= \left[\frac{2x^2 - 2x}{n\pi}\cos(n\pi x)\right]_{x=0}^{x=1} - \int_0^1 \frac{(4x-2)}{n\pi}\cos(n\pi x)\,dx \\ &= \int_0^1 \frac{4}{n^2\pi^2}\sin(n\pi x)\,dx \\ &= \begin{cases} 0 & \text{if } n \text{ is even} \\ \frac{8}{n^3\pi^3} & \text{if } n \text{ is odd} \end{cases} \end{aligned}$$

Therefore,

$$v(x,t) \sim \sum_{n=1}^{\infty} \frac{8}{(2n-1)^3\pi^3}\sin((2n-1)\pi x).$$

(d) Solution to the original initial boundary value problem:

$$u(x,t) \sim 1 - x + \frac{xt}{2} + \sum_{n=1}^{\infty} \frac{8}{(2n-1)^3\pi^3} e^{-(2n-1)^2\pi^2 t}\sin((2n-1)\pi x).$$

4.3.2 If $u(x,t)$ solves the heat equation $u_t = \kappa\, u_{xx}$ then $v(x,t) = -u(x,t)$ also solves the heat equation. By the Maximum Principle

$$\max_{(x,t)\in S} v(x,t) = \max_{(x,t)\in\partial R} v(x,t) \iff \max_{(x,t)\in S}(-u(x,t)) = \max_{(x,t)\in\partial R}(-u(x,t)) \iff \min_{(x,t)\in S} u(x,t) = \min_{(x,t)\in\partial R} u(x,t).$$

4.3.4 Set $w(x,t) = u(x,t) - v(x,t)$, then w is a solution of

$$u_t = \kappa\, u_{xx} \text{ for } 0 < x < L \text{ and } t > 0$$

and the assumptions on u and v and imply that

$$w(0,t) \le 0 \text{ and } w(L,t) \le 0 \text{ for } t > 0$$

and

$$w(x,0) \le 0 \text{ for } 0 \le x \le L.$$

These inequalities imply that for any fixed $T > 0$, the maximum of w on the set ∂R consisting of the portions $x = 0$, $x = L$, or $t = 0$ of the region $[0,L]\times[0,t]$ is less than or equal to 0. By the Maximum Principle then

$$w(x,t) \le 0 \text{ for } 0 \le x \le L \text{ and } 0 \le t \le T.$$

Since $T > 0$ is arbitrary, the proof is complete.
4.3.6 Since $-6.29218 \leq u(x,0) \leq 6.29218$ for $0 \leq x \leq \pi$ then by the Maximum and Minimum Principles $m = -6.29218 \leq u(x,y) \leq 6.29218 = M$.
4.3.8 Define function $u(x,t) = (1 - e^{-t}) \sin x$. Note that,

$$\begin{aligned} u_t &= e^{-t} \sin x \\ u_{xx} &= -(1 - e^{-t}) \sin x \\ u_t - u_{xx} &= \sin x \text{ for } 0 < x < 1 \text{ and } t > 0 \end{aligned}$$

On the boundaries, $u(0,t) = 0$ and $u(1,t) = 0$ and at $t = 0$ it is the case that $u(x,0) = 0 \leq \sin x$ for $0 < x < 1$. Hence by the Comparison Principle $u(x,t) \leq v(x,t)$ for all $0 \leq x \leq 1$ and $t \geq 0$.
4.4.2

(a) Taking the derivative with respect to r produces,

$$f'(r) = \frac{-c^3\sqrt{\pi r}e^{-c^2 r} - c\sqrt{\pi}e^{-c^2 r}/(2\sqrt{r})}{c^2 \pi r} = \frac{-(1+2c^2 r)e^{-c^2 r}}{2c\sqrt{\pi}r^{3/2}} < 0.$$

(b) Evaluate the limit:

$$\lim_{r\to\infty} f(r) = \lim_{r\to\infty} \frac{1}{c\sqrt{\pi r}e^{c^2 r}} = 0.$$

4.4.4 Use integration by substitution.

$$\begin{aligned} \operatorname{erf}(-x) &= \frac{2}{\sqrt{\pi}} \int_0^{-x} e^{-t^2}\,dt \\ &= -\frac{2}{\sqrt{\pi}} \int_0^{x} e^{-u^2}\,du \\ &= -\operatorname{erf}(x) \end{aligned}$$

where $u = -t$.
4.4.6 Using the fundamental solution to the heat equation,

$$\begin{aligned} u(x,t) &= \frac{1}{\sqrt{4\pi\kappa t}} \int_{-\infty}^{\infty} e^{-(x-y)^2/(4\kappa t)} u(y,0)\,dy \\ &= \frac{1}{\sqrt{4\pi\kappa t}} \int_{-1}^{1} e^{-(x-y)^2/(4\kappa t)}\,dy \\ &= \frac{1}{\sqrt{\pi}} \int_{\frac{x-1}{\sqrt{4\kappa t}}}^{\frac{x+1}{\sqrt{4\kappa t}}} e^{-u^2}\,du \\ &= \frac{1}{2}\operatorname{erf}\left(\frac{x+1}{\sqrt{4\kappa t}}\right) - \frac{1}{2}\operatorname{erf}\left(\frac{x-1}{\sqrt{4\kappa t}}\right). \end{aligned}$$

4.4.8 Starting with the definition of the cumulative distribution function for the standard normal random variable,

$$\begin{aligned} \Phi(x) &= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{0} e^{-t^2/2}\,dt + \frac{1}{\sqrt{2\pi}} \int_0^{x} e^{-t^2/2}\,dt \\ &= \frac{1}{2} + \frac{1}{\sqrt{\pi}} \int_0^{x/\sqrt{2}} e^{-z^2}\,dz \\ &= \frac{1}{2} + \frac{1}{2} \cdot \frac{2}{\sqrt{\pi}} \int_0^{x/\sqrt{2}} e^{-z^2}\,dz \\ &= \frac{1}{2} + \frac{1}{2}\operatorname{erf}\left(\frac{x}{\sqrt{2}}\right). \end{aligned}$$

Solving for $\mathrm{erf}(z)$ leads to another relationship,

$$\mathrm{erf}(x) = 2\,\Phi\left(\sqrt{2}x\right) - 1.$$

4.4.10 Using Thm. 4.7,

$$\begin{aligned}
u(x,t) &= \int_0^\infty (U(x-y,t) + U(x+y,t))u(y,0)\,dy \\
&= \frac{1}{\sqrt{4\pi\kappa t}} \int_0^1 \left(e^{-\frac{(x-y)^2}{4\kappa t}} + e^{-\frac{(x+y)^2}{4\kappa t}}\right) dy \\
&= \frac{1}{2}\left(\mathrm{erf}\left(\frac{x+1}{2\sqrt{\kappa t}}\right) - \mathrm{erf}\left(\frac{x-1}{2\sqrt{\kappa t}}\right)\right)
\end{aligned}$$

4.4.12 Suppose $u(x,t) = e^{-\alpha t}v(x,t)$ solves the differential equation.

$$\begin{aligned}
-\alpha e^{-\alpha t}v + e^{-\alpha t}v_t &= \kappa e^{-\alpha t}v_{xx} - \alpha e^{-\alpha t}v \\
v_t &= \kappa v_{xx}
\end{aligned}$$

If $u(x,0) = f(x)$ then likewise $v(x,0) = f(x)$.

$$\begin{aligned}
v(x,t) &= \frac{1}{\sqrt{4\pi\kappa t}} \int_{-\infty}^\infty e^{-\frac{(x-y)^2}{4\kappa t}} f(y)\,dy \\
u(x,t) &= \frac{e^{-\alpha t}}{\sqrt{4\pi\kappa t}} \int_{-\infty}^\infty e^{-\frac{(x-y)^2}{4\kappa t}} f(y)\,dy
\end{aligned}$$

4.5.2 The formal solution to the initial boundary value problem is the double Fourier sine series

$$u(x,y,t) = \sum_{n=1}^\infty \sum_{m=1}^\infty b_{n,m} e^{-\kappa(n^2+m^2)\pi^2 t} \sin(n\pi x)\sin(m\pi y).$$

Setting $t = 0$ and equating coefficients in the initial condition and the infinite series reveals $b_{1,3} = 1$ and $b_{n,m} = 0$ otherwise. Thus

$$u(x,y,t) = e^{-10\kappa\pi^2 t}\sin(\pi x)\sin(3\pi y).$$

4.5.4 The initial boundary value problem can be summarized as follows.

$$\begin{aligned}
u_t &= u_{xx} + u_{yy} \text{ for } (x,y) \in (0,1)\times(0,1) \text{ and } t > 0 \\
u(0,y,t) &= u(1,y,t) = 0 \text{ for } 0 < y < 1 \text{ and } t > 0 \\
u(x,0,t) &= 0 \text{ for } 0 < x < 1 \text{ and } t > 0 \\
u_y(x,1,t) &= 0 \text{ for } 0 < x < 1 \text{ and } t > 0 \\
u(x,y,0) &= \sin(2\pi x)\sin(\pi y) \text{ for } (x,y) \in (0,1)\times(0,1)
\end{aligned}$$

The product solutions to this IBVP take the form:

$$u_{m,n}(x,y,t) = e^{-\left(m^2 + \frac{(2n-1)^2}{4}\right)\pi^2 t} \sin(m\pi x)\sin\left(\frac{(2n-1)\pi y}{2}\right).$$

for m, $n \in \mathbb{N}$. The solution to the initial boundary value problem can be expressed as the double sum,

$$u(x,y,t) = \sum_{m=1}^\infty \sum_{n=1}^\infty b_{m,n} e^{-\left(m^2 + \frac{(2n-1)^2}{4}\right)\pi^2 t} \sin(m\pi x)\sin\left(\frac{(2n-1)\pi y}{2}\right).$$

For $m,\, n \in \mathbb{N}$,

$$\begin{aligned}
b_{m,n} &= 4\int_0^1\int_0^1 \sin(2\pi x)\sin(\pi y)\sin(m\pi x)\sin\left(\frac{(2n-1)\pi y}{2}\right)\,dx\,dy \\
&= 4\int_0^1 \sin(2\pi x)\sin(m\pi x)\,dx\cdot\int_0^1 \sin(\pi y)\sin\left(\frac{(2n-1)\pi y}{2}\right)\,dy \\
&= \begin{cases} \frac{8(-1)^n}{\pi(4n^2-4n-3)} & \text{if } m=2 \\ 0 & \text{otherwise.}\end{cases}
\end{aligned}$$

Therefore the solution can be expressed as

$$u(x,y,t) = \sum_{n=1}^{\infty} \frac{8(-1)^n\sin(2\pi x)}{\pi(4n^2-4n-3)} e^{-\left(4+\frac{(2n-1)^2}{4}\right)\pi^2 t}\sin\left(\frac{(2n-1)\pi y}{2}\right).$$

4.5.6 The initial boundary value problem can be summarized as follows.

$$\begin{aligned}
u_t &= \frac{1}{4}(u_{xx}+u_{yy}) \text{ for } (x,y)\in(0,2)\times(0,1) \text{ and } t>0 \\
u(0,y,t) &= 0 \text{ for } 0<y<1 \text{ and } t>0 \\
u(x,0,t) &= 0 \text{ for } 0<x<2 \text{ and } t>0 \\
u_x(2,y,t) &= 0 \text{ for } 0<y<1 \text{ and } t>0 \\
u_y(x,1,t) &= 0 \text{ for } 0<x<2 \text{ and } t>0 \\
u(x,y,0) &= \begin{cases} 0 & \text{for } 0<x<1 \text{ and } 0<y<1 \\ 100 & \text{for } 1<x<2 \text{ and } 0<y<1 \end{cases}
\end{aligned}$$

The product solutions to this IBVP take the form:

$$u_{m,n}(x,y,t) = e^{-\frac{1}{4}\left(\frac{(2m-1)^2}{16}+\frac{(2n-1)^2}{4}\right)\pi^2 t}\sin\left(\frac{(2m-1)\pi x}{4}\right)\sin\left(\frac{(2n-1)\pi y}{2}\right)$$

for $m,\, n\in\mathbb{N}$. The solution to the initial boundary value problem can be expressed as the double sum,

$$u(x,y,t) = \sum_{m=1}^{\infty}\sum_{n=1}^{\infty} b_{m,n} e^{-\frac{1}{4}\left(\frac{(2m-1)^2}{16}+\frac{(2n-1)^2}{4}\right)\pi^2 t}\sin\left(\frac{(2m-1)\pi x}{4}\right)\sin\left(\frac{(2n-1)\pi y}{2}\right).$$

For $m,\, n\in\mathbb{N}$

$$\begin{aligned}
b_{m,n} &= 2\int_0^1\int_0^2 u(x,y,0)\sin\left(\frac{(2m-1)\pi x}{4}\right)\sin\left(\frac{(2n-1)\pi y}{2}\right)\,dx\,dy \\
&= 2\int_0^1\int_1^2 100\sin\left(\frac{(2m-1)\pi x}{4}\right)\sin\left(\frac{(2n-1)\pi y}{2}\right)\,dx\,dy \\
&= 200\int_0^1 \sin\left(\frac{(2n-1)\pi y}{2}\right)\,dy\cdot\int_1^2 \sin\left(\frac{(2m-1)\pi x}{4}\right)\,dx \\
&= 200\,\frac{4\sin\left(\frac{m\pi}{2}+\frac{\pi}{4}\right)}{\pi(2m-1)}\,\frac{2}{\pi(2n-1)} \\
&= \frac{1600\sin\left(\frac{m\pi}{2}+\frac{\pi}{4}\right)}{\pi^2(2m-1)(2n-1)}.
\end{aligned}$$

5.1.1 The formal solution can be expressed as

$$u(x,t) = \sum_{n=1}^{\infty}(a_n\cos(nt)+b_n\sin(nt))\sin(nx),$$

where

$$a_n = \frac{2}{\pi}\int_0^\pi \sin x \cos x \sin(nx)\,dx = \begin{cases} 1/2 & \text{if } n = 2 \\ 0 & \text{if } n \neq 2 \end{cases}$$

$$b_n = 0 \text{ for all } n.$$

Thus the solution to the initial boundary value problem is

$$u(x,t) = \frac{1}{2}\cos(2t)\sin(2x).$$

5.1.3 The formal solution can be expressed as

$$u(x,t) = \sum_{n=1}^{\infty} (a_n \cos(nt) + b_n \sin(nt))\sin(nx)$$

where

$$a_n = 0 \text{ for all } n$$

$$b_n = \frac{2}{n\pi}\int_0^\pi \sin(3x)\sin(nx)\,dx = \begin{cases} 1/3 & \text{if } n = 3 \\ 0 & \text{if } n \neq 3. \end{cases}$$

Thus the solution to the initial boundary value problem is

$$u(x,t) = \frac{1}{3}\sin(3t)\sin(3x).$$

5.1.5 The formal solution can be expressed as

$$u(x,t) = \sum_{n=1}^{\infty} (a_n \cos(nt) + b_n \sin(nt))\sin(nx)$$

where

$$a_n = 0 \text{ for all } n$$

$$b_n = \frac{2}{n\pi}\int_0^\pi x(x-\pi)\sin(nx)\,dx = \frac{-4}{n^4\pi}(1-(-1)^n).$$

Thus the solution to the initial boundary value problem can be formally expressed as

$$u(x,t) = -\frac{4}{\pi}\sum_{n=1}^{\infty} \frac{1-(-1)^n}{n^4}\sin(nt)\sin(nx).$$

5.1.7 The formal solution can be expressed as

$$u(x,t) = \sum_{n=1}^{\infty} (a_n \cos(nt) + b_n \sin(nt))\sin(nx)$$

where

$$a_n = \frac{2}{\pi}\int_0^\pi x(x-\pi)\sin(nx)\,dx = \frac{-4}{n^3\pi}(1-(-1)^n)$$

$$b_n = \frac{2}{n\pi}\int_0^\pi \sin(nx)\,dx = \frac{2}{n^2\pi}(1-(-1)^n).$$

Thus the solution to the initial boundary value problem can be formally expressed as

$$u(x,t) = \frac{2}{\pi}\sum_{n=1}^{\infty}\left[\frac{(1-(-1)^n)}{n^2}\sin(nt) - \frac{2(1-(-1)^n)}{n^3}\cos(nt)\right]\sin(nx).$$

5.1.9

(a) The function $u_n(x,t)$ satisfies the boundary conditions since

$$u_n(0,t) = (c_n \cos(\lambda_n ct) + d_n \sin(\lambda_n ct)) \sin(\lambda_n 0) = 0$$

and

$$\begin{aligned}(u_n)_x(x,t) &= \lambda_n \left(c_n \cos(\lambda_n ct) + d_n \sin(\lambda_n ct)\right) \cos(\lambda_n x) \\ (u_n)_x(L,t) &= \lambda_n \left(c_n \cos(\lambda_n ct) + d_n \sin(\lambda_n ct)\right) \cos\left(\frac{(2n-1)\pi}{2}\right) = 0\end{aligned}$$

for $n = 1, 2, \ldots$. The product solutions solve the wave equation since

$$\begin{aligned}(u_n)_{tt}(x,t) &= -\lambda_n^2 c^2 \left(c_n \cos(\lambda_n ct) + d_n \sin(\lambda_n ct)\right) \sin(\lambda_n x) \\ (u_n)_{xx}(x,t) &= -\lambda_n^2 \left(c_n \cos(\lambda_n ct) + d_n \sin(\lambda_n ct)\right) \sin(\lambda_n x)\end{aligned}$$

and thus $(u_n)_{tt} = c^2 (u_n)_{xx}$.

(b) By the Principle of Superposition

$$u(x,t) = \sum_{n=1}^{\infty} (c_n \cos(\lambda_n ct) + d_n \sin(\lambda_n ct)) \sin(\lambda_n x)$$

satisfies the wave equation and the boundary conditions. When $t = 0$ then

$$u(x,0) = f(x) = \sum_{n=1}^{\infty} c_n \sin\left(\frac{(2n-1)\pi x}{2L}\right).$$

Since

$$\int_0^L \sin(\lambda_n x) \sin(\lambda_m x)\, dx = \frac{L}{2} \delta_{mn}$$

where δ_{mn} is the Kronecker delta, then

$$c_n = \frac{2}{L} \int_0^L f(x) \sin(\lambda_n x)\, dx.$$

Likewise when $t = 0$ then

$$u_t(x,0) = g(x) = \sum_{n=1}^{\infty} \lambda_n c d_n \sin\left(\frac{(2n-1)\pi x}{2L}\right),$$

where

$$d_n = \frac{2}{c\lambda_n L} \int_0^L g(x) \sin(\lambda_n x)\, dx.$$

(c) In this case $d_n = 0$ for $n = 1, 2, \ldots$ and

$$\begin{aligned}c_n &= 2 \int_0^1 f(x) \sin\left(\frac{(2n-1)\pi x}{2}\right) dx \\ &= 2 \int_{1/4}^{1/2} \left(x - \frac{1}{4}\right) \sin\left(\frac{(2n-1)\pi x}{2}\right) dx + 2 \int_{1/2}^{3/4} (3/4 - x) \sin\left(\frac{(2n-1)\pi x}{2}\right) dx \\ &= \frac{8\cos((2n+3)\pi/8) - 8\cos((2n+1)\pi/4) - (2n-1)\pi \sin((2n+1)\pi/4)}{(2n-1)^2\pi^2} \\ &\quad + \frac{8\cos((6n+1)\pi/8) - 8\cos((2n+1)\pi/4) - (2n-1)\pi \sin((2n+1)\pi/4)}{(2n-1)^2\pi^2} \\ &= \frac{8}{(2n-1)^2\pi^2} \left[2\sin\left(\frac{(2n-1)\pi}{4}\right) - \sin\left(\frac{3(2n-1)\pi}{8}\right) - \sin\left(\frac{(2n+1)\pi}{8}\right)\right].\end{aligned}$$

Thus the formal solution can be expressed as

$$u(x,t) = \frac{8}{\pi^2}\sum_{n=1}^{\infty}\frac{1}{(2n-1)^2}\left[2\sin\left(\frac{(2n-1)\pi}{4}\right) - \sin\left(\frac{3(2n-1)\pi}{8}\right) - \sin\left(\frac{(2n+1)\pi}{8}\right)\right]$$
$$* \cos\left(\frac{(2n-1)\pi ct}{2}\right)\sin\left(\frac{(2n-1)\pi x}{2}\right).$$

5.2.1 According to Eq. (5.19),

$$u(x,t) = \frac{1}{2}\left[\sin(2(x-ct)) + \sin(2(x+ct))\right].$$

5.2.3 The solution to the struck string problem is

$$u(x,t) = \frac{1}{2c}\int_{x-ct}^{x+ct}\sinh s\, ds = \frac{1}{2c}\left[\cosh(x+ct) - \cosh(x-ct)\right].$$

5.2.5 This is another example of a plucked string. The solution takes the form of half of the initial displacement moving left plus half of the initial displacement moving right.

$$\begin{aligned} u(x,t) &= \frac{1}{2}\begin{cases} 1-|x-ct| & \text{if } -1 < x-ct < 1 \\ 0 & \text{otherwise} \end{cases} + \frac{1}{2}\begin{cases} 1-|x+ct| & \text{if } -1 < x+ct < 1 \\ 0 & \text{otherwise} \end{cases} \\ &= \begin{cases} 0 & \text{if } x+ct < -1 \\ 1-\frac{1}{2}(|x-ct|+|x+ct|) & \text{if } -1 < x-ct \text{ and } x+ct < 1 \\ 0 & \text{if } x-ct > 1 \\ \frac{1}{2}(1-|x-ct|) & \text{if } x+ct > 1 \text{ and } -1 < x-ct < 1 \\ 0 & \text{if } x+ct > 1 \text{ and } x-ct < -1 \\ \frac{1}{2}(1-|x+ct|) & \text{if } -1 < x+ct < 1 \text{ and } x-ct < -1 \end{cases} \end{aligned}$$

5.2.7 Taking derivatives of the function produces,

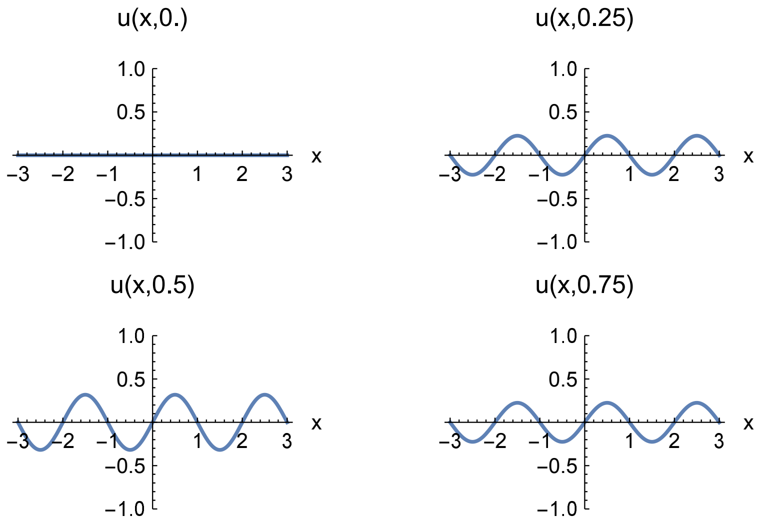
$$\begin{aligned} u_{tt} &= \sigma^2 e^{\sigma t + ikx} \\ u_{xx} &= -k^2 e^{\sigma t + ikx}. \end{aligned}$$

Substituting these expressions into the wave equation results in the following.

$$\begin{aligned} \sigma^2 e^{\sigma t+ikx} &= -c^2k^2e^{\sigma t+ikx} \\ \sigma^2 &= -c^2k^2 \\ \sigma &= \pm ick \end{aligned}$$

5.3.2 Since $u_t(x,0) = g(x)$ is an odd, 2-periodic function d'Alembert's solution is

$$u(x,t) = \frac{1}{2}\int_{x-t}^{x+t}\sin(\pi s)\, ds = \frac{1}{2\pi}\left[\cos(\pi(x-t)) - \cos(\pi(x+t))\right].$$



5.3.4 Consider the solution evaluated at $x = L/2$.

$$\begin{aligned} u(L/2, t) &= \sum_{n=1}^{\infty} c_n \cos\left(\frac{n\pi ct}{L}\right) \sin\frac{n\pi}{2} \\ &= \sum_{k=1}^{\infty} c_{2k-1} \cos\left(\frac{(2k-1)\pi ct}{L}\right) \sin\frac{(2k-1)\pi}{2} \end{aligned}$$

The last equation is true since $\sin(n\pi/2) = 0$ for n even. Thus $u(L/2, t) = 0$ for all $t > 0$ if $c_{2k-1} = 0$ for $k \in \mathbb{N}$.

5.3.6 Let $f_o(x)$ be the odd, 2-periodic extension of $f(x)$.

$$f_o(x) = \begin{cases} x(1-x) & \text{if } 0 \le x \le 1 \\ (x-1)(x-2) & \text{if } 1 < x \le 2 \end{cases}$$

The d'Alembertian solution can be expressed as

$$u(x,t) = \frac{1}{2}\left(f_o((x-t) \bmod 2) + f_o((x+t) \bmod 2)\right).$$

5.3.8 The initial displacement is already an odd, 2-periodic function. The initial velocity is an odd function which can be extended 2-periodically to the real line. Define

$$g_o(x) = \begin{cases} x & \text{if } 0 \le x \le 1, \\ x-2 & \text{if } 1 < x \le 2, \end{cases}$$

and

$$G(x) = \int_0^x g_o(s)\,ds = \begin{cases} \frac{x^2}{2} & \text{if } 0 \le x \le 1 \\ \frac{(x-2)^2}{2} & \text{if } 1 < x \le 2. \end{cases}$$

The d'Alembertian form of the solution can be expressed as follows.

$$u(x,t) = \frac{1}{2}\left[\sin(\pi(x+t)) + \sin(\pi(x-t))\right] + \frac{1}{2}\left[G((x+t) \bmod 2) - G((x-t) \bmod 2)\right]$$

5.4.2 Assume the solution has the form

$$u(x,t) = \sum_{n=1}^{\infty} T_n(t) \sin(n\pi x).$$

The function $T_n(t)$ must satisfy the initial value problem

$$\begin{aligned} T_n''(t) + (n\pi)^2 T_n(t) &= 2\int_0^1 t\sin(n\pi x)\,dx = \frac{2t}{n\pi}(1-(-1)^2) \\ T_n(0) &= T_n'(0) = 0. \end{aligned}$$

Using variation of parameters

$$T_n(t) = \frac{2}{n^4\pi^4}(1-(-1)^n)(n\pi t - \sin(n\pi t)).$$

Thus the formal solution to the initial boundary value problem is

$$u(x,t) = \frac{2}{\pi^4}\sum_{n=1}^{\infty}\frac{(1-(-1)^n)}{n^4}(n\pi t - \sin(n\pi t))\sin(n\pi x).$$

5.4.4 Following the steps of the solution outlined for Exercise 5.4.3 let $v(x,t) = e^t u(x,t)$. Differentiating and substituting into the initial value problem produce a new initial value problem.

$$\begin{aligned} v_{tt} &= v_{xx} \text{ for } -\infty < x < \infty \text{ and } t > 0, \\ v(x,0) &= e^x \text{ for } -\infty < x < \infty, \\ v_t(x,0) &= -e^x \text{ for } -\infty < x < \infty. \end{aligned}$$

The d'Alembertian solution to this initial value problem is

$$\begin{aligned} v(x,t) &= \frac{1}{2}\left(e^{x+t} + e^{x-t}\right) + \frac{1}{2}\int_{x-t}^{x+t}(-e^s)\,ds \\ &= e^{x-t}. \end{aligned}$$

Thus the solution to the telegraph equation is

$$u(x,t) = e^{-t}v(x,t) = e^{x-2t}.$$

5.4.6 Assume the solution can be written as $u(x,t) = t + v(x,t)$, then $v(x,t)$ must satisfy the initial boundary value problem

$$\begin{aligned} v_{tt} &= v_{xx} \text{ for } 0 < x < 1 \text{ and } t > 0, \\ v(0,t) &= v(1,t) = 0 \text{ for } t > 0, \\ v(x,0) &= 0 \text{ and } v_t(x,0) = -1 \text{ for } 0 < x < 1. \end{aligned}$$

Formally the solution can be written as

$$v(x,t) = \sum_{n=1}^{\infty}(a_n\cos(n\pi t) + b_n\sin(n\pi t))\sin(n\pi x)$$

where

$$\begin{aligned} a_n &= 0 \text{ for all } n \\ b_n &= \frac{2}{n\pi}\int_0^1(-1)\sin(n\pi x)\,dx = \frac{2}{n^2\pi^2}((-1)^n - 1). \end{aligned}$$

Thus $u(x,t)$ can be expressed formally as

$$u(x,t) = t + \frac{2}{\pi^2}\sum_{n=1}^{\infty}\frac{((-1)^n-1)}{n^2}\sin(n\pi t)\sin(n\pi x).$$

5.4.8 Assume the solution can be written as $u(x,t) = (1-x)\sin(\pi t) + v(x,t)$, then $v(x,t)$ must satisfy the initial boundary value problem

$$\begin{aligned} v_{tt} &= v_{xx} + \pi^2(1-x)\sin(\pi t) \text{ for } 0 < x < 1 \text{ and } t > 0, \\ v(0,t) &= v(1,t) = 0 \text{ for } t > 0, \\ v(x,0) &= 0 \text{ and } v_t(x,0) = \pi(x-1) \text{ for } 0 < x < 1. \end{aligned}$$

Let $v(x,t)$ be represented by the series

$$v(x,t) = \sum_{n=1}^{\infty} T_n(t)\sin(n\pi x)$$

where $T_n(t)$ satisfies the initial value problem

$$\begin{aligned} T_n''(t) + (n\pi)^2 T_n(t) &= 2\int_0^1 \pi^2(1-x)\sin(\pi t)\sin(n\pi x)\,dx = \frac{2\pi}{n}\sin(\pi t) \\ T_n(0) &= 0 \\ T_n'(0) &= 2\int_0^1 \pi(x-1)\sin(n\pi x)\,dx = -\frac{2}{n}. \end{aligned}$$

Using the method of undetermined coefficients, the solutions are

$$\begin{aligned} T_1(t) &= \frac{-1}{\pi}\sin(\pi t) - t\cos(\pi t) \\ T_n(t) &= \frac{2}{n(n^2-1)\pi}\left[\sin(\pi t) - n\sin(n\pi t)\right] \text{ for } n > 1. \end{aligned}$$

Formally the solution can be written as

$$u(x,t) = (1-x)\sin(\pi t) - \left(t\cos(\pi t) + \frac{\sin(\pi t)}{\pi}\right)\sin(\pi x) + \frac{2}{\pi}\sum_{n=2}^{\infty}\frac{\sin(\pi t) - n\sin(n\pi t)}{n(n^2-1)}\sin(n\pi x).$$

5.4.10 If $u(x,t) = e^{-t}v(x,t)$ then differentiating and substituting into the initial value problem produce the following initial value problem for the unknown function $v(x,t)$.

$$\begin{aligned} v_{tt} &= v_{xx} - 2e^t\sin t\sin x + e^t\cos t\sin x \text{ for } -\infty < x < \infty \text{ and } t > 0, \\ v(x,0) &= \sin x \text{ for } -\infty < x < \infty, \\ v_t(x,0) &= \sin x \text{ for } -\infty < x < \infty. \end{aligned}$$

Suppose $v(x,t) = T(t)\sin x$ then substituting this into the initial value problem results in

$$\begin{aligned} T''(t) &= -T(t) - 2e^t\sin t + e^t\cos t \text{ for } t > 0, \\ T(0) &= 1 \\ T'(0) &= 1. \end{aligned}$$

The solution to this nonhomogeneous ordinary differential equation is

$$T(t) = e^t\cos t.$$

Therefore

$$u(x,t) = \cos t\sin x.$$

5.5.2 Since the total energy is constant and $u_x(x,0) = f'(x)$ and $u_t(x,0) = 0$ then according to Eq. (5.48),

$$\begin{aligned}
E(t) &= \frac{1}{2}\int_0^L T_0 (f'(x))^2\,dx \\
&= \frac{T_0}{2}\int_0^L \left[\frac{\pi}{L}\cos\left(\frac{\pi x}{L}\right)\right]^2\,dx \\
&= \frac{T_0\pi^2}{2L^2}\int_0^L \cos^2\left(\frac{\pi x}{L}\right)\,dx \\
&= \frac{T_0\pi^2}{4L^2}\int_0^L \left[1+\cos\left(\frac{2\pi x}{L}\right)\right]\,dx \\
&= \frac{T_0\pi^2}{4L}.
\end{aligned}$$

5.5.4

$$E'(t) = T_0\left[u_t(L,t)u_x(L,t) - u_t(0,t)u_x(0,t)\right] = T_0\left[(0)u_x(L,t) - u_t(0,t)(0)\right] = 0$$

5.5.6 Suppose that $u_1(x,t)$ and $u_2(x,t)$ are solutions to the initial boundary value problem and define $v(x,t) = u_1(x,t) - u_2(x,t)$. The function $v(x,t)$ solves the following initial boundary value problem.

$$\begin{aligned}
v_{tt} &= c^2 v_{xx} \text{ for } 0 < x < L \text{ and } t > 0, \\
v(0,t) &= v(L,t) = 0 \text{ for } t > 0, \\
v(x,0) &= 0 \text{ for } 0 < x < L, \\
v_t(x,0) &= 0 \text{ for } 0 < x < L.
\end{aligned}$$

Note that $v(x,t) \equiv 0$ is a solution to this initial boundary value problem. According to Thm. 5.1, this solution is unique and thus $u_1(x,t) = u_2(x,t)$ for $0 \le x \le L$ and $t \ge 0$. Hence the original nonhomogeneous initial boundary value problem has a unique solution as well.

6.1.2

(a) If $u(x,y) = x^3 - 3xy^2$, then

$$\begin{aligned}
\Delta u &= \frac{\partial^2}{\partial x^2}\left[x^3 - 3xy^2\right] + \frac{\partial^2}{\partial y^2}\left[x^3 - 3xy^2\right] \\
&= \frac{\partial}{\partial x}\left[3x^2 - 3y^2\right] + \frac{\partial}{\partial y}\left[-6xy\right] \\
&= 6x - 6x = 0.
\end{aligned}$$

(b) If $u(x,y) = \arctan\frac{y}{x}$, then

$$\begin{aligned}
\Delta u &= \frac{\partial^2}{\partial x^2}\left[\arctan\frac{y}{x}\right] + \frac{\partial^2}{\partial y^2}\left[\arctan\frac{y}{x}\right] \\
&= \frac{\partial}{\partial x}\left[\frac{-y}{x^2+y^2}\right] + \frac{\partial}{\partial y}\left[\frac{x}{x^2+y^2}\right] \\
&= \frac{2xy}{(x^2+y^2)^2} + \frac{-2xy}{(x^2+y^2)^2} = 0.
\end{aligned}$$

6.1.4

(a) If $u(x,y,z) = 2x^3 - 3xy^2 - 3xz^2$, then

$$\begin{aligned}
\Delta u &= \frac{\partial^2}{\partial x^2}\left[2x^3 - 3xy^2 - 3xz^2\right] + \frac{\partial^2}{\partial y^2}\left[2x^3 - 3xy^2 - 3xz^2\right] + \frac{\partial^2}{\partial z^2}\left[2x^3 - 3xy^2 - 3xz^2\right] \\
&= \frac{\partial}{\partial x}\left[6x^2 - 3y^2 - 3z^2\right] + \frac{\partial}{\partial y}\left[-6xy\right] + \frac{\partial^2}{\partial z^2}\left[-6xz\right] \\
&= 12x + (-6x) + (-6x) \\
&= 0.
\end{aligned}$$

(b) If $u(x,y,z) = \cosh((a^2+b^2)^{1/2}x)\sin(ay)\sin(bz)$, then

$$\begin{aligned}
\Delta u &= \frac{\partial^2}{\partial x^2}\left[\cosh((a^2+b^2)^{1/2}x)\sin(ay)\sin(bz)\right] + \frac{\partial^2}{\partial y^2}\left[\cosh((a^2+b^2)^{1/2}x)\sin(ay)\sin(bz)\right] \\
&\quad + \frac{\partial^2}{\partial z^2}\left[\cosh((a^2+b^2)^{1/2}x)\sin(ay)\sin(bz)\right] \\
&= \frac{\partial}{\partial x}\left[(a^2+b^2)^{1/2}\sinh((a^2+b^2)^{1/2}x)\sin(ay)\sin(bz)\right] + \frac{\partial}{\partial y}\left[a\cosh((a^2+b^2)^{1/2}x)\cos(ay)\sin(bz)\right] \\
&\quad + \frac{\partial}{\partial z}\left[b\cosh((a^2+b^2)^{1/2}x)\sin(ay)\cos(bz)\right] \\
&= (a^2+b^2)\cosh((a^2+b^2)^{1/2}x)\sin(ay)\sin(bz) + (-a^2)\cosh((a^2+b^2)^{1/2}x)\sin(ay)\sin(bz) \\
&\quad + (-b^2)\cosh((a^2+b^2)^{1/2}x)\sin(ay)\sin(bz) \\
&= 0.
\end{aligned}$$

6.1.6 Define $\xi = x\cos\theta - y\sin\theta$ and $\eta = x\sin\theta + y\cos\theta$.

$$\begin{aligned}
u_x &= u_\xi \xi_x + u_\eta \eta_x = u_\xi \cos\theta + u_\eta \sin\theta \\
u_{xx} &= u_{\xi\xi}\cos^2\theta + 2u_{\xi\eta}\cos\theta\sin\theta + u_{\eta\eta}\sin^2\theta \\
u_y &= u_\xi \xi_y + u_\eta \eta_y = -u_\xi \sin\theta + u_\eta \cos\theta \\
u_{yy} &= u_{\xi\xi}\sin^2\theta - 2u_{\xi\eta}\cos\theta\sin\theta + u_{\eta\eta}\cos^2\theta
\end{aligned}$$

Add the second partial derivatives.

$$\begin{aligned}
u_{xx} + u_{yy} &= u_{\xi\xi}\cos^2\theta + 2u_{\xi\eta}\cos\theta\sin\theta + u_{\eta\eta}\sin^2\theta + u_{\xi\xi}\sin^2\theta - 2u_{\xi\eta}\cos\theta\sin\theta + u_{\eta\eta}\cos^2\theta \\
&= u_{\xi\xi} + u_{\eta\eta} \\
&= 0
\end{aligned}$$

6.2.2 The formal solution is of the form

$$u(x,y) = \sum_{n=1}^{\infty} a_n \sinh(n\pi x)\sin(n\pi y)$$

where

$$a_n = \frac{2}{\sinh(n\pi)}\int_0^1 y(1-y)\sin(n\pi y)\,dy = \frac{4(1-(-1)^n)}{n^3\pi^3\sinh(n\pi)}.$$

6.2.4 Let function $u_1(x,y)$ be formally

$$u_1(x,y) = \sum_{n=1}^{\infty} a_n \sin(n\pi x)\sinh(n\pi(1-y))$$

where

$$a_n = \frac{2}{\sinh(n\pi)} \int_0^1 \sin(\pi x)\sin(n\pi x)\,dx = \begin{cases} 1/\sinh\pi & \text{if } n = 1 \\ 0 & \text{if } n \neq 1. \end{cases}$$

Thus function $u_1(x,y)$ can be written as

$$u_1(x,y) = \frac{\sin(\pi x)\sinh(\pi(1-y))}{\sinh\pi}.$$

Similarly let function $u_3(x,y)$ be represented by the Fourier series

$$u_3(x,y) = \sum_{n=1}^{\infty} b_n \sin(n\pi x)\sinh(n\pi y))$$

where

$$b_n = \frac{2}{\sinh(n\pi)} \int_0^1 (1)\sin(n\pi x)\,dx = \frac{2(1-(-1)^n)}{n\pi\sinh(n\pi)}.$$

The desired solution to the boundary value problem can be expressed formally as $u(x,y) = u_1(x,y)+u_3(x,y)$.
6.2.6 Let function $u_1(x,y)$ be formally

$$u_1(x,y) = \sum_{n=1}^{\infty} a_n \sin(n\pi x)\sinh(n\pi(1-y))$$

where

$$a_n = \frac{2}{\sinh(n\pi)} \int_0^1 (1)\sin(n\pi x)\,dx = \frac{2(1-(-1)^n)}{n\pi\sinh(n\pi)}.$$

Similarly defined function $u_2(x,y)$ formally as

$$u_2(x,y) = \sum_{n=1}^{\infty} b_n \sinh(n\pi x)\sin(n\pi y)$$

where

$$\begin{aligned} b_n &= \frac{2}{\sinh(n\pi)} \int_0^1 \cos(2\pi y)\sin(n\pi y)\,dy \\ &= \begin{cases} 2n(1-(-1)^n)/(\pi(n^2-4)\sinh(n\pi)) & \text{if } n \neq 2 \\ 0 & \text{if } n = 2. \end{cases} \end{aligned}$$

Finally let function $u_3(x,y)$ be represented by the Fourier series

$$u_3(x,y) = \sum_{n=1}^{\infty} c_n \sin(n\pi x)\sinh(n\pi y))$$

where

$$c_n = \frac{2}{\sinh(n\pi)} \int_0^1 (1)\sin(n\pi x)\,dx = \frac{2(1-(-1)^n)}{n\pi\sinh(n\pi)}.$$

The desired solution to the boundary value problem can be expressed formally as $u(x,y) = u_1(x,y) + u_2(x,y) + u_3(x,y)$.
6.3.2 As was done in Exercise 6.3.1 we arrive at the two equations

$$\begin{aligned} e^{\alpha}\cos\beta &= r\cos\theta = x \\ e^{\alpha}\sin\beta &= r\sin\theta = y. \end{aligned}$$

Dividing the second equation by the first yields

$$\tan\beta = \frac{y}{x} \implies \beta = \operatorname{Im}(w) = \operatorname{Im}(\ln z) = \tan^{-1}\frac{y}{x}.$$

6.3.4 The general solution was found to be

$$u(r,\theta) = \frac{a_0}{2} + \sum_{n=1}^{\infty}\left(\frac{r}{2}\right)^n (a_n\cos(n\theta) + b_n\sin(n\theta)).$$

To satisfy the boundary conditions let $a_n = 0$ for all n, $b_3 = 1$, and $b_n = 0$ for all $n \neq 3$. Thus the solution to the boundary value problem is

$$u(r,\theta) = \frac{r^3}{8}\sin(3\theta).$$

6.3.6 The general solution was found to be

$$u(r,\theta) = \frac{a_0}{2} + \sum_{n=1}^{\infty} r^n(a_n\cos(n\theta) + b_n\sin(n\theta)),$$

where

$$\begin{aligned}
a_0 &= \frac{1}{\pi}\int_{-\pi}^{\pi} f(\theta)\,d\theta = \frac{1}{\pi}\int_0^{\pi/2}(1)\,d\theta = \frac{1}{2}\\
a_n &= \frac{1}{\pi}\int_{-\pi}^{\pi} f(\theta)\cos(n\theta)\,d\theta = \frac{1}{\pi}\int_0^{\pi/2}\cos(n\theta)\,d\theta = \frac{1}{n\pi}\sin\left(\frac{n\pi}{2}\right)\\
b_n &= \frac{1}{\pi}\int_{-\pi}^{\pi} f(\theta)\sin(n\theta)\,d\theta = \frac{1}{\pi}\int_0^{\pi/2}\sin(n\theta)\,d\theta = \frac{1}{n\pi}\left(1-\cos\left(\frac{n\pi}{2}\right)\right).
\end{aligned}$$

6.3.8 The general solution was found to be

$$u(r,\theta) = \frac{a_0}{2} + \sum_{n=1}^{\infty}\left(\frac{r}{2}\right)^n (a_n\cos(n\theta) + b_n\sin(n\theta)),$$

where

$$\begin{aligned}
a_0 &= \frac{1}{\pi}\int_{-\pi}^{\pi} f(\theta)\,d\theta = \frac{1}{\pi}\int_{-\pi}^{-\pi/2}\frac{-2(\theta+\pi)}{\pi}\,d\theta + \frac{1}{\pi}\int_0^{\pi/2}\frac{2\theta}{\pi}\,d\theta = 0\\
a_n &= \frac{1}{\pi}\int_{-\pi}^{\pi} f(\theta)\cos(n\theta)\,d\theta\\
&= -\frac{1}{\pi}\int_{-\pi}^{-\pi/2}\frac{2(\theta+\pi)}{\pi}\cos(n\theta)\,d\theta + \frac{1}{\pi}\int_0^{\pi/2}\frac{2\theta}{\pi}\cos(n\theta)\,d\theta = \frac{2((-1)^n-1)}{n^2\pi^2} + \frac{2}{n\pi}\sin\left(\frac{n\pi}{2}\right)\\
b_n &= \frac{1}{\pi}\int_{-\pi}^{\pi} f(\theta)\sin(n\theta)\,d\theta\\
&= -\frac{1}{\pi}\int_{-\pi}^{-\pi/2}\frac{2(\theta+\pi)}{\pi}\sin(n\theta)\,d\theta + \frac{1}{\pi}\int_0^{\pi/2}\frac{2\theta}{\pi}\sin(n\theta)\,d\theta = \frac{4}{n^2\pi^2}\sin\left(\frac{n\pi}{2}\right).
\end{aligned}$$

6.4.2 The general solution to this boundary value problem has the form,

$$u(r,\theta) = c_0\ln r + d_0 + \sum_{n=1}^{\infty}[(c_n^+ r^n + c_n^- r^{-n})\cos(n\theta) + (d_n^+ r^n + d_n^- r^{-n})\sin(n\theta)].$$

The coefficients in the series solution must satisfy the set of equations below.

$$\sin\theta = c_0 \ln 1 + d_0 + \sum_{n=1}^{\infty}[(c_n^+ + c_n^-)\cos(n\theta) + (d_n^+ + d_n^-)\sin(n\theta)]$$
$$\cos 2\theta = c_0 \ln 2 + d_0 + \sum_{n=1}^{\infty}[(c_n^+ 2^n + c_n^- 2^{-n})\cos(n\theta) + (d_n^+ 2^n + d_n^- 2^{-n})\sin(n\theta)]$$

By equating coefficients of the trigonometric functions on both sides of the equations,

$$c_2^+ + c_2^- = 0$$
$$c_2^+ 2^2 + c_n^- 2^{-2} = 1$$

which implies $c_2^+ = \frac{4}{15}$ and $c_2^- = -\frac{4}{15}$. Similarly,

$$d_1^+ + d_1^- = 1$$
$$d_1^+ 2^1 + d_1^- 2^{-1} = 0$$

which implies $d_1^+ = -\frac{1}{3}$ and $d_1^- = \frac{4}{3}$. All other coefficients vanish. Hence the solution to the boundary value problem is

$$u(r,\theta) = \left(\frac{4}{15}r^2 - \frac{4}{15}r^{-2}\right)\cos(2\theta) + \left(-\frac{1}{3}r + \frac{4}{3}r^{-1}\right)\sin\theta.$$

6.4.4 The formal solution can be written as

$$u(r,\theta) = \sum_{n=1}^{\infty} a_n r^{2n}\sin(2n\theta).$$

On the boundary where $r = 1$,

$$u(1,\theta) = \sin(4\theta) = \sum_{n=1}^{\infty} a_n \sin(2n\theta).$$

Thus by equating coefficients of the sine functions, $a_2 = 1$ and $a_n = 0$ for $n \neq 2$. Hence the solution to the boundary value problem is

$$u(r,\theta) = r^4\sin(4\theta).$$

6.4.6 The formal solution can be written as

$$u(r,\theta) = \sum_{n=1}^{\infty}(a_n^+ r^n + a_n^- r^{-n})\sin(n\theta)$$

where a_n^+ and a_n^- must solve the system of equations

$$a_n^+(1^n) + a_n^-(1^{-n}) = \frac{2}{\pi}\int_0^{\pi} u(1,\theta)\sin(n\theta)\,d\theta = \frac{2}{\pi}\int_0^{\pi/2}\sin(n\theta)\,d\theta = \frac{2}{n\pi}\left(1-\cos\frac{n\pi}{2}\right)$$
$$a_n^+(2^n) + a_n^-(2^{-n}) = \frac{2}{\pi}\int_0^{\pi} u(2,\theta)\sin(n\theta)\,d\theta = 0.$$

Thus

$$a_n^+ = \frac{-2}{(4^n-1)n\pi}\left(1-\cos\frac{n\pi}{2}\right)$$
$$a_n^- = \frac{2^{2n+1}}{(4^n-1)n\pi}\left(1-\cos\frac{n\pi}{2}\right).$$

6.4.8 The formal solution can be written as

$$u(r,\theta) = \sum_{n=1}^{\infty} (a_n^+ r^{2n/3} + a_n^- r^{-2n/3}) \sin\left(\frac{2n\theta}{3}\right)$$

where a_n^+ and a_n^- must solve the system of equations

$$\begin{aligned}
a_n^+(1^{2n/3}) + a_n^-(1^{-2n/3}) &= \frac{4}{3\pi}\int_0^{3\pi/2} u(1,\theta)\sin\left(\frac{2n\theta}{3}\right) d\theta = \frac{4}{3\pi}\int_0^{3\pi/2} \sin\left(\frac{2n\theta}{3}\right) d\theta \\
&= \frac{2}{n\pi}(1-(-1)^n) \\
a_n^+(2^{2n/3}) + a_n^-(2^{-2n/3}) &= \frac{4}{3\pi}\int_0^{3\pi/2} u(2,\theta)\sin\left(\frac{2n\theta}{3}\right) d\theta = \frac{4}{3}\int_0^{\pi} \theta\sin\left(\frac{2n\theta}{3}\right) d\theta \\
&= \frac{1}{n^2}\left(3\sin\frac{2n\pi}{3} - 2n\pi\cos\frac{2n\pi}{3}\right).
\end{aligned}$$

Thus

$$\begin{aligned}
a_n^+ &= \frac{-2(1-(-1)^n)n - 2^{(2n+3)/3}n\pi^2\cos\frac{2n\pi}{3} + 2^{2n/3}(3\pi)\sin\frac{2n\pi}{3}}{n^2\pi(4^{n/3}-1)} \\
a_n^- &= \frac{2^{2n/3}\left(2^{(2n+3)/3}(1-(-1)^n)n + 2n\pi^2\cos\frac{2n\pi}{3} - 3\pi\sin\frac{2n\pi}{3}\right)}{n^2\pi(4^{n/3}-1)}.
\end{aligned}$$

6.5.2 Define the formal solution $u(x,y)$ as follows.

$$u(x,y) = a_0 + \sum_{n=1}^{\infty} a_n \cos(n\pi x)\cosh(n\pi(y-1)).$$

This solution is harmonic and satisfies the boundary conditions $u_x(0,y) = u_x(1,y) = 0$ for $0 < y < 1$ and the boundary condition $u_y(x,1) = 0$ for $0 < x < 1$. Differentiate the solution with respect to y and set $y = 0$.

$$\begin{aligned}
u_y(x,y) &= \sum_{n=1}^{\infty} a_n n\pi\cos(n\pi x)\sinh(n\pi(y-1)) \\
u_y(x,0) &= \sum_{n=1}^{\infty} a_n(-n\pi)\sinh(n\pi)\cos(n\pi x) \\
\sin(\pi x) &= \sum_{n=1}^{\infty} a_n(-n\pi)\sinh(n\pi)\cos(n\pi x)
\end{aligned}$$

Multiply both sides of the last equation by $\cos(m\pi x)$ and integrate with respect to x for $0 \le x \le 1$. Since the solution can be determined only up to an additive constant, the value of a_0 is undetermined. When $n > 0$,

$$\begin{aligned}
\int_0^1 \cos(m\pi x)\sin(\pi x)\,dx &= \sum_{n=1}^{\infty} a_n(-n\pi)\sinh(n\pi)\int_0^1 \cos(m\pi x)\cos(n\pi x)\,dx \\
\frac{1+(-1)^n}{\pi(1-n^2)} &= -\frac{1}{2}a_n n\pi\sinh(n\pi)
\end{aligned}$$

if $m = n \neq 1$. For the case when $m = 1$, the definite integral on the left-hand side of the equation is 0. In fact whenever $m = n$ is an odd integer, the definite integral vanishes. Therefore

$$a_n = \begin{cases} 0 & \text{if } n \text{ is odd} \\ \frac{-4}{n(1-n^2)\pi^2\sinh(n\pi)} & \text{if } n \text{ is even.} \end{cases}$$

Thus the solution can be represented as

$$u(x,y) = a_0 + \sum_{n=1}^{\infty} \frac{-2\cos(2n\pi x)\cosh(2n\pi(y-1))}{n(1-4n^2)\pi^2\sinh(2n\pi)}.$$

6.5.4 Suppose $u(x,y) = ax^2 + bx + cy^2 + dy$ then

$$\Delta u = 2a + 2c = 0 \iff a = -c.$$

Hence look for a solution of the form $u(x,y) = a(x^2 - y^2) + bx + dy$. To satisfy the boundary conditions consider the system of equations.

$$\begin{aligned} b &= 1 \\ 2a + b &= 0 \\ d &= -1 \\ -2a + d &= 0 \end{aligned}$$

The solution to the system is $a = -\frac{1}{2}$, $b = 1$, $d = -1$ and therefore

$$u(x,y) = -\frac{1}{2}(x^2 - y^2) + x - y.$$

6.5.6 The additive constant a_0, if present, is the average of the boundary condition on that boundary. In this case,

$$\frac{1}{1}\int_0^1 \sin(2\pi x)\,dx = \left[-\frac{1}{2\pi}\cos(2\pi x)\right]_{x=0}^{x=1} = 0$$

so no additive constant term is present in the series solution.
6.5.8 The solution to Laplace's equation can be written as $u(x,y) = u_1(x,y) + u_2(x,y)$ where $u_1(x,y)$ solves Laplace's equation with boundary conditions:

$$\begin{aligned} &(u_1)_x(0,y) = 0 \text{ and } (u_1)_x(1,y) = 0 \text{ for } 0 < y < 1 \\ &(u_1)_y(x,0) = \sin(\pi x) \text{ and } (u_1)_y(x,1) = 0 \text{ for } 0 < x < 1 \end{aligned}$$

and $u_2(x,y)$ solves Laplace's equation with boundary conditions:

$$\begin{aligned} &(u_2)_x(0,y) = \sin(2\pi y) \text{ and } (u_2)_x(1,y) = 0 \text{ for } 0 < y < 1 \\ &(u_2)_y(x,0) = 0 \text{ and } (u_2)_y(x,1) = 0 \text{ for } 0 < x < 1. \end{aligned}$$

These solutions were found in Exercises 6.5.2 and 6.5.7 respectively. Therefore the solution to Laplace's equation with the two nonhomogeneous Neumann boundary conditions can be written as follows.

$$\begin{aligned} u(x,y) = a_0 &+ \sum_{n=1}^{\infty} \frac{-2\cos(2n\pi x)\cosh(2n\pi(y-1))}{n(1-4n^2)\pi^2\sinh(2n\pi)} \\ &+ \sum_{n=1}^{\infty} \frac{8\cosh((2n-1)\pi(x-1))\cos((2n-1)\pi y)}{\pi^2(2n-1)((2n-1)^2-4)\sinh((2n-1)\pi)} \end{aligned}$$

6.6.2 The general solution can be written formally as

$$u(r,\theta) = a_0 + \sum_{n=1}^{\infty} \left(\frac{r}{3}\right)^n (a_n\cos(n\theta) + b_n\sin(n\theta))$$

where a_0 is an arbitrary constant, $a_5 = 3/5$, $a_n = 0$ for all $n \neq 5$, $b_n = 0$ for all n. Thus the solution to the boundary value problem is

$$u(r,\theta) = a_0 + \frac{3}{5}\left(\frac{r}{3}\right)^5 \cos(5\theta).$$

6.6.4 The general solution can be written formally as

$$u(r,\theta) = a_0 + \sum_{n=1}^{\infty} a_n \left(\frac{r}{2}\right)^n \cos(n\theta).$$

Differentiating with respect to r and setting $r = 2$ result in

$$u_r(2,\theta) = \cos\theta = \sum_{n=1}^{\infty} \frac{n}{2} a_n \cos(n\theta).$$

Equating coefficients implies $a_1 = 2$ and $a_n = 0$ for $n > 1$. Thus the solution to the boundary value problem is

$$u(r,\theta) = a_0 + r\cos(\theta).$$

6.6.6 The general solution can be written formally as

$$u(r,\theta) = a_0 + \sum_{n=1}^{\infty} r^{-n}[a_n \cos(n\theta) + b_n \sin(n\theta)].$$

Differentiating with respect to r and setting $r = 1$ result in

$$u_r(1,\theta) = \theta^2 - \pi^2 = \sum_{n=1}^{\infty}(-n)[a_n \cos(n\theta) + b_n \sin(n\theta)].$$

The Fourier coefficients are found using the Euler-Fourier integral formulas.

$$\begin{aligned} -na_n &= \frac{1}{\pi}\int_{-\pi}^{\pi} (\theta^2 - \pi^2)\cos(n\theta)\,d\theta = \frac{4(-1)^n}{n^2} \\ -nb_n &= \frac{1}{\pi}\int_{-\pi}^{\pi} (\theta^2 - \pi^2)\sin(n\theta)\,d\theta = 0 \end{aligned}$$

Hence the solution to the boundary value problem can be expressed as

$$u(r,\theta) = a_0 + 4\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n^3 r^n}\cos(n\theta).$$

6.6.8 The formal series solution has the form,

$$u(r,\theta) = d_0 + \sum_{n=1}^{\infty} \left(r^n + r^{-n}\right)[a_n \cos(n\theta) + b_n \sin(n\theta)].$$

The function $u(r,\theta)$ is harmonic and 2π-periodic on the annulus. The function also satisfies the homogeneous Neumann boundary condition when $r = 1$. Differentiating the series with respect to r and setting $r = 2$ produce,

$$\begin{aligned} u_r(r,\theta) &= \sum_{n=1}^{\infty} \left(nr^{n-1} - nr^{-n-1}\right)[a_n \cos(n\theta) + b_n \sin(n\theta)] \\ u_r(2,\theta) &= \sum_{n=1}^{\infty} n\left(2^{n-1} - 2^{-n-1}\right)[a_n \cos(n\theta) + b_n \sin(n\theta)] \\ \cos(2\theta) &= \sum_{n=1}^{\infty} n\left(2^{n-1} - 2^{-n-1}\right)[a_n \cos(n\theta) + b_n \sin(n\theta)] \end{aligned}$$

Equating coefficients of the trigonometric functions on both sides of the last equation suggests $b_n = 0$ for $n \in \mathbb{N}$, $a_n = 0$ for $n \neq 2$ and when $n = 2$, $a_2 = \frac{4}{15}$. Hence the solution can be expressed as follows.

$$u(r,\theta) = d_0 + \frac{4}{15}\left(r^2 + r^{-2}\right)\cos(2\theta).$$

6.7.2 Suppose $m \neq n$ then

$$\int_0^1 \cos\left(\frac{(2m-1)\pi y}{2}\right)\cos\left(\frac{(2n-1)\pi y}{2}\right)dy = \left[\frac{\sin((m-n)\pi y)}{2\pi(m-n)} + \frac{\sin((m+n-1)\pi)y}{2\pi(m+n-1)}\right]_{y=0}^{y=1}$$
$$= 0.$$

When $m = n$ then

$$\int_0^1 \cos^2\left(\frac{(2m-1)\pi y}{2}\right)dy = \left[\frac{y}{2} + \frac{\sin((2n-1)\pi y)}{2\pi(2n-1)}\right]_{y=0}^{y=1} = \frac{1}{2}.$$

Therefore the eigenfunctions are orthogonal.

6.7.4 The general solution to the boundary value problem takes the form,

$$u(x,y) = \sum_{n=1}^{\infty} a_n \sinh(n\pi x)\cos(n\pi y).$$

This general solution is harmonic and satisfies the homogeneous Neumann boundary conditions when $y = 0$ or $y = 1$. The homogeneous Dirichlet boundary condition at $x = 0$ is also met. When $x = 2$,

$$u(2,y) = \sum_{n=1}^{\infty} a_n \sinh(2n\pi)\cos(n\pi y) = -\frac{1}{2} + \sin^2(\pi y) = -\frac{1}{2}\cos(2\pi y).$$

Multiply the boundary condition and the infinite series by $\cos(n\pi y)$ and integrate over the interval $0 \leq y \leq 1$.

$$\sum_{n=1}^{\infty} a_n \sinh(2n\pi)\int_0^1 \cos^2(n\pi y)\,dy = \int_0^1 \frac{-1}{2}\cos(2\pi y)\cos(n\pi y)\,dy$$
$$\frac{1}{2}a_n \sinh(2n\pi) = \begin{cases} -\frac{1}{4} & \text{if } n = 2, \\ 0 & \text{otherwise} \end{cases}$$

Hence $a_2 = \frac{-1}{2\sinh(4\pi)}$ and $a_0 = 0$ for $n \neq 2$. Therefore,

$$u(x,y) = \frac{-\sinh(2\pi x)\cos(2\pi y)}{2\sinh(4\pi)}.$$

6.7.6 Define the solution $u(x,y)$ formally as

$$u(x,y) = \sum_{n=1}^{\infty} a_n \sinh(n\pi x)\sin\left(\frac{(2n-1)\pi y}{2}\right),$$

where

$$a_n = \frac{2}{n\pi\cosh(2n\pi)}\int_0^1 \left(y - \frac{1}{2}\right)\sin\left(\frac{(2n-1)\pi y}{2}\right)dy = \frac{-2((2n-1)\pi + 4(-1)^n)}{n\pi^3(2n-1)^2\cosh(2n\pi)}.$$

6.7.8 The general solution to this boundary value problem takes the form,

$$u(x,y) = a_0 + \sum_{n=1}^{\infty} a_n \cosh(n\pi x)\cos(n\pi y).$$

This function is harmonic on Ω and satisfies the Neumann boundary conditions when $x = 0$, $y = 0$, or $y = 1$. When $x = 1$,

$$u(1, y) = a_0 + \sum_{n=1}^{\infty} a_n \cosh(n\pi)\cos(n\pi y) = \cos(\pi y).$$

Equating coefficients of the trigonometric functions on both sides of the equation produces $a_n = 0$ for $n \neq 1$ and $a_1 = \frac{1}{\cosh \pi}$. The solution is written as

$$u(x, y) = \frac{\cosh(\pi x)\cos(\pi y)}{\cosh \pi}.$$

6.8.2 By Thm. 6.2 the value of u at the center of the disk is the average value of u on the boundary of the disk. This average value is

$$\frac{1}{2\pi}\int_{-\pi}^{\pi}(2\sin(3\theta) - 1)\,d\theta = \frac{1}{2\pi}\left[\frac{-2}{3}\cos(3\theta) - \theta\right]_{\theta=-\pi}^{\theta=\pi} = -1.$$

6.8.4 Following the approach of the last problem, first find a function $v(x, y)$ such that $\Delta v = x\,y$. The right hand side of the equation suggests looking for a function of the form of $v(x, y) = Ax^3y + Bxy^3$. For this choice

$$\Delta v = 6Axy + 6Bxy.$$

Thus choose A and B such that $6A + 6B = 1$. There are many choices of A and B. However, considering the underlying domain of the boundary value problem is a disk, it is convenient to choose $A = B = 1/12$. Next, define $w(x, y) = u(x, y) - v(x, y)$, then u is a solution of the original boundary value problem if and only if w satisfies

$$\Delta w = 0 \text{ for } x^2 + y^2 < a^2$$
$$w(x, y) = -\frac{1}{12}a^2 x\,y \text{ for } x^2 + y^2 = a^2.$$

Note that $w(x, y) = -a^2xy/12$ is a solution of this boundary value problem. Therefore $u(x, y) = \dfrac{1}{12}x^3y + \dfrac{1}{12}xy^3 - \dfrac{1}{2}a^2xy$.

6.8.6 By Euler's identity, $\sin(nx) = (e^{inx} - e^{-inx})/(2i)$ which implies

$$\begin{aligned}\sum_{n=1}^{\infty}\rho^n\sin(nx) &= \frac{1}{2i}\sum_{n=1}^{\infty}\left[\rho^n e^{inx} - \rho^n e^{-inx}\right] = \frac{1}{2i}\sum_{n=1}^{\infty}\left[\left(\rho e^{ix}\right)^n - \left(\rho e^{-ix}\right)^n\right] \\ &= \frac{1}{2i}\left[\frac{\rho e^{ix}}{1-\rho e^{ix}} - \frac{\rho e^{-ix}}{1-\rho e^{-ix}}\right] = \frac{1}{2i}\left[\frac{\rho e^{ix} - \rho e^{-ix}}{1 - \rho e^{ix} - \rho e^{-ix} + \rho^2}\right] \\ &= \frac{\rho\sin x}{1 - 2\rho\cos x + \rho^2}.\end{aligned}$$

6.8.8 Decompose the last boundary value problem into three problems such that each has only one nonzero boundary condition on the boundary $y = 0$, $x = \pi$, and $y = \pi$. Denote the solutions of each of the new boundary value problems by w_1, w_2, and w_3 respectively. For the boundary value problem

$$\Delta w_1 = 0 \text{ for } 0 < x < \pi \text{ and } 0 < y < \pi$$
$$w_1(x, 0) = -x^2 \text{ for } 0 < x < \pi$$

the solution is

$$\begin{aligned}w_1(x, y) &= \sum_{n=1}^{\infty}\frac{2}{\pi\sinh n\pi}\left[\int_0^{\pi}(-x^2)\sin(nx)\,dx\right]\sinh(n(\pi - y))\sin(nx) \\ &= \sum_{n=1}^{\infty}\frac{2}{\pi\sinh n\pi}\left(\frac{(-1)^n\pi^2}{n} - \frac{2((-1)^n - 1)}{n^3}\right)\sinh(n(\pi - y))\sin(nx).\end{aligned}$$

For the boundary value problem

$$\Delta w_2 = 0 \text{ for } 0 < x < \pi \text{ and } 0 < y < \pi$$
$$w_2(\pi, y) = -\pi^2 \text{ for } 0 < y < \pi$$

the solution is

$$\begin{aligned} w_2(x,y) &= \sum_{n=1}^{\infty} \frac{2}{\pi \sinh n\pi} \left[\int_0^{\pi} (-\pi^2) \sin(ny)\, dy \right] \sinh(nx) \sin(ny) \\ &= \sum_{n=1}^{\infty} \frac{2((-1)^n - 1)\pi}{n \sinh n\pi} \sinh(nx) \sin(ny). \end{aligned}$$

Finally for the boundary value problem

$$\Delta w_3 = 0 \text{ for } 0 < x < \pi \text{ and } 0 < y < \pi$$
$$w_3(x, \pi) = -x^2 \text{ for } 0 < x < \pi$$

the solution is

$$\begin{aligned} w_3(x,y) &= \sum_{n=1}^{\infty} \frac{2}{\pi \sinh n\pi} \left[\int_0^{\pi} (-x^2) \sin(nx)\, dx \right] \sinh(ny) \sin(nx) \\ &= \sum_{n=1}^{\infty} \frac{2}{\pi \sinh n\pi} \left(\frac{(-1)^n \pi^2}{n} - \frac{2((-1)^n - 1)}{n^3} \right) \sin(nx) \sinh(ny). \end{aligned}$$

Thus the solution to the original boundary value problem is

$$u(x,y) = x^2 + w_1(x,y) + w_2(x,y) + w_3(x,y).$$

6.9.2 The maximum and minimum of the solution to the boundary value problem are found on the boundary of the disk. The boundary condition is $u(\theta) = 2\sin(3\theta) - 1$. Hence,

$$\min_{(x,y) \in \overline{\Omega}} u(x,y) = \min_{-\pi \leq \theta \pi} (2\sin(3\theta) - 1) = -3$$

and

$$\max_{(x,y) \in \overline{\Omega}} u(x,y) = \max_{-\pi \leq \theta \pi} (2\sin(3\theta) - 1) = 1$$

6.9.4 Suppose functions $u_1(x,y)$ and $u_2(x,y)$ both satisfy Poisson's equation for $(x,y) \in \Omega$ and $u_1(x,y) = u_2(x,y) = \phi(x,y)$ for $(x,y) \in \partial\Omega$. Define the function $v(x,y) = u_1(x,y) - u_2(x,y)$. Then $\Delta v = f(x,y) - f(x,y) = 0$ for $(x,y) \in \Omega$ and $v(x,y) = 0$ for $(x,y) \in \partial\Omega$. By Cor. 6.2 then $v(x,y) = 0$ for $(x,y) \in \overline{\Omega}$. Consequently $u_1(x,y) = u_2(x,y)$ for all $(x,y) \in \overline{\Omega}$.

6.9.6 Define $v(x,y) = u_1(x,y) - u_2(x,y)$ for all $(x,y) \in \Omega \cup \partial\Omega$, then v satisfies the following boundary value problem.

$$\Delta v = 0 \text{ for } (x,y) \in \Omega$$
$$v(x,y) = \phi_1(x,y) - \phi_2(x,y) \text{ for } (x,y) \in \partial\Omega$$

By assumption $-\epsilon \leq v(x,y) = \phi_1(x,y) - \phi_2(x,y) \leq \epsilon$ for all $(x,y) \in \partial\Omega$. According to Cor. 6.2,

$$\min_{(x,y) \in \Omega \cup \partial\Omega} v(x,y) = \min_{(x,y) \in \partial\Omega} v(x,y) \geq -\epsilon$$

and thus $v(x,y) \geq -\epsilon$ for all $(x,y) \in \Omega$ and

$$\max_{(x,y) \in \Omega \cup \partial\Omega} v(x,y) = \max_{(x,y) \in \partial\Omega} v(x,y) \leq \epsilon$$

and thus $v(x,y) \le \epsilon$ for all $(x,y) \in \Omega$. Therefore $|v(x,y)| \le \epsilon$ for all $(x,y) \in \Omega$.
7.1.2 Integrate both sides of the equation twice with respect to x.

$$\begin{aligned} y' + \frac{1}{2}x^2 &= A \\ y + \frac{1}{6}x^3 &= Ax + B \\ y(x) &= -\frac{1}{6}x^3 + Ax + B \end{aligned}$$

If $x = 0$ then $y(0) = 1 = B$. The other boundary condition implies

$$2 = -\frac{\pi^3}{6} + A\pi + 1 - \frac{\pi^2}{2} + A$$

and thus $A = (6 + \pi^3 + 3\pi^2)/(6(\pi + 1))$. Consequently

$$y(x) = -\frac{1}{6}x^3 + \frac{(6 + \pi^3 + 3\pi^2)x}{6(\pi + 1)} + 1.$$

7.1.4 The general solution to the ordinary differential equation is $y(x) = A\cos(3x) + B\sin(3x)$. Imposing the boundary conditions produces the following system of equations.

$$\begin{aligned} 3B &= 0 \\ A\cos\pi &= 0 \end{aligned}$$

This system of equations has no nonzero solutions for A and B and thus there are no nontrivial solutions to the original boundary value problem.
7.1.6 Proceed using the definition of the inner product, the definition of the operator and integrate by parts twice.

$$\begin{aligned} \langle L[f], g\rangle &= \int_a^b \left([p(x)f'(x)]' + q(x)f(x)\right)\overline{g(x)}\,dx \\ &= \int_a^b [p(x)f'(x)]'\overline{g(x)}\,dx + \int_a^b q(x)f(x)\overline{g(x)}\,dx \\ &= \Big[p(x)f'(x)\overline{g(x)}\Big]_{x=a}^{x=b} - \int_a^b p(x)f'(x)\overline{g'(x)}\,dx + \int_a^b q(x)f(x)\overline{g(x)}\,dx \\ &= \Big[p(x)f'(x)\overline{g(x)}\Big]_{x=a}^{x=b} - \Big[p(x)f(x)\overline{g'(x)}\Big]_{x=a}^{x=b} + \int_a^b \left(p'(x)\overline{g'(x)} + p(x)\overline{g''(x)}\right)f(x)\,dx + \int_a^b q(x)f(x)\overline{g(x)}\,dx \\ &= \Big[p(x)\Big(f'(x)\overline{g(x)} - f(x)\overline{g'(x)}\Big)\Big]_{x=a}^{x=b} + \int_a^b \Big([p(x)\overline{g'(x)}]' + q(x)\overline{g(x)}\Big)f(x)\,dx \\ &= \Big[p(x)\Big(f'(x)\overline{g(x)} - f(x)\overline{g'(x)}\Big)\Big]_{x=a}^{x=b} + \int_a^b f(x)\overline{\left([p(x)g'(x)]' + q(x)g(x)\right)}\,dx \\ &= \Big[p(x)\Big(f'(x)\overline{g(x)} - f(x)\overline{g'(x)}\Big)\Big]_{x=a}^{x=b} + \langle f, L[g]\rangle \end{aligned}$$

The complex conjugation of $L[g](x)$ is possible due to $p(x)$ and $q(x)$ being real-valued. Now consider the first term on the right-hand side of the last equation. If f and g satisfy the separated boundary conditions of Eq. (7.11) then

$$\Big[p(x)\Big(f'(x)\overline{g(x)} - f(x)\overline{g'(x)}\Big)\Big]_{x=a}^{x=b} = p(b)\Big(f'(b)\overline{g(b)} - f(b)\overline{g'(b)}\Big) - p(a)\Big(f'(a)\overline{g(a)} - f(a)\overline{g'(a)}\Big).$$

If none of the scalars α_1, α_2, β_1, and β_2 are zero then $f'(a) = -\alpha_1 f(a)/\beta_1$, $f'(b) = -\alpha_2 f(b)/\beta_2$, $g'(a) = -\alpha_1 g(a)/\beta_1$, and $g'(b) = -\alpha_2 g(b)/\beta_2$. Substituting into the expression above,

$$\begin{aligned}\left[p(x)\left(f'(x)\overline{g(x)} - f(x)\overline{g'(x)}\right)\right]_{x=a}^{x=b} &= p(b)\left(-\frac{\alpha_2}{\beta_2}f(b)\overline{g(b)} + f(b)\frac{\alpha_2}{\beta_2}\overline{g(b)}\right) - p(a)\left(-\frac{\alpha_1}{\beta_1}f(a)\overline{g(a)} + f(a)\frac{\alpha_1}{\beta_1}\overline{g(a)}\right)\\ &= 0\end{aligned}$$

The expression is likewise zero in the cases in which one or more of the coefficients in the separated boundary conditions are zero. Thus $\langle L[f], g\rangle = \langle f, L[g]\rangle$ and the operator is self-adjoint.

7.1.8 Let $u(x,t) = X(x)T(t)$ then

$$\begin{aligned}X(x)T'(t) &= kX''(x)T(t) - C_0X'(x)T(t)\\ \frac{X(x)T'(t)}{kX(x)T(t)} &= \frac{kX''(x)T(t)}{kX(x)T(t)} - \frac{C_0X'(x)T(t)}{kX(x)T(t)}\\ \frac{T'(t)}{kT(t)} &= \frac{X''(x)}{X(x)} - \frac{C_0X'(x)}{kX(x)} = -\lambda\end{aligned}$$

where λ is a constant. Thus function $X(x)$ must solve the ordinary differential equation

$$X''(x) - \frac{C_0}{k}X'(x) + \lambda X(x) = 0.$$

If $p(x) = e^{-C_0x/k}$ and both sides of the ordinary differential equation are multiplied by $p(x)$ then

$$0 = e^{-C_0x/k}X''(x) - \frac{C_0}{k}e^{-C_0x/k}X'(x) + \lambda e^{-C_0x/k}X(x) = \left[e^{-C_0x/k}X'(x)\right]' + \lambda e^{-C_0x/k}X(x),$$

which is the self-adjoint form.

7.1.10 Differentiate to find

$$\begin{aligned}\phi_n'(x) &= \frac{1}{(x+1)^{3/2}}\left[\frac{n\pi}{\ln 2}\cos\left(\frac{n\pi\ln(x+1)}{\ln 2}\right) - \frac{1}{2}\sin\left(\frac{n\pi\ln(x+1)}{\ln 2}\right)\right]\\ (1+x)^2\phi_n'(x) &= (1+x)^{1/2}\left[\frac{n\pi}{\ln 2}\cos\left(\frac{n\pi\ln(x+1)}{\ln 2}\right) - \frac{1}{2}\sin\left(\frac{n\pi\ln(x+1)}{\ln 2}\right)\right]\\ \frac{d}{dx}\left[(1+x)^2\phi_n'(x)\right] &= \frac{1}{\sqrt{x+1}}\left[\frac{n\pi}{2\ln 2}\cos\left(\frac{n\pi\ln(x+1)}{\ln 2}\right) - \frac{1}{4}\sin\left(\frac{n\pi\ln(x+1)}{\ln 2}\right)\right]\\ &\quad - \frac{1}{\sqrt{x+1}}\left[\left(\frac{n\pi}{\ln 2}\right)^2\sin\left(\frac{n\pi\ln(x+1)}{\ln 2}\right) + \frac{n\pi}{2\ln 2}\cos\left(\frac{n\pi\ln(x+1)}{\ln 2}\right)\right]\\ &= -\left[\left(\frac{n\pi}{\ln 2}\right)^2 + \frac{1}{4}\right]\frac{1}{\sqrt{x+1}}\sin\left(\frac{n\pi\ln(x+1)}{\ln 2}\right) = -\left[\left(\frac{n\pi}{\ln 2}\right)^2 + \frac{1}{4}\right]\phi_n(x).\end{aligned}$$

Thus $\phi_n(x)$ solves the ordinary differential equation.

$$\begin{aligned}\phi_n(0) &= \frac{1}{\sqrt{1}}\sin\left(\frac{n\pi\ln(1)}{\ln 2}\right) = 0\\ \phi_n(1) &= \frac{1}{\sqrt{2}}\sin\left(\frac{n\pi\ln(2)}{\ln 2}\right) = 0\end{aligned}$$

Thus $\phi_n(x)$ satisfies the boundary conditions.

7.1.12

(a) $f(x) = x^2 - 6x + 9$

$$L[x^2 - 6x + 9] = 2 - 6(2x-6) + 9(x^2 - 6x + 9) = 9x^2 - 66x + 119$$

(b) $g(x) = e^{3x}$

$$L[e^{3x}] = 9e^{3x} - 6(3e^{3x}) + 9e^{3x} = 0$$

(c) $h(x) = xe^{3x}$

$$L[xe^{3x}] = 3e^{3x} + 3e^{3x} + 9xe^{3x} - 6(e^{3x} + 3xe^{3x}) + 9xe^{3x} = 0$$

(d) $k(x) = \cosh(3x)$

$$L[\cosh(3x)] = 9\cosh(3x) - 6(3\sinh(3x)) + 9\cosh(3x) = 18\cosh(3x) - 18\sinh(3x)$$

7.2.1 By Eq. (7.15)

$$u\,L[v] - v\,L[u] = \frac{d}{dx}\left[p(x)\left(u(x)\frac{dv}{dx} - v(x)\frac{du}{dx}\right)\right],$$

thus

$$\int_a^b (uL[v] - vL[u])\;dx = \left[p(x)\left(u(x)\frac{dv}{dx} - v(x)\frac{du}{dx}\right)\right]_{x=a}^{x=b}.$$

7.2.3 By assumption

$$\begin{aligned} \alpha_1 u(a) + \beta_1 u'(a) = \alpha_2 u(b) + \beta_2 u'(b) = 0 \\ \alpha_1 v(a) + \beta_1 v'(a) = \alpha_2 v(b) + \beta_2 v'(b) = 0 \end{aligned}$$

and thus

$$\left[p(x)\left(u(x)\frac{dv}{dx} - v(x)\frac{du}{dx}\right)\right]_{x=a}^{x=b} = p(b)\left(u(b)v'(b) - v(b)u'(b)\right) - p(a)\left(u(a)v'(a) - v(a)u'(a)\right).$$

If $\alpha_1 \neq 0$ and $\alpha_2 \neq 0$ then

$$\beta_1 u'(a)/\alpha_1 = u(a) \text{ and } \beta_2 u'(b)/\alpha_2 = u(b).$$

Likewise

$$\beta_1 v'(a)/\alpha_1 = v(a) \text{ and } \beta_2 v'(b)/\alpha_2 = v(b).$$

Substituting into the equation above yields

$$p(b)\left(\frac{\beta_2}{\alpha_2}u'(b)v'(b) - \frac{\beta_2}{\alpha_2}u'(b)v'(b)\right) - p(a)\left(\frac{\beta_1}{\alpha_1}u'(a)v'(a) - \frac{\beta_1}{\alpha_1}u'(a)v'(a)\right) = 0.$$

If $\alpha_1 \neq 0$ and $\beta_2 \neq 0$ then

$$\beta_1 u'(a)/\alpha_1 = u(a) \text{ and } \alpha_2 u(b)/\beta_2 = u'(b).$$

Likewise

$$\beta_1 v'(a)/\alpha_1 = v(a) \text{ and } \alpha_2 v(b)/\beta_2 = v'(b).$$

Substituting once again produces

$$\begin{aligned} &p(b)\left(u(b)v'(b) - v(b)u'(b)\right) - p(a)\left(u(a)v'(a) - v(a)u'(a)\right) \\ &= p(b)\left(\frac{\alpha_2}{\beta_2}u(b)v(b) - \frac{\alpha_2}{\beta_2}u(b)v(b)\right) - p(a)\left(\frac{\beta_1}{\alpha_1}u'(a)v'(a) - \frac{\beta_1}{\alpha_1}u'(a)v'(a)\right) = 0. \end{aligned}$$

Similar results hold if $\beta_1 \neq 0$ and $\alpha_2 \neq 0$. Finally if $\beta_1 \neq 0$ and $\beta_2 \neq 0$ then

$$\alpha_1 u(a)/\beta_1 = u'(a) \text{ and } \alpha_2 u(b)/\beta_2 = u'(b).$$

Likewise

$$\alpha_1 v(a)/\beta_1 = v'(a) \text{ and } \alpha_2 v(b)/\beta_2 = v'(b).$$

Upon substitution the following equation results.

$$\begin{aligned}
&p(b)\left(u(b)v'(b) - v(b)u'(b)\right) - p(a)\left(u(a)v'(a) - v(a)u'(a)\right) \\
&= p(b)\left(\frac{\alpha_2}{\beta_2}u(b)v(b) - \frac{\alpha_2}{\beta_2}u(b)v(b)\right) - p(a)\left(\frac{\alpha_1}{\beta_1}u(a)v(a) - \frac{\alpha_1}{\beta_1}u(a)v(a)\right) = 0
\end{aligned}$$

7.2.5 There are three cases to consider.

Case $\lambda = 0$**:** which implies $y(x) = Ax + B$. The boundary conditions imply $A = B = 0$.

Case $\lambda = -\gamma^2$**:** where $\gamma > 0$. In this situation the general solution to the ordinary differential equation is $y(x) = A\cosh(\gamma x) + B\sinh(\gamma x)$. The boundary conditions imply $A = B = 0$.

Case $\lambda = \gamma^2$**:** where $\gamma > 0$. In this situation the general solution to the ordinary differential equation is $y(x) = A\cos(\gamma x) + B\sin(\gamma x)$. The boundary conditions require the following two equations be satisfied.

$$\begin{aligned}
0 &= A + B\gamma \\
0 &= (B - A\gamma)\sin\gamma
\end{aligned}$$

If $\gamma = n\pi$ with $n \in \mathbb{N}$ then if $A = -Bn\pi$ the function

$$\phi_n(x) = -Bn\pi\cos(n\pi x) + B\sin(n\pi x)$$

is an eigenfunction corresponding to the eigenvalue $\lambda_n = n^2\pi^2$.

7.2.7 Suppose $m \neq n$ then

$$\begin{aligned}
\int_1^e \sin(m\pi\ln x)\sin(n\pi\ln x)\frac{1}{x}\,dx &= \int_0^1 \sin(m\pi z)\sin(n\pi z)\,dz \\
&= \frac{1}{2}\int_0^1 \left[\cos((m-n)\pi z) - \cos((m+n)\pi z)\right]\,dz = 0.
\end{aligned}$$

If $m = n$ then

$$\begin{aligned}
\int_1^e \sin^2(n\pi\ln x)\frac{1}{x}\,dx &= \int_0^1 \sin^2(n\pi z)\,dz \\
&= \frac{1}{2}\int_0^1 \left[1 - \cos(2n\pi z)\right]\,dz = \frac{1}{2}.
\end{aligned}$$

7.2.9 Using the hint,

$$\begin{aligned}
y(x) &= \frac{z(x)}{x} \\
y'(x) &= \frac{xz'(x) - z(x)}{x^2} \\
y''(x) &= \frac{x^2z''(x) - 2xz'(x) + 2z(x)}{x^3}.
\end{aligned}$$

Substituting these expressions into the ordinary differential equation produces,

$$\begin{aligned}
x^2\frac{x^2z''(x) - 2xz'(x) + 2z(x)}{x^3} + 2x\frac{xz'(x) - z(x)}{x^2} + \gamma x^2\frac{z(x)}{x} &= 0 \\
xz''(x) - 2z'(x) + \frac{2z(x)}{x} + 2z'(x) - \frac{2z(x)}{x} + \gamma xz(x) &= 0 \\
z''(x) + \gamma z(x) &= 0.
\end{aligned}$$

If $\gamma = -\lambda^2 < 0$ with $\lambda > 0$ then $z(x) = c_1 e^{-\lambda x} + c_2 e^{\lambda x}$ and thus $y(x) = c_1 \frac{e^{-\lambda x}}{x} + c_2 \frac{e^{\lambda x}}{x}$. If such as solution is to be bounded as $x \to 0^+$ then $c_2 = 0$. If $y'(1) = 0$ this implies $\lambda = -1 < 0$ which is a contradiction of the assumption that $\lambda > 0$. Therefore there are no negative eigenvalues with nontrivial eigenfunctions.

If $\gamma = 0$ then $z(x) = c_1 x + c_2$ and hence $y(x) = c_1 + \frac{c_2}{x}$. If such as solution is to be bounded as $x \to 0^+$ then $c_2 = 0$. Therefore $y_0(x) = a_0$ (a constant) is an eigenfunction corresponding to the eigenvalue $\gamma_0 = 0$.

If $\gamma = \lambda^2 > 0$ with $\lambda > 0$ then $z(x) = c_1 \cos(\lambda x) + c_2 \sin(\lambda x)$ and thus $y(x) = c_1 \frac{\cos(\lambda x)}{x} + c_2 \frac{\sin(\lambda x)}{x}$. If such as solution is to be bounded as $x \to 0^+$ then $c_1 = 0$. If $y'(1) = 0$ this implies $\lambda = \tan \lambda > 0$. If λ_n is the nth positive solution to this equation, then λ_n^2 is an eigenvalue with corresponding eigenfunction $y_n(x) = \frac{\sin(\lambda_n x)}{x}$.

7.2.11

(a) $\langle f+g, h\rangle_\rho = \langle f, h\rangle_\rho + \langle g, h\rangle_\rho$

$$\begin{aligned}\langle f+g, h\rangle_\rho &= \int_a^b (f+g)(x)h(x)\rho(x)\,dx = \int_a^b (f(x)h(x)\rho(x) + g(x)h(x)\rho(x))\,dx \\ &= \int_a^b f(x)h(x)\rho(x)\,dx + \int_a^b g(x)h(x)\rho(x))\,dx = \langle f, h\rangle_\rho + \langle g, h\rangle_\rho\end{aligned}$$

(b) $\langle \gamma f, g\rangle_\rho = \gamma\langle f, g\rangle_\rho$

$$\langle \gamma f, g\rangle_\rho = \int_a^b \gamma f(x)g(x)\rho(x)\,dx = \gamma\int_a^b f(x)g(x)\rho(x)\,dx = \gamma\langle f, g\rangle_\rho$$

(c) $\langle f, g\rangle_\rho = \langle g, f\rangle_\rho$

$$\langle f, g\rangle_\rho = \int_a^b f(x)g(x)\rho(x)\,dx = \int_a^b g(x)f(x)\rho(x)\,dx = \langle g, f\rangle_\rho$$

(d) $\langle f, f\rangle_\rho > 0$ if and only if $f(x) \not\equiv 0$ on (a, b).

If $f(x) \equiv 0$ on (a, b) then

$$\langle f, f\rangle_\rho = \int_a^b (0)^2\rho(x)\,dx = 0.$$

If $f(x) \not\equiv 0$ on (a, b) then there exists $a < z < b$ such that $f(z) \neq 0$. Without loss of generality assume $f(z) > 0$. Since f is continuous then there exists $\epsilon > 0$ such that $f(x) > 0$ for $a < z - \epsilon < x < z + \epsilon < b$.

$$\langle f, f\rangle_\rho = \int_a^b (f(x))^2\rho(x)\,dx \geq \int_{z-\epsilon}^{z+\epsilon} (f(x))^2\rho(x)\,dx \geq 0$$

7.3.1

(a) Note that

$$y'(x) = \frac{v'(x)}{x^{1/2}} - \frac{v(x)}{2x^{3/2}} \text{ and } y''(x) = \frac{v''(x)}{x^{1/2}} - \frac{v'(x)}{x^{3/2}} + \frac{3v(x)}{4x^{5/2}}.$$

Substituting these expressions into Bessel's equation produces

$$\begin{aligned}
0 &= x^2\left(\frac{v''(x)}{x^{1/2}} - \frac{v'(x)}{x^{3/2}} + \frac{3v(x)}{4x^{5/2}}\right) + x\left(\frac{v'(x)}{x^{1/2}} - \frac{v(x)}{2x^{3/2}}\right) + (x^2-\nu^2)\frac{v(x)}{x^{1/2}} \\
&= x^{3/2}v''(x) - x^{1/2}v'(x) + \frac{3}{4x^{1/2}}v(x) + x^{1/2}v'(x) - \frac{v(x)}{2x^{1/2}} + x^{3/2}v(x) - \frac{\nu^2}{x^{1/2}}v(x) \\
&= x^{3/2}v''(x) + \left(\frac{3}{4x^{1/2}} - \frac{1}{2x^{1/2}} + x^{3/2} - \frac{\nu^2}{x^{1/2}}\right)v(x) \\
&= x^{3/2}v''(x) + \left(\frac{3-2+4x^2-4\nu^2}{4x^{1/2}}\right)v(x) \\
0 &= v''(x) + \left(\frac{1+4x^2-4\nu^2}{4x^2}\right)v(x).
\end{aligned}$$

(b) Suppose $0 < \nu < 1/2$ and define the linear operator $L[y] = y''+y$. A solution to the ordinary differential equation $L[u] + (0)u = 0$ is $u(x) = \sin x$ which has a root at every integer multiple of π. Now define $s(x) = (1-4\nu^2)/(4x^2)$. Since $0 < r(x) < 1/(4x^2)$ then by the Sturm Comparison Theorem if $v(x)$ is a solution to $L[v] + s(x)v = 0$ then $v(x)$ has a root in the interval $((n-1)\pi, n\,\pi)$ where $n \in \mathbb{N}$.

(c) Suppose $0 < \nu < 1/2$, then since $v(x) = x^{1/2}y(x)$ has a root in the interval $((n-1)\pi, n\,\pi)$ where $n \in \mathbb{N}$ and $x^{1/2} > 0$ for $x > 0$ then $y(x)$ has a root in the interval $((n-1)\pi, n\,\pi)$.

7.3.3 Two solutions are $y_1(x) = \sin x$ and $y_2(x) = \cos x$. The zeros of y_1 are the set $\{n\pi\}_{n\in\mathbb{Z}}$ while the zeros of y_2 are the set $\{\frac{(2n+1)\pi}{2}\}_{n\in\mathbb{Z}}$.

7.3.5 First consider the case of the interval $[-1,1]$. Let $q_1(x) = x^2-1$ and $q_2(x) = 0$ and note that $q_1(x) \leq q_2(x)$ on $[-1,1]$. The constant function $\psi(x) = 1$ is a nontrivial solution to the ODE $y''+q_2(x)y = 0$. Since $q_1(x) \leq q_2(x)$ on $[-1,1]$ if $y(x)$ is a nontrivial solution to $y''+q_1(x)y = 0$ with two zeros in $[-1,1]$ then $\psi(x)$ must have a zero in $[-1,1]$ according to the Sturm Comparison Theorem. This is a contradiction since $\psi(x) > 0$ always.

Now consider the interval $(1,\infty)$. For $x \geq \sqrt{2}$, $x^2-1 \geq 1$ and the function $\phi(x) = \sin x$ is a nontrivial solution to the ODE $y''+(1)y = 0$ which has infinitely many roots in $(1,\infty)$. By the Sturm Comparison Theorem then any nontrivial solution to $y''+(x^2-1)y = 0$ will also have infinitely many zeros in $(1,\infty)$. The same reasoning applies to the interval $(-\infty,-1)$.

7.3.7 Let $u(x)$ be a nontrivial solution to $u''+g(x)u = 0$ with consecutive zeros at $z_1 < z_2$. Note that $m(x) = \sin(\sqrt{A}(x-z_1))$ is a solution to the ordinary differential equation

$$y'' + A\,y = 0$$

and that $m(z_1) = 0 = m(z_1+\pi/\sqrt{A})$. Since $0 < A < g(x)$ the Sturm Comparison Theorem implies that between any two zeros of $m(x)$ there is a zero of $u(x)$. Thus there exists $z_1 < z_2 < z_1+\pi/\sqrt{A}$ for which $u(z_2) = 0$. Without loss of generality we may assume u has no other zeros in the interval (z_1,z_2). Thus $z_2 - z_1 < \pi/\sqrt{A}$.

Likewise note that $M(x) = \sin(\sqrt{B}(x-z_1))$ is a solution to the ordinary differential equation

$$y'' + B\,y = 0$$

with $M(z_1) = 0 = M(z_1+\pi/\sqrt{B})$. If $z_1 < z_2$ are consecutive zeros of u and $g(x) < B$, the Sturm Comparison Theorem implies there exists a zero of M between z_1 and z_2. Since z_1 and $z_1+\pi/\sqrt{B}$ are consecutive zeros of M then $z_1 < z_1+\pi/\sqrt{B} < z_2$ which implies $\pi/\sqrt{B} < z_2 - z_1$.

7.4.2 Evaluate the definite integral below.

$$\begin{aligned}
\langle y_n(x), y_n(x)\rangle_\rho &= \int_0^L \sin^2\left(\frac{(2n-1)\pi x}{2L}\right) dx \\
&= \frac{1}{2}\int_0^L \left[1 - \cos\left(\frac{(2n-1)\pi x}{L}\right)\right] dx \\
&= \frac{L}{2}
\end{aligned}$$

Hence the normalized eigenfunctions are

$$\phi_n(x) = \sqrt{\frac{2}{L}} \sin\left(\frac{(2n-1)\pi x}{2L}\right)$$

for $n \in \mathbb{N}$.

7.4.4 The roots of the characteristic equation are $r_1 = -1 - \sqrt{1-\lambda}$ and $r_2 = -1 + \sqrt{1-\lambda}$. When $\lambda < 1$ the general solution to the ODE has the form,

$$\begin{aligned}
y(x) &= Ae^{-x(1+\sqrt{1-\lambda})} + Be^{-x(1-\sqrt{1-\lambda})} \\
y'(x) &= -A(1+\sqrt{1-\lambda})e^{-x(1+\sqrt{1-\lambda})} - B(1-\sqrt{1-\lambda})e^{-x(1-\sqrt{1-\lambda})}.
\end{aligned}$$

When $x = 0$,

$$\begin{aligned}
0 &= -A(1+\sqrt{1-\lambda}) - B(1-\sqrt{1-\lambda}) \\
-B &= \frac{1+\sqrt{1-\lambda}}{1-\sqrt{1-\lambda}} A.
\end{aligned}$$

When $x = L$,

$$\begin{aligned}
0 &= -A(1+\sqrt{1-\lambda})e^{-L(1+\sqrt{1-\lambda})} - B(1-\sqrt{1-\lambda})e^{-L(1-\sqrt{1-\lambda})} \\
&= -A(1+\sqrt{1-\lambda})e^{-L(1+\sqrt{1-\lambda})} + A(1+\sqrt{1-\lambda})e^{-L(1-\sqrt{1-\lambda})} \\
-\sqrt{1-\lambda} &= \sqrt{1-\lambda}
\end{aligned}$$

which has no solution. Hence there are no eigenvalues less than 1.

If $\lambda = 1$ the general solution to the ODE is

$$\begin{aligned}
y(x) &= (A + Bx)e^{-x} \\
y'(x) &= (B - A - Bx)e^{-x}.
\end{aligned}$$

When $x = 0$,

$$0 = B - A \iff A = B.$$

When $X = L$,

$$\begin{aligned}
0 &= (B - A - BL)e^{-L} \\
&= A - A - AL
\end{aligned}$$

which implies $A = B = 0$ and hence there are no nontrivial solutions when $\lambda = 1$.

If $\lambda > 1$ the general solution to the ODE is

$$\begin{aligned}
y(x) &= e^{-x}(A\cos(\sqrt{\lambda-1}x) + B\sin(\sqrt{\lambda-1}x)) \\
y'(x) &= e^{-x}((B\sqrt{\lambda-1} - A)\cos(\sqrt{\lambda-1}x) + (B - A\sqrt{\lambda-1})\sin(\sqrt{\lambda-1}x)).
\end{aligned}$$

When $x = 0$,

$$\begin{aligned} 0 &= B\sqrt{\lambda - 1} - A \\ A &= B\sqrt{\lambda - 1}. \end{aligned}$$

When $x = L$,

$$\begin{aligned} 0 &= e^{-L}((B\sqrt{\lambda - 1} - A)\cos(\sqrt{\lambda - 1}L) + (B - A\sqrt{\lambda - 1})\sin(\sqrt{\lambda - 1}L)) \\ &= (B - B(\lambda - 1))\sin(\sqrt{\lambda - 1}L)). \end{aligned}$$

If $B = 0$ there are no nontrivial solutions, so assume $B \neq 0$. Therefore,

$$\sin(\sqrt{\lambda - 1}L) = 0 \iff \lambda_n = 1 + \frac{n^2\pi^2}{L^2}$$

for $n \in \mathbb{N}$. The corresponding eigenfunctions are

$$y_n(x) = e^{-x}\left(\frac{n\pi}{L}\cos\left(\frac{n\pi x}{L}\right) + \sin\left(\frac{n\pi x}{L}\right)\right)$$

for $n \in \mathbb{N}$. The eigenfunctions are normalized as follows.

$$\begin{aligned} \langle y_n(x), y_n(x)\rangle &= \int_0^L e^{-2x}\left(\frac{n^2\pi^2}{L^2}\cos^2\left(\frac{n\pi x}{L}\right) + \frac{n\pi}{L}\sin\left(\frac{2n\pi x}{L}\right) + \sin^2\left(\frac{n\pi x}{L}\right)\right)dx \\ &= \frac{(1 - e^{-2L})n^2\pi^2(n^2\pi^2 + 5L^2)}{4L^2(n^2\pi^2 + L^2)} \end{aligned}$$

Thus the normalized eigenfunctions are

$$\phi_n(x) = \frac{2L\sqrt{n^2\pi^2 + L^2}}{n\pi\sqrt{(1 - e^{-2L})(n^2\pi^2 + 5L^2)}}e^{-x}\left(\frac{n\pi}{L}\cos\left(\frac{n\pi x}{L}\right) + \sin\left(\frac{n\pi x}{L}\right)\right)$$

for $n \in \mathbb{N}$.

7.4.6 There are no nontrivial solutions to the boundary value problem for $\lambda \leq 0$. For $\lambda > 0$ the general solution to the ordinary differential equation has the form,

$$y(x) = A\cos(\sqrt{\lambda}\ln x) + B\sin(\sqrt{\lambda}\ln x).$$

When $x = 1$, $A = 0$ and when $x = L$

$$0 = B\sin(\sqrt{\lambda}\ln L).$$

If $B = 0$ there are no nontrivial solutions, so assume $B \neq 0$. Therefore,

$$\sin(\sqrt{\lambda}\ln L) = 0 \iff \lambda_n = \frac{n^2\pi^2}{(\ln L)^2}$$

for $n \in \mathbb{N}$. The corresponding eigenfunctions are

$$y_n(x) = \sin\left(\frac{n\pi\ln x}{\ln L}\right)$$

for $n \in \mathbb{N}$. The eigenfunctions are normalized as follows.

$$\begin{aligned} \langle y_n(x), y_n(x)\rangle &= \int_1^L \sin^2\left(\frac{n\pi\ln x}{\ln L}\right)dx \\ &= \frac{2(L - 1)n^2\pi^2}{4n^2\pi^2 + (\ln L)^2} \end{aligned}$$

Thus the normalized eigenfunctions are

$$\phi_n(x) = \frac{\sqrt{4n^2\pi^2 + (\ln L)^2}}{n\pi\sqrt{2(L-1)}} \sin\left(\frac{n\pi \ln x}{\ln L}\right)$$

for $n \in \mathbb{N}$.

7.5.2 If $\dfrac{d\phi(a)}{dx} = \dfrac{d\phi(b)}{dx} = 0$, then

$$p(b)\phi(b)\frac{d\phi}{dx}(b) - p(a)\phi(a)\frac{d\phi}{dx}(a) = 0.$$

7.5.4 Various answers are possible, for example if $y(x) = x$ then the boundary conditions are satisfied.

$$R[y] = \frac{-\int_0^1 x(0)\,dx}{\int_0^1 x^2\,dx} = 0$$

Thus $\lambda_1 \leq 0$ (in fact, since $y(x)$ is an eigenfunction, $\lambda_1 = 0$).

7.5.6 Consider the trial function $y(x) = \sin(\pi x)$.

$$R[y] = \frac{-\int_0^1 \sin(\pi x)(-\pi^2 \sin(\pi x))\,dx}{\int_0^1 \left(1 + \frac{x}{100}\right)\sin^2(\pi x)\,dx} = \frac{\pi^2/2}{201/400} \approx 9.8205$$

7.5.8 If $\lambda = 0$ the solution to the ODE is a linear function $y(x) = Ax + B$. If $x = 0$ then $y(0) = B = 0$ implies $y(x) = Ax$. When $x = 1$ the boundary condition implies,

$$2A + 37A = 0 \iff A = 0$$

and hence there are no nontrivial solutions. If $\lambda > 0$ the oscillatory solutions to the ODE are of the form,

$$y(x) = A\cos(\sqrt{\lambda}x) + B\sin(\sqrt{\lambda}x).$$

If $y(0) = 0$ this implies $A = 0$. At $x = 1$,

$$\begin{aligned} 0 &= 37B\sin(\sqrt{\lambda}) + 2B\sqrt{\lambda}\cos(\sqrt{\lambda}) \\ &= 37\sin(\sqrt{\lambda}) + 2\sqrt{\lambda}\cos(\sqrt{\lambda}). \end{aligned}$$

Plotting the right-hand side of the last equation as a function of λ reveals infinitely many positive zeros. Using Newton's method the four smallest positive zeros are approximately

$$\begin{aligned} \lambda_1 &\approx 8.89107 \\ \lambda_2 &\approx 35.6527 \\ \lambda_3 &\approx 80.5178 \\ \lambda_4 &\approx 143.791 \end{aligned}$$

7.6.2 Use the piecewise-defined function,

$$\theta = \begin{cases} \arctan\frac{y}{x} & \text{if } x > 0, \\ \pi + \arctan\frac{y}{x} & \text{if } x < 0. \end{cases}$$

7.6.4 Differentiate $y(x)$.

$$\begin{aligned} y'(x) &= -\frac{A}{p(x)}\cos\left(B - \int_a^x \frac{1}{p(u)}\,du\right) \\ y''(x) &= -\frac{A}{(p(x))^2}\sin\left(B - \int_a^x \frac{1}{p(u)}\,du\right) + \frac{Ap'(x)}{(p(x))^2}\cos\left(B - \int_a^x \frac{1}{p(u)}\,du\right) \end{aligned}$$

The ordinary differential equation takes the following form.

$$\begin{aligned} L[y](x)+\frac{1}{p(x)}y(x) &= p(x)y''(x)+p'(x)y'(x)+\frac{1}{p(x)}y(x) \\ &= -\frac{A}{p(x)}\sin\left(B-\int_a^x \frac{1}{p(u)}\,du\right)+\frac{Ap'(x)}{p(x)}\cos\left(B-\int_a^x \frac{1}{p(u)}\,du\right) \\ &\quad -\frac{Ap'(x)}{p(x)}\cos\left(B-\int_a^x \frac{1}{p(u)}\,du\right)+\frac{A}{p(x)}\sin\left(B-\int_a^x \frac{1}{p(u)}\,du\right) \\ &= 0 \end{aligned}$$

This demonstrates $y(x)$ solves Eq. (7.10).
7.6.6 Let $K=0$ and $L=1$ in the result of Exercise 7.6.5, then $f(x)=0$.
8.1.2 By definition $\Gamma(p+1)=\int_0^\infty e^{-x}x^p\,dx$. If $u^2=x$ then $2u\,du=dx$ and

$$\Gamma(p+1)=\int_0^\infty e^{-u^2}(u^2)^p(2u)\,du=2\int_0^\infty e^{-u^2}u^{2p+1}\,du.$$

8.1.4

$$\Gamma\left(n+\frac{1}{2}\right)=\frac{1}{2}\left(\frac{3}{2}\right)\cdots\left(\frac{2n-1}{2}\right)\sqrt{\pi}=\frac{(1)(3)\cdots(2n-1)\sqrt{\pi}}{2^n}=\frac{(2n!)\sqrt{\pi}}{(2)(4)\cdots(2n)2^n}=\frac{(2n)!\sqrt{\pi}}{2^{2n}n!}$$

8.1.6

(a) Let $\Gamma(p)=2\int_0^\infty e^{-u^2}u^{2(p-1)+1}\,du$ and $\Gamma(q)=2\int_0^\infty e^{-v^2}v^{2(q-1)+1}\,dv$, then

$$\Gamma(p)\Gamma(q)=\left(2\int_0^\infty e^{-u^2}u^{2p-1}\,du\right)\left(2\int_0^\infty e^{-v^2}v^{2q-1}\,dv\right)=4\int_0^\infty\int_0^\infty e^{-u^2-v^2}u^{2p-1}v^{2q-1}\,du\,dv.$$

(b) Let $u=r\cos t$ and $v=r\sin t$ then

$$\begin{aligned} \Gamma(p)\Gamma(q) &= 4\int_0^\infty\int_0^\infty e^{-u^2-v^2}u^{2p-1}v^{2q-1}\,du\,dv=4\int_0^{\pi/2}\int_0^\infty e^{-r^2}(r\cos t)^{2p-1}(r\sin t)^{2q-1}r\,dr\,dt \\ &= \left(2\int_0^{\pi/2}\cos^{2p-1}t\sin^{2q-1}t\,dt\right)\left(2\int_0^\infty e^{-r^2}r^{2(p+q)-1}\,dr\right)=2\Gamma(p+q)\int_0^{\pi/2}\cos^{2p-1}t\sin^{2q-1}t\,dt. \end{aligned}$$

8.1.8

(a) $\displaystyle\int_0^{\pi/2}\sin^{2n}t\,dt=\frac{\pi}{2}\frac{(2n)!}{2^{2n}(n!)^2}=\int_0^{\pi/2}\cos^{2n}t\,dt$ for $n=0,1,2,\ldots$.

$$\int_0^{\pi/2}\sin^{2n}t\,dt=\int_0^{\pi/2}\cos^{2n}t\,dt=\frac{\Gamma(1/2)\Gamma(n+1/2)}{2\Gamma(n+1)}=\frac{\sqrt{\pi}\frac{(2n)!\sqrt{\pi}}{2^{2n}n!}}{2(n!)}=\frac{\pi}{2}\frac{(2n)!}{2^{2n}(n!)^2}$$

(b) $\displaystyle\int_0^{\pi/2}\sin^{2n+1}t\,dt=\frac{2^{2n}(n!)^2}{(2n+1)!}=\int_0^{\pi/2}\cos^{2n+1}t\,dt$ for $n=0,1,2,\ldots$.

$$\int_0^{\pi/2}\sin^{2n+1}t\,dt=\int_0^{\pi/2}\cos^{2n+1}t\,dt=\frac{\Gamma(1/2)\Gamma(n+1)}{2\Gamma(n+1+1/2)}=\frac{\sqrt{\pi}n!}{(2)\frac{(2n+2)!\sqrt{\pi}}{2^{2n+2}(n+1)!}}=\frac{2^{2n}(n!)^2}{(2n+1)!}$$

8.1.10 If $\ln x = -t$ then $x = e^{-t}$ and $dx = -e^{-t}\,dt$. As $x \to 0^+$ then $t \to \infty$ while as $x \to 1^-$, $t \to 0$.

$$\begin{aligned}\int_0^1 (\ln x)^r\,dx &= \int_\infty^0 (-t)^r(-e^{-t})\,dt \\ &= (-1)^n \int_0^\infty t^r e^{-t}\,dt \\ &= (-1)^n \int_0^\infty t^{(r+1)-1} e^{-t}\,dt \\ &= (-1)^n \Gamma(r+1)\end{aligned}$$

8.2.2

$$\cos(x\sin\theta - n\theta) = \cos(x\sin\theta)\cos(n\theta) + \sin(x\sin\theta)\sin(n\theta)$$

If $n = 2m-1$ is odd $\cos(x\sin\theta)\cos((2m-1)\theta)$ exhibits odd symmetry about $\theta = \pi/2$ and $\sin(x\sin\theta)\sin((2m-1)\theta)$ exhibits even symmetry about $\theta = \pi/2$. Thus

$$\int_0^\pi \cos(x\sin\theta)\cos((2m-1)\theta)\,d\theta = 0$$

and

$$\int_0^\pi \sin(x\sin\theta)\sin((2m-1)\theta)\,d\theta = 2\int_0^{\pi/2} \sin(x\sin\theta)\sin((2m-1)\theta)\,d\theta$$

If $n = 2m$ is even $\cos(x\sin\theta)\cos(2m\theta)$ exhibits even symmetry about $\theta = \pi/2$ and $\sin(x\sin\theta)\sin(2m\theta)$ exhibits odd symmetry about $\theta = \pi/2$. Thus

$$\int_0^\pi \sin(x\sin\theta)\sin(2m\theta)\,d\theta = 0$$

and

$$\int_0^\pi \cos(x\sin\theta)\cos(2m\theta)\,d\theta = 2\int_0^{\pi/2} \cos(x\sin\theta)\cos(2m\theta)\,d\theta.$$

8.2.4

(a) $f(x) = \begin{cases} 1 & \text{if } 0 < x < 1/2, \\ 0 & \text{if } 1/2 < x < 1. \end{cases}$ with $p = 0$.

The nth coefficient in the series is

$$a_n = \frac{2\int_0^1 f(x)J_0(\lambda_{0,n}x)x\,dx}{(J_1(\lambda_{0,n}))^2} = \frac{2\int_0^{1/2} J_0(\lambda_{0,n}x)x\,dx}{(J_1(\lambda_{0,n}))^2}.$$

Numerically approximated the first five coefficients are $a_1 \approx 0.769756$, $a_2 \approx 0.661472$, $a_3 \approx -0.282963$, $a_4 \approx -0.464336$, and $a_5 \approx 0.198712$. Thus $f(x) = \sum_{n=1}^\infty a_n J_0(\lambda_{0,n}x)$.

(b) $f(x) = x^2$ with $p = 2$.

The nth coefficient in the series is

$$a_n = \frac{2\int_0^1 f(x)J_2(\lambda_{2,n}x)x\,dx}{(J_3(\lambda_{2,n}))^2} = \frac{2\int_0^1 J_2(\lambda_{2,n}x)x^3\,dx}{(J_3(\lambda_{2,n}))^2}.$$

Numerically approximated the first five coefficients are $a_1 \approx 1.14652$, $a_2 \approx -0.875544$, $a_3 \approx 0.74048$, $a_4 \approx -0.654457$, and $a_5 \approx 0.593202$. Thus $f(x) = \sum_{n=1}^\infty a_n J_2(\lambda_{2,n}x)$.

(c) $f(x) = x^3$ with $p = 3$.

The nth coefficient in the series is

$$a_n = \frac{2\int_0^1 f(x) J_3(\lambda_{3,n} x) x\, dx}{(J_4(\lambda_{3,n}))^2} = \frac{2\int_0^1 J_3(\lambda_{3,n} x) x^4\, dx}{(J_4(\lambda_{3,n}))^2}.$$

Numerically approximated the first five coefficients are $a_1 \approx 1.05095$, $a_2 \approx -0.821503$, $a_3 \approx 0.703991$, $a_4 \approx -0.627577$, and $a_5 \approx 0.572301$. Thus $f(x) = \sum_{n=1}^{\infty} a_n J_3(\lambda_{3,n} x)$.

(d) $f(x) = x^4$ with $p = 4$.

The nth coefficient in the series is

$$a_n = \frac{2\int_0^1 f(x) J_4(\lambda_{4,n} x) x\, dx}{(J_5(\lambda_{4,n}))^2} = \frac{2\int_0^1 J_4(\lambda_{4,n} x) x^5\, dx}{(J_5(\lambda_{4,n}))^2}.$$

Numerically approximated the first five coefficients are $a_1 \approx 0.982109$, $a_2 \approx -0.779533$, $a_3 \approx 0.674312$, $a_4 \approx -0.605009$, and $a_5 \approx 0.554342$. Thus $f(x) = \sum_{n=1}^{\infty} a_n J_4(\lambda_{4,n} x)$.

8.2.6 When $n = 0$, Eq. (8.35) gives

$$x^{0+1}\sqrt{\frac{2}{\pi x}}\frac{\cos x}{x} = \sqrt{\frac{2}{\pi x}}\cos x = J_{-1/2}(x).$$

For $n = 1$, Eq. (8.35) gives

$$x^{1+1}\sqrt{\frac{2}{\pi x}}\frac{d}{x\,dx}\left[\frac{\cos x}{x}\right] = -\sqrt{\frac{2}{\pi x}}\left(\frac{1}{x}\cos x - \sin x\right) = J_{-3/2}(x).$$

To use the principle of mathematical induction, suppose the claim has been established for some $m \in \mathbb{N}$. Then Eq. (8.24) with $p = -m - 1/2$ can be written as

$$J_{-m-3/2}(x) = \frac{-m - 1/2}{x} J_{-m-1/2}(x) + J'_{-m-1/2}(x).$$

Substituting the expression from the right-hand side of Eq. (8.35) into the right-hand side of the last equation yields

$$\begin{aligned}
J_{-m-3/2}(x) &= \frac{-m-1/2}{x} x^{m+1}\sqrt{\frac{2}{\pi x}}\left(\frac{d}{x\,dx}\right)^m \frac{\cos x}{x} + \frac{d}{dx}\left[x^{m+1}\sqrt{\frac{2}{\pi x}}\left(\frac{d}{x\,dx}\right)^m \frac{\cos x}{x}\right] \\
&= (-m - \frac{1}{2}) x^m \sqrt{\frac{2}{\pi x}}\left(\frac{d}{x\,dx}\right)^m \frac{\cos x}{x} + (m + \frac{1}{2}) x^m \sqrt{\frac{2}{\pi x}}\left(\frac{d}{x\,dx}\right)^m \frac{\cos x}{x} \\
&\quad + x^{m+1}\sqrt{\frac{2}{\pi x}}\frac{d}{dx}\left[\left(\frac{d}{x\,dx}\right)^m \frac{\cos x}{x}\right] \\
&= x^{m+2}\sqrt{\frac{2}{\pi x}}\left(\frac{d}{x\,dx}\right)^{m+1}\frac{\cos x}{x},
\end{aligned}$$

which is Eq. (8.35) with n replaced by $m + 1$.

8.2.8

(a) $\dfrac{d}{dx}\left[x^{n+1} j_n(x)\right] = x^{n+1} j_{n-1}(x)$

$$\begin{aligned}
\frac{d}{dx}\left[x^{n+1} j_n(x)\right] &= \frac{d}{dx}\left[x^{n+1}\sqrt{\frac{\pi}{2x}} J_{n+1/2}(x)\right] \\
&= (n+1)x^n\sqrt{\frac{\pi}{2x}} J_{n+1/2}(x) - \frac{x^n}{2}\sqrt{\frac{\pi}{2x}} J_{n+1/2}(x) \\
&\quad + x^{n+1}\sqrt{\frac{\pi}{2x}}\left(J_{n-1/2}(x) - \frac{n+1/2}{x} J_{n+1/2}(x)\right) \\
&= x^{n+1}\sqrt{\frac{\pi}{2x}} J_{n-1/2}(x) = x^{n+1} j_{n-1}(x)
\end{aligned}$$

(b) $\dfrac{d}{dx}\left[x^{-n} j_n(x)\right] = -x^{-n} j_{n+1}(x)$

$$\begin{aligned}
\frac{d}{dx}\left[x^{-n} j_n(x)\right] &= \frac{d}{dx}\left[x^{-n}\sqrt{\frac{\pi}{2x}} J_{n+1/2}(x)\right] \\
&= -nx^{-n-1}\sqrt{\frac{\pi}{2x}} J_{n+1/2}(x) - \frac{x^{-n-1}}{2}\sqrt{\frac{\pi}{2x}} J_{n+1/2}(x) \\
&\quad + x^{-n}\sqrt{\frac{\pi}{2x}}\left(\frac{n+1/2}{x} J_{n+1/2}(x) - J_{n+3/2}(x)\right) \\
&= -x^{-n}\sqrt{\frac{\pi}{2x}} J_{n+3/2}(x) = -x^{-n} j_{n+1}(x)
\end{aligned}$$

(c) $j_{n-1}(x) + j_{n+1}(x) = \dfrac{2n+1}{x} j_n(x)$

$$\begin{aligned}
j_{n-1}(x) + j_{n+1}(x) &= \sqrt{\frac{\pi}{2x}}\left(J_{n-1/2}(x) + J_{n+3/2}(x)\right) \\
&= \sqrt{\frac{\pi}{2x}}\left(J'_{n+1/2}(x) + \frac{n+1/2}{x} J_{n+1/2}(x) + -J_{n+1/2}(x) + \frac{n+1/2}{x} J_{n+1/2}(x)\right) \\
&= \frac{2n+1}{x}\sqrt{\frac{\pi}{2x}} J_{n+1/2}(x) = \frac{2n+1}{x} j_n(x)
\end{aligned}$$

(d) $n\, j_{n-1}(x) - (n+1) j_{n+1}(x) = (2n+1) j'_n(x)$

$$
\begin{aligned}
(2n+1)j_n'(x) &= (2n+1)\left(-\frac{1}{2x}\sqrt{\frac{\pi}{2x}}J_{n+1/2}(x) + \sqrt{\frac{\pi}{2x}}J'_{n+1/2}(x)\right) \\
&= \frac{-(2n+1)}{2x}\sqrt{\frac{\pi}{2x}}J_{n+1/2}(x) \\
&\quad + \frac{(2n+1)}{2}\sqrt{\frac{\pi}{2x}}\left(\frac{n+1/2}{x}J_{n+1/2}(x) - J_{n+3/2}(x) + J_{n-1/2}(x) - \frac{n+1/2}{x}J_{n+1/2}(x)\right) \\
&= \frac{-(2n+1)}{2x}\sqrt{\frac{\pi}{2x}}J_{n+1/2}(x) + \frac{(2n+1)}{2}\sqrt{\frac{\pi}{2x}}\left(J_{n-1/2}(x) - J_{n+3/2}(x)\right) \\
&= \frac{-(2n+1)}{2x}j_n(x) + \frac{(2n+1)}{2}(j_{n-1}(x) - j_{n+1}(x)) \\
&= -\frac{1}{2}(j_{n-1}(x) + j_{n+1}(x)) + \frac{(2n+1)}{2}(j_{n-1}(x) - j_{n+1}(x)) \\
&= n\, j_{n-1}(x) - (n+1)j_{n+1}(x)
\end{aligned}
$$

8.2.10 Let $x = \rho/a$ then

$$
\begin{aligned}
\int_0^a j_n\left(\frac{\lambda_{n+1/2,k}\rho}{a}\right) j_n\left(\frac{\lambda_{n+1/2,l}\rho}{a}\right)\rho^2\,d\rho &= a^3\int_0^1 j_n(\lambda_{n+1/2,k}x)j_n(\lambda_{n+1/2,l}x)x^2\,dx \\
&= \frac{a^3}{2}\left(j_{n+1}(\lambda_{n+1/2,k})\right)^2\delta_{kl},
\end{aligned}
$$

according to Cor. 8.2.
8.2.12 Using Eq. (8.42)

$$
G(J_n(x);1) = \sum_{n=-\infty}^{\infty} J_n(x)(1)^n = e^{\frac{x}{2}(1-1^{-1})} = e^0 = 1.
$$

8.2.14 Making use of the hint,

$$
G(J_n(x);z)G(J_n(x);z^{-1}) = e^{\frac{x}{2}(z-z^{-1})}e^{\frac{x}{2}(z^{-1}-z)} = e^0 = 1.
$$

Therefore we may write,

$$
\begin{aligned}
1 &= \sum_{m=-\infty}^{\infty} J_m(x)z^m \sum_{n=-\infty}^{\infty} J_n(x)z^{-n} \\
&= \sum_{k=-\infty}^{\infty}\left(\sum_{m-n=k} J_m(x)J_n(x)\right)z^{m-n} \\
&= \sum_{k=-\infty}^{\infty}\left(\sum_{n=-\infty}^{\infty} J_{n+k}(x)J_n(x)\right)z^k \\
&= \sum_{n=-\infty}^{\infty} J_n^2(x)z^0 + \sum_{\substack{k\in\mathbb{Z}\\k\neq 0}}\left(\sum_{n=-\infty}^{\infty} J_{n+k}(x)J_n(x)\right)z^k.
\end{aligned}
$$

When $z = 0$ then

$$
1 = \sum_{n=-\infty}^{\infty} J_n^2(x).
$$

8.2.16 In Exercise 8.2.14 we derived that

$$1 = \sum_{n=-\infty}^{\infty} J_n^2(x) + \sum_{\substack{k\in\mathbb{Z}\\k\neq 0}}^{\infty} \left(\sum_{n=-\infty}^{\infty} J_{n+k}(x) J_n(x) \right) z^k$$

$$1 = 1 + \sum_{\substack{k\in\mathbb{Z}\\k\neq 0}}^{\infty} \left(\sum_{n=-\infty}^{\infty} J_{n+k}(x) J_n(x) \right) z^k$$

$$0 = \sum_{\substack{k\in\mathbb{Z}\\k\neq 0}}^{\infty} \left(\sum_{n=-\infty}^{\infty} J_{n+k}(x) J_n(x) \right) z^k$$

Equating powers like powers of z on both sides of the last equation reveals,

$$\sum_{n=-\infty}^{\infty} J_{n+k}(x) J_n(x) = 0.$$

8.3.2 Making use of the hints,

$$\begin{aligned}
\int_{-1}^{1} f(x)P_n(x)\,dx &= \int_{-1}^{1} f(x)\frac{1}{2^n n!}\frac{d^n}{dx^n}\left[(x^2-1)^n\right]\,dx \\
&= \frac{1}{2^n n!}\left(\left[f(x)\frac{d^{n-1}}{dx^{n-1}}\left[(x^2-1)^n\right]\right]_{x=-1}^{x=1} - \int_{-1}^{1} f'(x)\frac{d^{n-1}}{dx^{n-1}}\left[(x^2-1)^n\right]\,dx\right) \\
&= \frac{-1}{2^n n!}\int_{-1}^{1} f'(x)\frac{d^{n-1}}{dx^{n-1}}\left[(x^2-1)^n\right]\,dx \\
&\ \vdots \\
&= \frac{(-1)^n}{2^n n!}\int_{-1}^{1} f^{(n)}(x)(x^2-1)^n\,dx.
\end{aligned}$$

8.3.4

(a) $\displaystyle\int_{-1}^{1}(1-x)P_2(x)\,dx$

$$\int_{-1}^{1}(1-x)P_2(x)\,dx = 0$$

since $\deg(1-x) = 1 < 2$.

(b) $\displaystyle\int_{-1}^{1}(1-x^2)P_2(x)\,dx$

$$\int_{-1}^{1}(1-x^2)P_2(x)\,dx = \frac{(-1)^2}{2^2 2!}\int_{-1}^{1}(-2)(x^2-1)^2\,dx = -\frac{4}{15}$$

(c) $\displaystyle\int_{-1}^{1} x^n P_n(x)\,dx$

$$\int_{-1}^{1} x^n P_n(x)\,dx = \frac{(-1)^n}{2^n n!}\int_{-1}^{1} n!(x^2-1)^n\,dx = \frac{(-1)^n}{2^n}\int_{-1}^{1}(x^2-1)^n\,dx$$
$$= \frac{(-1)^n}{2^n}\int_{-1}^{1}\sum_{k=0}^{n}\frac{n!x^{2k}(-1)^{n-k}}{k!(n-k)!}\,dx = \frac{n!}{2^n}\sum_{k=0}^{n}\frac{2(-1)^k}{k!(n-k)!}\int_0^1 x^{2k}\,dx$$
$$= \frac{n!}{2^{n-1}}\sum_{k=0}^{n}\frac{(-1)^k}{(2k+1)k!(n-k)!}$$

(d) $\int_{-1}^{1} x^n P_{n-1}(x)\,dx$

$$\int_{-1}^{1} x^n P_{n-1}(x)\,dx = \frac{(-1)^{n-1}}{2^{n-1}(n-1)!}\int_{-1}^{1} n!x(x^2-1)^{n-1}\,dx = \frac{(-1)^{n-1}}{2^{n-1}}\int_{-1}^{1} x(x^2-1)^{n-1}\,dx = 0$$

8.3.6

(a) $f(x) = \begin{cases} -1 & \text{if } -1 < x < 0, \\ 1 & \text{if } 0 < x < 1. \end{cases}$

$$a_n = -\frac{2n+1}{2}\int_{-1}^{0} P_n(x)\,dx + \frac{2n+1}{2}\int_0^1 P_n(x)\,dx = \begin{cases} 0 & \text{if } n = 0 \\ \frac{(2n+1)\sqrt{\pi}}{2\Gamma(1-n/2)\Gamma((3+n)/2)} & \text{if } n \geq 1 \end{cases}$$
$$f(x) \sim \sum_{n=1}^{\infty}\frac{(2n+1)\sqrt{\pi}}{2\Gamma(1-n/2)\Gamma((3+n)/2)}P_n(x)$$

(b) $f(x) = \begin{cases} 0 & \text{if } -1 < x < 0, \\ x & \text{if } 0 < x < 1. \end{cases}$

$$a_n = \frac{2n+1}{2}\int_0^1 xP_n(x)\,dx = \frac{(2n+1)\sqrt{\pi}}{8\Gamma(2+n/2)\Gamma((3-n)/2)}$$
$$f(x) \sim \sum_{n=0}^{\infty}\frac{(2n+1)\sqrt{\pi}}{8\Gamma(2+n/2)\Gamma((3-n)/2)}P_n(x)$$

(c) $f(x) = |x|$.

$$a_n = -\frac{2n+1}{2}\int_{-1}^{0} xP_n(x)\,dx + \frac{2n+1}{2}\int_0^1 xP_n(x)\,dx = \begin{cases} 0 & \text{if } n \text{ is odd} \\ \frac{(2n+1)\sqrt{\pi}}{4\Gamma(2+n/2)\Gamma((3-n)/2)} & \text{if } n \text{ is even} \end{cases}$$
$$f(x) \sim \sum_{n=0}^{\infty}\frac{(4n+1)\sqrt{\pi}}{4(n+1)!\Gamma(3/2-n)}P_{2n}(x)$$

(d) $f(x) = \sin\frac{\pi x}{2}$.

$$
\begin{aligned}
a_n &= \frac{2n+1}{2}\int_{-1}^{1}\sin\left(\frac{\pi x}{2}\right)P_n(x)\,dx = \frac{(2n+1)(-1)^n}{2^{n+1}n!}\int_{-1}^{1}\left[\sin\frac{\pi x}{2}\right]^{(n)}(x^2-1)^n\,dx \\
&= \begin{cases} 0 & \text{if } n \text{ is even} \\ \dfrac{(4k-1)(-1)^k\pi^{2k-1}}{2^{4k-2}(2k-1)!}\displaystyle\int_0^1 \cos\frac{\pi x}{2}(x^2-1)^{2k-1}\,dx & \text{if } n=2k-1 \text{ is odd} \end{cases} \\
f(x) &\sim \sum_{k=1}^{\infty} a_{2k-1}P_{2k-1}(x)
\end{aligned}
$$

8.3.8

$$
\begin{aligned}
P'_{n+1}(x) - x\,P'_n(x) &= (n+1)P_n(x) \\
nP_n(x) &= P'_{n+1}(x) - x\,P'_n(x) - P_n(x) = P'_{n+1}(x) - \frac{d}{dx}\left[xP_n(x)\right] \\
n\int_0^1 P_n(x)\,dx &= \int_0^1 P'_{n+1}(x)\,dx - \int_0^1 \frac{d}{dx}\left[xP_n(x)\right]\,dx \\
&= P_{n+1}(1) - P_{n+1}(0) - P_n(1) = -P_{n+1}(0)
\end{aligned}
$$

8.3.10 Recall that if $n=0$ then $P_0(x)=1$ and $P'_0(x)=0$. Substituting into Eq. (8.76) produces

$$
\int_t^1 P_m(x)\,dx = \frac{(1-t^2)P'_m(t)}{m(m+1)}.
$$

8.3.12

(a) $n=2$ and $m=-1$

$$
P_2^{-1}(x) = \frac{(2-1)!}{(2+1)!}(1-x^2)^{1/2}\frac{d}{dx}\left[P_2(x)\right] = \frac{1}{6}(1-x^2)^{1/2}(3x) = \frac{x}{2}(1-x^2)^{1/2}
$$

(b) $n=3$ and $m=0$

$$
P_3^0(x) = P_3(x) = \frac{1}{2}(5x^3-3x)
$$

(c) $n=4$ and $m=2$

$$
P_4^2(x) = (1-x^2)\frac{d^2}{dx^2}\left[P_4(x)\right] = (1-x^2)\frac{1}{8}(420x^2-60) = \frac{15}{2}(x^2-1)(1-7x^2)
$$

(d) $n=4$ and $m=-2$

$$
P_4^{-2}(x) = \frac{(4-2)!}{(4+2)!}(1-x^2)\frac{d^2}{dx^2}\left[P_4(x)\right] = \frac{2}{720}(1-x^2)\frac{1}{8}(420x^2-60) = \frac{1}{48}(x^2-1)(1-7x^2)
$$

8.3.14 When n is even, $a_n = 0$.

$$a_{2n-1} = \frac{4n-1}{2}\,\frac{(2n-2)!}{(2n)!}\int_{-1}^{1}(1-x^2)P_{2n-1}^{1}(x)\,dx \text{ for } n \in \mathbb{N}$$
$$a_1 = -\frac{9\pi}{32}$$
$$a_3 = \frac{7\pi}{256}$$
$$a_5 = \frac{11\pi}{4096}$$
$$a_7 = \frac{45\pi}{65536}$$
$$a_9 = \frac{133\pi}{524288}$$

8.3.16

$$G(P_n(-1);z) = \frac{1}{\sqrt{1+2z+z^2}}$$
$$\sum_{n=0}^{\infty} P_n(-1)z^n = \frac{1}{1+z}$$
$$= \sum_{n=0}^{\infty}(-1)^n z^n$$

Equating coefficients of the powers of reveals $P_n(-1) = (-1)^n$ for $n = 0, 1, 2, \ldots$.

8.3.18 Suppose $n \in \{0\} \cup \mathbb{N}$ and $m \in \{0, 1, \ldots, n\}$. Find the mth derivative of $G(P_n(x);z)$ with respect to x.

$$\frac{d^0}{dx^0}\left[(1-2xz+z^2)^{-1/2}\right] = (1-2xz+z^2)^{-1/2}$$
$$\frac{d^1}{dx^1}\left[(1-2xz+z^2)^{-1/2}\right] = (1)z(1-2xz+z^2)^{-3/2}$$
$$\frac{d^2}{dx^2}\left[(1-2xz+z^2)^{-1/2}\right] = (1)(3)z^2(1-2xz+z^2)^{-5/2}$$
$$\frac{d^3}{dx^3}\left[(1-2xz+z^2)^{-1/2}\right] = (1)(3)(5)z^3(1-2xz+z^2)^{-7/2}$$
$$\vdots$$
$$\frac{d^m}{dx^m}\left[(1-2xz+z^2)^{-1/2}\right] = (1)(3)(5)\cdots(2m-1)z^m(1-2xz+z^2)^{-m-1/2}$$
$$= \frac{(2m)!}{2^m m!}\,\frac{z^m}{(1-2xz+z^2)^{m+1/2}}$$

This is equivalent to differentiating inside the infinite series.

$$\frac{(2m)!}{2^m m!}\,\frac{z^m}{(1-2xz+z^2)^{m+1/2}} = \sum_{n=0}^{\infty} P_n^{(m)}(x)z^n$$

Multiply both sides of this equation by $(-1)^m(1-x^2)^{m/2}$ to obtain the generating function,

$$\begin{aligned} G(P_n^m(x); z) &= \sum_{n=0}^{\infty} P_n^m(x) z^n \\ &= \frac{(2m)!}{2^m m!} \frac{(-1)^m (1-x^2)^{m/2} z^m}{(1-2xz+z^2)^{m+1/2}}. \end{aligned}$$

8.4.1 Make the substitution $x = \cos\varphi$, then

$$\int_{-\pi}^{\pi}\int_{0}^{\pi} e^{i\,m\,\theta} P_n^m(\cos\varphi) e^{-i\,\hat{m}\,\theta} P_{\hat{n}}^{\hat{m}}(\cos\varphi) \sin\varphi \, d\varphi \, d\theta = \int_{-\pi}^{\pi} e^{i(m-\hat{m})\theta} \, d\theta \int_{-1}^{1} P_n^m(x) P_{\hat{n}}^{\hat{m}}(x) \, dx$$

If $m \neq \hat{m}$ the first integral on the right-hand side becomes

$$\int_{-\pi}^{\pi} e^{i(m-\hat{m})\theta} \, d\theta = \left[\frac{1}{i(m-\hat{m})} e^{i(m-\hat{m})\theta}\right]_{\theta=-\pi}^{\theta=\pi} = \frac{1}{i(m-\hat{m})}\left(e^{i(m-\hat{m})\pi} - e^{-i(m-\hat{m})\pi}\right) = 0$$

by Euler's identity. Now suppose $n \neq \hat{n}$. If $m \neq \hat{m}$ then the result has already been proved, so without loss of generality assume $m = \hat{m}$.

$$\int_{-\pi}^{\pi}\int_{0}^{\pi} e^{i\,m\,\theta} P_n^m(\cos\varphi) e^{-i\,\hat{m}\,\theta} P_{\hat{n}}^{\hat{m}}(\cos\varphi) \sin\varphi \, d\varphi \, d\theta = 2\pi \int_{-1}^{1} P_n^m(x) P_{\hat{n}}^m(x) \, dx = 0$$

by Thm. 8.10.

Now suppose $m = \hat{m}$ and $n = \hat{n}$, then

$$\int_{-\pi}^{\pi}\int_{0}^{\pi} e^{i\,m\,\theta} P_n^m(\cos\varphi) e^{-i\,\hat{m}\,\theta} P_{\hat{n}}^{\hat{m}}(\cos\varphi) \sin\varphi \, d\varphi \, d\theta = 2\pi \int_{-1}^{1} \left(P_n^m(x)\right)^2 \, dx = \frac{4\pi}{2n+1} \frac{(n+m)!}{n-m)!}$$

by Thm. 8.11.

8.4.3

$$Y_n^n(\varphi, \theta) = \sqrt{\frac{2n+1}{4\pi} \frac{(n-n)!}{(n+n)!}} e^{in\theta} P_n^n(\cos\varphi)$$

By definition, $P_n^n(x) = (-1)^n(1-x^2)^{n/2} P_n^{(n)}(x)$. Since $P_n(x)$ is a polynomial of degree n with leading coefficient $\frac{(2n)!}{2^n (n!)^2}$ then $P_n^n(x) = (-1)^n(1-x^2)^{n/2} \frac{(2n)!}{2^n n!}$. This implies

$$\begin{aligned} Y_n^n(\varphi, \theta) &= \sqrt{\frac{2n+1}{4\pi(2n)!}} e^{in\theta} (-1)^n (1-\cos^2\varphi)^{n/2} \frac{(2n)!}{2^n n!} \\ &= \frac{(-1)^n}{2^n n!} \sqrt{\frac{(2n+1)!}{4\pi(2n)!}} e^{in\theta} \sin^n \varphi. \end{aligned}$$

8.4.5

$$\begin{aligned}
Y_n^m(\pi-\varphi,\pi+\theta) &= \sqrt{\frac{2n+1}{4\pi}\frac{(n-m)!}{(n+m)!}}e^{im(\pi+\theta)}P_n^m(\cos(\pi-\varphi)) \\
&= e^{im\pi}\sqrt{\frac{2n+1}{4\pi}\frac{(n-m)!}{(n+m)!}}e^{im\theta}P_n^m(\cos\pi\cos\varphi+\sin\pi\sin\varphi) \\
&= (\cos(m\pi)+i\sin(m\pi))\sqrt{\frac{2n+1}{4\pi}\frac{(n-m)!}{(n+m)!}}e^{im\theta}P_n^m(-\cos\varphi) \\
&= (-1)^m\sqrt{\frac{2n+1}{4\pi}\frac{(n-m)!}{(n+m)!}}e^{im\theta}P_n^m(-\cos\varphi) \\
&= (-1)^m(-1)^{m+n}\sqrt{\frac{2n+1}{4\pi}\frac{(n-m)!}{(n+m)!}}e^{im\theta}P_n^m(\cos\varphi) \text{ (by Lemma 8.11)} \\
&= (-1)^n Y_n^m(\varphi,\theta)
\end{aligned}$$

8.5.1 Using integration by parts,

$$\begin{aligned}
\int_0^\infty f(x)L_n(x)e^{-x}\,dx &= \frac{1}{n!}\int_0^\infty f(x)\frac{d^n}{dx^n}\left[e^{-x}x^n\right]\,dx \\
&= \frac{1}{n!}\left[f(x)\frac{d^{n-1}}{dx^{n-1}}(e^{-x}x^n)\right]_{x=0}^{x\to\infty} - \frac{1}{n!}\int_0^\infty f'(x)\frac{d^{n-1}}{dx^{n-1}}\left[e^{-x}x^n\right]\,dx \\
&= \frac{-1}{n!}\int_0^\infty f'(x)\frac{d^{n-1}}{dx^{n-1}}\left[e^{-x}x^n\right]\,dx \\
&\vdots \\
&= \frac{(-1)^n}{n!}\int_0^\infty f^{(n)}(x)e^{-x}x^n\,dx.
\end{aligned}$$

8.5.3

$$\begin{aligned}
a_n &= \int_0^\infty J_0(2\sqrt{x})L_n(x)e^{-x}\,dx = \int_0^\infty L_n(x)e^{-x}\sum_{k=0}^\infty \frac{(-1)^k}{(k!)^2}x^k\,dx \\
&= \sum_{k=0}^\infty \frac{(-1)^k}{(k!)^2}\int_0^\infty x^k L_n(x)e^{-x}\,dx = \sum_{k=0}^\infty \frac{(-1)^{k+n}}{(k!)^2(n!)}\int_0^\infty \left[x^k\right]^{(n)} x^n e^{-x}\,dx \\
&= \sum_{k=n}^\infty \frac{(-1)^{k+n}}{(k!)^2(n!)}\int_0^\infty k(k-1)\cdots(k-n+1)x^{k-n}x^n e^{-x}\,dx = \sum_{k=n}^\infty \frac{(-1)^{k+n}}{(k!)(n!)(k-n)!}\int_0^\infty x^k e^{-x}\,dx \\
&= \sum_{k=n}^\infty \frac{(-1)^{k+n}}{(n!)(k-n)!} = \frac{1}{n!}\sum_{k=n}^\infty \frac{(-1)^{k-n}}{(k-n)!} = \frac{e^{-1}}{n!}
\end{aligned}$$

$$J_0(2\sqrt{x}) = e^{-1}\sum_{n=0}^\infty \frac{L_n(x)}{n!}$$

8.5.5 Multiply both sides of the equation by $x^\alpha e^{-x}$.

$$\begin{aligned}
xy'' + (\alpha+1-x)y' + ny &= 0 \\
x^{\alpha+1}e^{-x}y'' + (\alpha+1-x)x^\alpha e^{-x}y' + nx^\alpha e^{-x}y &= 0 \\
\left[x^{\alpha+1}e^{-x}y'\right]' + nx^\alpha e^{-x}y &= 0
\end{aligned}$$

8.5.7

$$\begin{aligned}
\frac{d}{dx}\left[L_n^\alpha(x)\right] &= \frac{d}{dx}\left[\frac{x^{-\alpha}e^x}{n!}\frac{d^n}{dx^n}(e^{-x}x^{n+\alpha})\right] = \frac{d}{dx}\left[\frac{x^{-\alpha}e^x}{n!}\sum_{k=0}^{n}\frac{n!}{k!(n-k)!}\left[e^{-x}\right]^{(k)}\left[x^{n+\alpha}\right]^{(n-k)}\right] \\
&= \frac{d}{dx}\left[\sum_{k=0}^{n}\frac{(-1)^k}{k!(n-k)!}(n+\alpha)(n-1+\alpha)\cdots(n-(n-k-1)+\alpha)x^k\right] \\
&= \sum_{k=1}^{n}\frac{(-1)^k}{(k-1)!(n-k)!}(n+\alpha)(n-1+\alpha)\cdots(n-(n-k-1)+\alpha)x^{k-1} \\
&= -\sum_{k=0}^{n-1}\frac{(-1)^k}{k!(n-k-1)!}(n+\alpha)(n-1+\alpha)\cdots(n-(n-k-2)+\alpha)x^k \\
&= -\sum_{k=0}^{n-1}\frac{(-1)^k}{k!((n-1)-k)!}\left(\prod_{j=0}^{n-k-2}((n-1)-j+(\alpha+1))\right)x^k \\
&= -\sum_{k=0}^{n-1}\frac{(-1)^k\Gamma((n-1)+1+(\alpha+1))}{k!((n-1)-k)!\Gamma(k+1+(\alpha+1))}x^k = -L_{n-1}^{\alpha+1}(x)
\end{aligned}$$

8.5.9 Starting with the Rodrigues' formula for $L_n^\alpha(x)$,

$$\begin{aligned}
L_n^\alpha(x) &= \frac{x^{-\alpha}e^x}{n!}\frac{d^n}{dx^n}\left[x^{n+\alpha}e^{-x}\right] = \frac{x^{-\alpha}e^x}{n!}\frac{d^{n-1}}{dx^{n-1}}\left[(n+\alpha)x^{n-1+\alpha}e^{-x} - x^{n+\alpha}e^{-x}\right] \\
&= \frac{(n+\alpha)}{n}\frac{x^{-\alpha}e^x}{(n-1)!}\frac{d^{n-1}}{dx^{n-1}}\left[x^{n-1+\alpha}e^{-x}\right] - \frac{x^{-\alpha}e^x}{n!}\frac{d^{n-1}}{dx^{n-1}}\left[x^{n+\alpha}e^{-x}\right] \\
&= \frac{(n+\alpha)}{n}L_{n-1}^\alpha(x) - \frac{x^{-\alpha}e^x}{n!}\frac{d^{n-1}}{dx^{n-1}}\left[x^{n+\alpha}e^{-x}\right].
\end{aligned}$$

Differentiate both sides of the equation.

$$\begin{aligned}
\frac{d}{dx}\left[L_n^\alpha(x)\right] &= \frac{(n+\alpha)}{n}\frac{d}{dx}\left[L_{n-1}^\alpha(x)\right] - \frac{-\alpha x^{-\alpha-1}e^x + x^{-\alpha}e^x}{n!}\frac{d^{n-1}}{dx^{n-1}}\left[x^{n+\alpha}e^{-x}\right] - \frac{x^{-\alpha}e^x}{n!}\frac{d^n}{dx^n}\left[x^{n+\alpha}e^{-x}\right] \\
&= \frac{(n+\alpha)}{n}\frac{d}{dx}\left[L_{n-1}^\alpha(x)\right] + \frac{\alpha}{n}\frac{x^{-(\alpha+1)}e^x}{(n-1)!}\frac{d^{n-1}}{dx^{n-1}}\left[x^{n-1+\alpha+1}e^{-x}\right] \\
&\quad - \frac{x}{n}\frac{x^{-(\alpha+1)}e^x}{(n-1)!}\frac{d^{n-1}}{dx^{n-1}}\left[x^{n-1+\alpha+1}e^{-x}\right] - L_n^\alpha(x) \\
&= \frac{(n+\alpha)}{n}\frac{d}{dx}\left[L_{n-1}^\alpha(x)\right] + \frac{\alpha}{n}L_{n-1}^{\alpha+1}(x) - \frac{x}{n}L_{n-1}^{\alpha+1}(x) - L_n^\alpha(x) \\
&= \frac{(n+\alpha)}{n}\frac{d}{dx}\left[L_{n-1}^\alpha(x)\right] - \frac{\alpha}{n}\frac{d}{dx}\left[L_n^\alpha(x)\right] + \frac{x}{n}\frac{d}{dx}\left[L_n^\alpha(x)\right] - L_n^\alpha(x)
\end{aligned}$$

Using the result of Exercise 8.5.8,

$$\begin{aligned}
\frac{d}{dx}\left[L_n^\alpha(x)\right] &= \left(1+\frac{\alpha}{n}\right)\frac{d}{dx}\left[L_{n-1}^\alpha(x)\right] + \frac{x-\alpha}{n}\frac{d}{dx}\left[L_n^\alpha(x)\right] - \frac{x}{n}\frac{d}{dx}\left[L_{n-1}^\alpha(x)\right] - \left(1+\frac{\alpha}{n}-\frac{x}{n}\right)L_{n-1}^\alpha(x) \\
&= \left(1+\frac{\alpha}{n}-\frac{x}{n}\right)\frac{d}{dx}\left[L_{n-1}^\alpha(x)\right] + \frac{x-\alpha}{n}\frac{d}{dx}\left[L_n^\alpha(x)\right] - \left(1+\frac{\alpha}{n}-\frac{x}{n}\right)L_{n-1}^\alpha(x) \\
&= \frac{d}{dx}\left[L_{n-1}^\alpha(x)\right] - L_{n-1}^\alpha(x) + \frac{x-\alpha}{n}\left(L_{n-1}^\alpha(x) + \frac{d}{dx}\left[L_n^\alpha(x)\right] - \frac{d}{dx}\left[L_{n-1}^\alpha(x)\right]\right) \\
0 &= \frac{x-\alpha-n}{n}\left(L_{n-1}^\alpha(x) + \frac{d}{dx}\left[L_n^\alpha(x)\right] - \frac{d}{dx}\left[L_{n-1}^\alpha(x)\right]\right)
\end{aligned}$$

which implies

$$L_{n-1}^{\alpha}(x) + \frac{d}{dx}\left[L_n^{\alpha}(x)\right] - \frac{d}{dx}\left[L_{n-1}^{\alpha}(x)\right] = 0$$

and the identity is established.
8.5.11 Using integration by parts,

$$\begin{aligned}
\int_0^\infty f(x)L_n^{\alpha}(x)x^{\alpha}e^{-x}\,dx &= \frac{1}{n!}\int_0^\infty f(x)x^{-\alpha}e^{x}\frac{d^n}{dx^n}\left[e^{-x}x^{n+\alpha}\right]x^{\alpha}e^{-x}\,dx \\
&= \frac{1}{n!}\int_0^\infty f(x)\frac{d^n}{dx^n}\left[e^{-x}x^{n+\alpha}\right]\,dx \\
&= \frac{1}{n!}\left[f(x)\frac{d^{n-1}}{dx^{n-1}}(e^{-x}x^{n+\alpha})\right]_{x=0}^{x\to\infty} - \frac{1}{n!}\int_0^\infty f'(x)\frac{d^{n-1}}{dx^{n-1}}[e^{-x}x^{n+\alpha}]\,dx \\
&= -\frac{1}{n!}\int_0^\infty f'(x)\frac{d^{n-1}}{dx^{n-1}}[e^{-x}x^{n+\alpha}]\,dx \\
&\vdots \\
&= \frac{(-1)^n}{n!}\int_0^\infty f^{(n)}(x)x^{n+\alpha}e^{-x}\,dx
\end{aligned}$$

8.5.13 Start by expanding the right-hand side of the proposed generating function as a Taylor series in z.

$$\begin{aligned}
\frac{1}{1-z}e^{-\frac{xz}{1-z}} &= \frac{1}{1-z}\sum_{n=0}^{\infty}\frac{1}{n!}\left(-\frac{xz}{1-z}\right)^n \\
&= \sum_{n=0}^{\infty}\frac{(-1)^n}{n!}\frac{x^n z^n}{(1-z)^{n+1}}
\end{aligned}$$

If $|z| < 1$ then $(1-z)^{-1} = \sum_{k=0}^{\infty} z^k$. Differentiating this geometric series n times produces

$$\frac{n!}{(1-z)^{n+1}} = \sum_{k=n}^{\infty} k(k-1)\cdots(k-n+1)z^{k-n} = \sum_{k=n}^{\infty}\frac{k!}{(k-n)!}z^{k-n} = \sum_{k=0}^{\infty}\frac{(k+n)!}{k!}z^k.$$

Substituting this into the Taylor series expansion above gives

$$\begin{aligned}
\frac{1}{1-z}e^{-\frac{xz}{1-z}} &= \sum_{n=0}^{\infty}\left(\frac{(-1)^n}{n!}x^n z^n\sum_{k=0}^{\infty}\frac{(k+n)!}{k!n!}z^k\right) \\
&= \sum_{n=0}^{\infty}\left(\frac{(-1)^n}{(n!)^2}x^n\sum_{k=0}^{\infty}\frac{(k+n)!}{k!}z^{n+k}\right) \\
&= \sum_{n=0}^{\infty}\sum_{k=0}^{\infty}\frac{(-1)^n}{(n!)^2}x^n\frac{(k+n)!}{k!}z^{n+k}.
\end{aligned}$$

To find the coefficient of the mth power of z in the double summation suppose $m = k+n$. The summation above can be rewritten as

$$\begin{aligned}
\frac{1}{1-z}e^{-\frac{xz}{1-z}} &= \sum_{m=0}^{\infty}\sum_{\substack{k+n=m\\ k,n\geq 0}}\frac{(-1)^n}{(n!)^2}x^n\frac{(k+n)!}{k!}z^m \\
&= \sum_{m=0}^{\infty}\left(m!\sum_{n=0}^{m}\frac{(-1)^n}{(n!)^2(m-n)!}x^n\right)z^m \\
&= \sum_{m=0}^{\infty}L_m(x)z^m = G(L_m(x);z).
\end{aligned}$$

8.5.15 Proceeding as in Exercise 8.5.13 we have

$$\frac{1}{(1-z)^{\alpha+1}}e^{-\frac{xz}{1-z}} = \sum_{n=0}^{\infty} \frac{(-1)^n}{n!} \frac{x^n z^n}{(1-z)^{n+\alpha+1}}.$$

The next step is to expand $(1-z)^{-(n+\alpha+1)}$ as a Taylor series about $z_0 = 0$.

$$\frac{1}{(1-z)^{n+\alpha+1}} = \sum_{k=0}^{\infty} \frac{(n+\alpha+1)(n+\alpha+2)\cdots(n+\alpha+k)z^k}{k!}$$

Substituting this into the Taylor series expansion above gives

$$\begin{aligned}
\frac{1}{(1-z)^{\alpha+1}}e^{-\frac{xz}{1-z}} &= \sum_{n=0}^{\infty}\sum_{k=0}^{\infty} \frac{(-1)^n}{n!} x^n \frac{(n+\alpha+1)\cdots(n+\alpha+k)}{k!} z^{n+k} \\
&= \sum_{n=0}^{\infty}\sum_{k=0}^{\infty} \frac{(-1)^n}{n!} x^n \frac{\Gamma(n+\alpha+k+1)}{k!\Gamma(n+\alpha+1)} z^{n+k},
\end{aligned}$$

by Lemma 8.3. To find the coefficient of the mth power of z in the double summation suppose $m = k+n$. The summation above can be rewritten as

$$\begin{aligned}
\frac{1}{(1-z)^{\alpha+1}}e^{-\frac{xz}{1-z}} &= \sum_{m=0}^{\infty} \left(\sum_{\substack{k+n=m \\ k,n\geq 0}} \frac{(-1)^n}{n!} x^n \frac{\Gamma(m+\alpha+1)}{k!\Gamma(n+\alpha+1)} \right) z^m \\
&= \sum_{m=0}^{\infty}\sum_{n=0}^{m} \left(\frac{(-1)^n \Gamma(m+\alpha+1) x^n}{n!(m-n)!\Gamma(n+\alpha+1)} \right) z^m \\
&= \sum_{m=0}^{\infty} L_m^{\alpha}(x) z^m = G(L_m^{\alpha}(x); z).
\end{aligned}$$

8.6.2

$$\begin{aligned}
H_{n+1}(x) &= (-1)^{n+1} e^{x^2} \frac{d^{n+1}}{dx^{n+1}} \left[e^{-x^2} \right] = (-1)^{n+1} e^{x^2} \frac{d^n}{dx^n} \left[-2x e^{-x^2} \right] \\
&= (-1)^n e^{x^2} \sum_{k=0}^{n} \frac{n!}{k!(n-k)!} [2x]^{(k)} \left[e^{-x^2} \right]^{(n-k)} \\
&= (-1)^n e^{x^2} \sum_{k=0}^{1} \frac{n!}{k!(n-k)!} [2x]^{(k)} \left[e^{-x^2} \right]^{(n-k)} \\
&= (-1)^n e^{x^2} \left(2x \left[e^{-x^2} \right]^{(n)} + 2n \left[e^{-x^2} \right]^{(n-1)} \right) \\
&= 2x(-1)^n e^{x^2} \frac{d^n}{dx^n} \left[e^{-x^2} \right] - 2n(-1)^{n+1} e^{x^2} \frac{d^{n-1}}{dx^{n-1}} \left[e^{-x^2} \right] = 2xH_n(x) - 2nH_{n-1}(x)
\end{aligned}$$

8.6.4 Since $f(x)$ is a quadratic function, it must be a linear combination of $H_0(x)$, $H_1(x)$, and $H_2(x)$.

$$x^2 + x + 1 = a(1) + b(2x) + c(4x^2 - 2) = 4cx^2 + 2bx + (a - 2c)$$

Equating coefficients results in $a = 3/2$, $b = 1/2$, and $c = 1/4$.

8.6.6

$$\begin{aligned}
a_n &= \frac{1}{2^n n!\sqrt{\pi}} \int_{-\infty}^{\infty} e^x H_n(x) e^{-x^2}\,dx = \frac{1}{2^n n!\sqrt{\pi}} \int_{-\infty}^{\infty} [e^x]^{(n)} e^{-x^2}\,dx \\
&= \frac{1}{2^n n!\sqrt{\pi}} \int_{-\infty}^{\infty} e^x e^{-x^2}\,dx = \frac{1}{2^n n!\sqrt{\pi}} \int_{-\infty}^{\infty} e^{-(x^2-x)}\,dx \\
&= \frac{e^{1/4}}{2^n n!\sqrt{\pi}} \int_{-\infty}^{\infty} e^{-(x^2-x+1/4)}\,dx = \frac{e^{1/4}}{2^n n!\sqrt{\pi}} \int_{-\infty}^{\infty} e^{-(x-1/2)^2}\,dx \\
&= \frac{e^{1/4}}{2^n n!} \\
f(x) &\sim e^{1/4} \sum_{n=0}^{\infty} \frac{1}{2^n n!} H_n(x)
\end{aligned}$$

8.6.8

$$\begin{aligned}
\frac{(-1)^n}{2^{2n+1} n!\sqrt{x}} H_{2n+1}(\sqrt{x}) &= \frac{(-1)^n}{2^{2n+1} n!\sqrt{x}} (2n+1)! \sum_{k=0}^{n} \frac{(-1)^k}{k!(2n+1-2k)!} (2\sqrt{x})^{2n+1-2k} \\
&= \frac{(2n+1)!}{n!} \sum_{k=0}^{n} \frac{(-1)^{n-k} x^{n-k}}{2^{2k} k!(2(n-k)+1)!} = \frac{(2n+1)!}{n!} \sum_{k=0}^{n} \frac{(-1)^k x^k}{2^{2(n-k)} (n-k)!(2k+1)!} \\
&= \sum_{k=0}^{n} \frac{(-1)^k 2^{2k} (2n+1)! x^k}{2^{2n} n!(n-k)!(2k+1)!} = \sum_{k=0}^{n} \frac{(-1)^k (n+1/2)(n-1/2)\cdots(3/2)(1/2) x^k}{k!(n-k)!(k+1/2)(k-1/2)\cdots(3/2)(1/2)} \\
&= \sum_{k=0}^{n} \frac{(-1)^k \Gamma(n+1+1/2) x^k}{k!(n-k)!\Gamma(k+1+1/2)} = L_n^{1/2}(x)
\end{aligned}$$

8.6.10

$$\begin{aligned}
G\left(\frac{1}{n!} H_n(-x); z\right) &= \sum_{n=0}^{\infty} \frac{1}{n!} H_n(-x) z^n \\
&= \sum_{n=0}^{\infty} \frac{1}{n!} (-1)^n H_n(x) z^n \\
&= \sum_{n=0}^{\infty} \frac{1}{n!} H_n(x) (-z)^n \\
&= G\left(\frac{1}{n!} H_n(x); -z\right)
\end{aligned}$$

Equating like powers of z on both sides of the equation implies $H_n(-x) = (-1)^n H_n(x)$ or equivalently $H_n(x) = (-1)^n H_n(-x)$ for $n \in \{0\} \cup \mathbb{N}$.

This can also be shown directly from the formula for the Hermite polynomials in Eq. (8.104).

8.6.12 When $n = 0$, $\left(2x - \frac{d}{dx}\right)^0 [1] = 1 = H_0(x)$. Likewise when $n = 1$, $\left(2x - \frac{d}{dx}\right)^1 [1] = 2x - 0 = 2x =$

$H_1(x)$. Suppose there exists an $n \in \mathbb{N}$ such that the formula holds for all $k \le n$. Consider the case of $n+1$.

$$\begin{aligned}\left(2x - \frac{d}{dx}\right)^{n+1}[1] &= \left(2x - \frac{d}{dx}\right)\left[\left(2x - \frac{d}{dx}\right)^{n}[1]\right] \\ &= \left(2x - \frac{d}{dx}\right)[H_n(x)] \\ &= 2xH_n(x) - H_n'(x) \\ &= 2xH_n(x) - 2nH_{n-1}(x) \text{ (by Exercise 8.6.3)} \\ &= H_{n+1}(x) \text{ (by Exercise 8.6.2)}\end{aligned}$$

Hence the formula holds for all $n \in \{0\} \cup \mathbb{N}$.

8.7.2

(a) $(x-1)[T_{2n+1}(x) - 1] = [T_{n+1}(x) - T_n(x)]^2$

Using Eq. (8.121) and the product-to-sum formula for the cosine

$$\begin{aligned}[T_{n+1}(x) - T_n(x)]^2 &= T_{n+1}(x)T_{n+1}(x) - 2T_{n+1}(x)T_n(x) + T_n(x)T_n(x) \\ &= \frac{1}{2}\left(T_{2n+2}(x) - 1\right) - \left(T_{2n+1}(x) + T_1(x)\right) + \frac{1}{2}\left(T_{2n}(x) - 1\right) \\ &= \frac{1}{2}T_{2n+2}(x) - T_{2n+1}(x) - T_1(x) + \frac{1}{2}T_{2n}(x) - 1 \\ &= T_1(x)T_{2n+1}(x) - T_{2n+1}(x) - T_1(x) - 1 \\ &= xT_{2n+1}(x) - T_{2n+1}(x) - x - 1 = (x-1)[T_{2n+1}(x) - 1].\end{aligned}$$

(b) $2(x^2-1)[T_{2n}(x) - 1] = [T_{n+1}(x) - T_{n-1}(x)]^2$

Using Eq. (8.121) and the product-to-sum formula for the cosine

$$\begin{aligned}[T_{n+1}(x) - T_{n-1}(x)]^2 &= T_{n+1}(x)T_{n+1}(x) - 2T_{n+1}(x)T_{n-1}(x) + T_{n-1}(x)T_{n-1}(x) \\ &= \frac{1}{2}\left(T_{2n+2}(x) - 1\right) - \left(T_{2n}(x) + T_2(x)\right) + \frac{1}{2}\left(T_{2n-2}(x) - 1\right) \\ &= \frac{1}{2}T_{2n+2}(x) - T_{2n}(x) - T_2(x) + \frac{1}{2}T_{2n-2}(x) - 1 \\ &= T_2(x)T_{2n}(x) - T_{2n}(x) - T_2(x) - 1 \\ &= (2x^2-1)T_{2n}(x) - T_{2n}(x) - (2x^2-1) - 1 \\ &= 2(x^2-1)T_{2n}(x) - 2(x^2-1) = 2(x^2-1)[T_{2n}(x) - 1].\end{aligned}$$

8.7.4 Since $p(x)$ is a polynomial of degree n, there exist constants c_k for $k = 0, 1, \ldots, n$ such that $p(x) = \sum_{k=0}^{n} c_k T_k(x)$. By assumption $p(x)$ is orthogonal to $T_0(x)$ on $[-1, 1]$ with respect to the weighting function $(1-x^2)^{-1/2}$.

$$0 = \int_{-1}^{1} T_0(x)p(x)(1-x^2)^{-1/2}\,dx = \sum_{k=0}^{n} c_k \int_{-1}^{1} \frac{T_k(x)T_0(x)}{\sqrt{1-x^2}}\,dx = \pi c_0 \iff c_0 = 0$$

Let $m \in \{1, 2, \ldots, n-1\}$, then since $p(x)$ is orthogonal to $T_m(x)$ on $[-1, 1]$ with respect to the weighting function $(1-x^2)^{-1/2}$,

$$0 = \int_{-1}^{1} T_m(x)p(x)(1-x^2)^{-1/2}\,dx = \sum_{k=0}^{n} c_k \int_{-1}^{1} \frac{T_m(x)T_k(x)}{\sqrt{1-x^2}}\,dx = \frac{\pi}{2}c_m \iff c_m = 0.$$

Since the degree of $p(x)$ is n, then $c_n \neq 0$ and $p(x) = c_n T_n(x)$.
8.7.6 The generating function takes the form,

$$\begin{aligned}\sum_{n=0}^{\infty} T_n(x) z^n &= \sum_{n=0}^{\infty} \cos(n\theta) z^n \\ &= \sum_{n=0}^{\infty} \text{Re}\left(e^{in\theta}\right) z^n \\ &= \text{Re}\left(\sum_{n=0}^{\infty} e^{in\theta} z^n\right) \\ &= \text{Re}\left(\sum_{n=0}^{\infty} (ze^{i\theta})^n\right) \\ &= \text{Re}\left(\frac{1}{1 - ze^{i\theta}}\right)\end{aligned}$$

if $|z| < 1$. Multiply numerator and denominator by the complex conjugate.

$$\begin{aligned}\sum_{n=0}^{\infty} T_n(x) z^n &= \text{Re}\left(\frac{1 - ze^{-i\theta}}{(1 - ze^{i\theta})(1 - ze^{-i\theta})}\right) \\ &= \text{Re}\left(\frac{1 - z\cos\theta + iz\sin\theta}{1 - 2z\cos\theta + z^2}\right) \\ &= \frac{1 - z\cos\theta}{1 - 2z\cos\theta + z^2} \\ &= \frac{1 - xz}{1 - 2xz + z^2}.\end{aligned}$$

8.7.8 If $n = 0$ then the extreme value of $T_0(x) = 1$. Suppose $n \geq 1$, then

$$\begin{aligned}T_n(x) &= \cos(n \arccos x) \\ T_n'(x) &= \frac{-n}{\sqrt{1 - x^2}} \sin(n \arccos x).\end{aligned}$$

Thus the function $T_n(x)$ has critical numbers at $x = \cos\frac{k\pi}{n}$ for $k = 1, 2, \ldots, n - 1$. By the Extreme Value Theorem the extrema of $T_n(x)$ occur either at the critical numbers or at $x = \pm 1$.

$$T_n\left(\cos\frac{k\pi}{n}\right) = \cos(k\pi) = (-1)^k$$

When $x = -1$, $T_n(x) = \cos(n \arccos(-1)) = \cos(n\pi) = (-1)^n$. When $x = 1$, $T_n(x) = \cos(n \arccos(1)) = \cos(0) = 1$. Hence the absolute minimum of $T_n(x)$ on $[-1, 1]$ is -1 while the absolute maximum of $T_n(x)$ on $[-1, 1]$ is 1.
8.7.10 Let $x = 0$ in Eq. (8.129).

$$\begin{aligned}G(T_n(0); z) &= \frac{1}{1 + z^2} \\ \sum_{n=0}^{\infty} T_n(0) z^n &= \sum_{m=0}^{\infty} (-1)^m z^{2m}\end{aligned}$$

Thus when n is odd, $T_n(0) = 0$ and when $n = 2m$ is even, $T_{2m}(0) = (-1)^m$.

9.1.2 If $\gamma = 0$ then Eq. (9.6) becomes

$$(L-x)X''(x) - X'(x) = 0 \text{ for } 0 < x < L$$
$$X(0) = 0.$$

Let $v(x) = X'(x)$ and divide both sides by $L - x$.

$$\begin{aligned} v'(x) - \frac{1}{L-x}v(x) &= 0 \\ \frac{v'(x)}{v(x)} &= \frac{1}{L-x} \\ v(x) &= \frac{c_1}{L-x} \\ X(x) &= c_2 - c_1 \ln(L-x) \end{aligned}$$

where c_1 and c_2 are constants. Since $X(0) = 0$ then $c_2 = c_1 \ln L$ which implies $X(x) = c_1 \ln \frac{L}{L-x}$. This solution is bounded as $x \to L^-$ only when $c_1 = 0$. Thus $X(x) = 0$.

If $\gamma > 0$, then $T''(t) - \gamma\, g\, T(t) = 0$ has solutions of the form $T(t) = Acosh((\gamma\, g)^{1/2} t) + B \sinh((\gamma\, g)^{1/2} t)$ which are unbounded except in the trivial case in which $A = B = 0$.

9.1.4

$$\begin{aligned} 0 &= (L-x)\frac{d^2X}{dx^2} - \frac{dX}{dx} - cX(x) \\ &= \alpha\xi^\beta \left(\frac{\xi^{1-\beta}}{\alpha\beta}\right)^2 \frac{d^2X}{d\xi^2} + \alpha\xi^\beta \frac{(1-\beta)\xi^{1-2\beta}}{\alpha^2\beta^2}\frac{dX}{d\xi} + \frac{\xi^{1-\beta}}{\alpha\beta}\frac{dX}{d\xi} - cX \\ &= \frac{1}{\alpha\beta^2}\xi^{2-\beta}\frac{d^2X}{d\xi^2} + \left(\frac{1-\beta}{\alpha\beta^2} + \frac{1}{\alpha\beta}\right)\xi^{1-\beta}\frac{dX}{d\xi} - cX \\ &= \xi^{2-\beta}\frac{d^2X}{d\xi^2} + \xi^{1-\beta}\frac{dX}{d\xi} - \alpha\beta^2 cX \end{aligned}$$

9.1.6 Since we are assuming $c < 0$ or equivalently $-c > 0$ let $-c = \lambda^2 > 0$ and consider the ordinary differential equation,

$$vX''(v) + X'(v) + \lambda^2 X(v) = 0.$$

The point $v = 0$ is a singular point for this ODE. The limits

$$\lim_{v\to 0} v\,\frac{1}{v} = 1 = p_0$$
$$\lim_{v\to 0} v^2\frac{\lambda^2}{v} = 0 = q_0$$

imply that $v = 0$ is a regular singular point. Assume an infinite series of the form $X(v) = \sum_{k=0}^\infty a_k v^{k+r}$ solves the ODE. Differentiating this series and substituting it into the ODE result in

$$\begin{aligned} 0 &= v\sum_{k=0}^\infty a_k(k+r-1)(k+r)v^{k+r-2} + \sum_{k=0}^\infty a_k(k+r)v^{k+r-1} + \lambda^2\sum_{k=0}^\infty a_k v^{k+r} \\ &= \sum_{k=0}^\infty a_k(k+r)^2 v^{k+r-1} + \lambda^2\sum_{k=0}^\infty a_k v^{k+r} \\ &= a_0 r^2 v^r + \sum_{k=0}^\infty \left(a_{k+1}(k+r+1)^2 + \lambda^2 a_k\right)v^{k+r}. \end{aligned}$$

If $r = 0$ the first term vanishes and the equation can be rewritten as

$$0 = \sum_{k=0}^{\infty} \left(a_{k+1}(k+1)^2 + \lambda^2 a_k \right) v^k.$$

This implies the recurrence relation for the series is

$$a_{k+1} = -\frac{\lambda^2}{(k+1)^2} a_k \text{ for } k \in \{0\} \cup \mathbb{N}.$$

If a_0 is arbitrary then by induction we see that

$$a_k = \frac{(-1)^k \lambda^{2k}}{(k!)^2} a_0 \text{ for } k \in \mathbb{N}.$$

Hence one solution to the ODE is

$$X_1(v) = \sum_{k=0}^{\infty} \frac{(-1)^k \lambda^{2k} v^k}{(k!)^2} = \sum_{k=0}^{\infty} \frac{(-1)^k (\lambda\sqrt{v})^{2k}}{(k!)^2} = \sum_{k=0}^{\infty} \frac{(-1)^k}{(k!)^2} \left(\frac{2\lambda\sqrt{v}}{2} \right)^{2k} = J_0(2\lambda\sqrt{v}).$$

A second linearly independent solution will be $X_2(t) = Y_0(2\lambda\sqrt{v})$. The general solution is then $X(v) = c_1 J_0(2\lambda\sqrt{v}) + c_2 Y_0(2\lambda\sqrt{v})$. If this solution satisfies the boundary condition $X(L) = 0$ then

$$X(v) = c_1 \left(J_0(2\lambda\sqrt{v}) - \frac{J_0(2\lambda\sqrt{L}) Y_0(2\lambda\sqrt{v})}{Y_0(2\lambda\sqrt{L})} \right).$$

9.1.8 According to Eq. (9.8) if the initial displacement of the chain is 0 then $A_n = 0$ for $n \in \mathbb{N}$. Differentiating Eq. (9.8) with respect to t and setting $t = 0$ produces,

$$x = \frac{1}{2}\sqrt{\frac{g}{L}} \sum_{n=1}^{\infty} B_n \lambda_{0,n} J_0 \left(\lambda_{0,n} \sqrt{1 - \frac{x}{L}} \right).$$

Make the change of variable

$$\xi = \sqrt{1 - \frac{x}{L}} \iff x = L(1 - \xi^2)$$

and then

$$L(1-\xi^2) = \frac{1}{2}\sqrt{\frac{g}{L}} \sum_{n=1}^{\infty} B_n \lambda_{0,n} J_0 \left(\lambda_{0,n} \xi \right)$$

$$2L\sqrt{\frac{L}{g}}(1-\xi^2) = \sum_{n=1}^{\infty} B_n \lambda_{0,n} J_0 \left(\lambda_{0,n} \xi \right).$$

Multiply both sides of this equation by $2\xi J_0(\lambda_{0,m}\xi)/(J_1(\lambda_{0,m}))^2$ and integrate over $[0, 1]$. Using the orthogonality of the Bessel function of the first kind of order 0, the only nonzero term of the right-hand side of the equation will be the one for which $n = m$.

$$B_n \lambda_{0,n} = 4L\sqrt{\frac{L}{g}} \frac{\int_0^1 (1-\xi^2) J_0(\lambda_{0,n}\xi)\xi \, d\xi}{(J_1(\lambda_{0,n}))^2}$$

$$B_n = \frac{16L^2}{\lambda_{0,n}^4 J_1(\lambda_{0,n})} \sqrt{\frac{L}{g}}$$

Thus the formal solution may be expressed as

$$u(x,t) = 16L^2\sqrt{\frac{L}{g}}\sum_{n=1}^{\infty}\frac{1}{\lambda_{0,n}^4 J_1(\lambda_{0,n})}\sin\left(\frac{\lambda_{0,n}}{2}\sqrt{\frac{g}{L}}t\right)J_0\left(\lambda_{0,n}\sqrt{1-\frac{x}{L}}\right).$$

9.1.10 Suppose the linear density of the chain is a constant ρ. At a distance of x from the axle, an infinitesimal chain element of length dx experiences the fictional centrifugal force $\omega^2 x\rho\,dx$. This will be balanced by the infinitesimal tension element in the chain denoted $-T'(x)\,dx$.

$$\begin{aligned}
-T'(x)\,dx &= \omega^2\rho x\,dx \\
-\int_x^L T'(v)\,dv &= \int_x^L \omega^2\rho v\,dv \\
T(x) - T(L) &= \frac{1}{2}\omega^2\rho(L^2 - x^2) \\
T(x) &= \frac{1}{2}\omega^2\rho(L^2 - x^2)
\end{aligned}$$

since $T(L) = 0$ (there is no tension at the end of the chain). Substituting this expression for the tension function in Eq. (9.3) yields,

$$\begin{aligned}
\rho u_{tt} &= \left(\frac{1}{2}\omega^2\rho(L^2 - x^2)u_x\right)_x \\
u_{tt} &= \frac{1}{2}\omega^2\left(L^2 - x^2 u_x\right)_x
\end{aligned}$$

since ρ is constant.

9.1.12 The product solutions to this initial, boundary value problem have the form,

$$u_n(x,t) = P_n\left(\frac{x}{L}\right)\left[A_n\cos\left(\omega t\sqrt{\frac{n(n+1)}{2}}\right) + B_n\sin\left(\omega t\sqrt{\frac{n(n+1)}{2}}\right)\right]$$

when n is an odd integer. Thus the series solution is

$$u(x,t) = \sum_{n=1}^{\infty} P_{2n-1}\left(\frac{x}{L}\right)\left[A_{2n-1}\cos\left(\omega t\sqrt{\frac{2n(2n-1)}{2}}\right) + B_{2n-1}\sin\left(\omega t\sqrt{\frac{2n(2n-1)}{2}}\right)\right].$$

When $t = 0$,

$$u(x,0) = f(x) = \sum_{n=1}^{\infty} A_{2n-1}P_{2n-1}\left(\frac{x}{L}\right).$$

Multiply both sides of the equation by $P_{2m-1}(x/L)$ and integrate over the interval $[0, L]$.

$$\begin{aligned}
\int_0^L f(x)P_{2m-1}\left(\frac{x}{L}\right)dx &= \sum_{n=1}^{\infty} A_{2n-1}\int_0^L P_{2m-1}\left(\frac{x}{L}\right)P_{2n-1}\left(\frac{x}{L}\right)dx \\
&= \sum_{n=1}^{\infty} A_{2n-1}L\int_0^1 P_{2m-1}(v)P_{2n-1}(v)\,dv
\end{aligned}$$

Recall that the set of Legendre polynomials is orthogonal on $[-1, 1]$. Since $P_{2m-1}(x)$ and $P_{2n-1}(x)$ are odd functions, their product is even and

$$\begin{aligned}
\int_0^1 P_{2m-1}(v)P_{2n-1}(v)\,dv &= \frac{1}{2}\int_{-1}^1 P_{2m-1}(v)P_{2n-1}(v)\,dv \\
&= \frac{1}{2}\,\frac{2}{2(2m-1)+1}\,\delta_{mn} = \frac{1}{4m-1}\,\delta_{mn},
\end{aligned}$$

where δ_{mn} is the Dirac delta function. Therefore

$$\sum_{n=1}^{\infty} A_{2n-1} L \int_0^1 P_{2m-1}(v) P_{2n-1}(v)\,dv = \frac{L}{4m-1} A_{2m-1}$$
$$\frac{4m-1}{L} \int_0^L f(x) P_{2m-1}\left(\frac{x}{L}\right)\,dx = A_{2m-1}.$$

Taking the derivative with respect to t of the series solution, evaluating that derivative at $t = 0$, and setting the sum of zero (since there is no initial displacement velocity) reveal that $B_{2n-1} = 0$ for all n. Hence the series solution may be expressed as follows.

$$u(x,t) = \sum_{m=1}^{\infty} A_{2m-1} P_{2m-1}\left(\frac{x}{L}\right) \cos\left(\omega t \sqrt{\frac{2m(2m-1)}{2}}\right).$$

9.2.2 When the initial displacement of the membrane is 0, a doubly-infinite series for the solution takes the form (with $r_0 = 1$ and $c = 1$):

$$u(r,\theta,t) = \sum_{m=0}^{\infty}\sum_{n=1}^{\infty} \left[J_m(\lambda_{m,n} r)\left(A_{m,n}\cos(m\theta) + B_{m,n}\sin(m\theta)\right)\sin(\lambda_{m,n} t)\right]$$

Since the initial and boundary conditions are independent of θ, the solution will be as well which enables the series solution to simplify by setting $m = 0$.

$$u(r,t) = \sum_{n=1}^{\infty} A_{0,n} J_0(\lambda_{0,n} r)\sin(\lambda_{0,n} t)$$

Calculate the partial derivative of this series with respect to t and set $t = 0$.

$$u_t(r,0) = \sum_{n=1}^{\infty} A_{0,n}\lambda_{0,n} J_0(\lambda_{0,n} r)$$
$$1 - r = \sum_{n=1}^{\infty} A_{0,n}\lambda_{0,n} J_0(\lambda_{0,n} r)$$
$$(1-r) J_0(\lambda_{0,\hat{n}} r) r = \sum_{n=1}^{\infty} A_{0,n}\lambda_{0,n} J_0(\lambda_{0,n} r) J_0(\lambda_{0,\hat{n}} r) r$$
$$\int_0^1 (r - r^2) J_0(\lambda_{0,\hat{n}} r)\,dr = \sum_{n=1}^{\infty} A_{0,n}\lambda_{n,0} \int_0^1 J_0(\lambda_{0,n} r) J_0(\lambda_{0,\hat{n}} r) r\,dr$$
$$\int_0^1 (r - r^2) J_0(\lambda_{0,\hat{n}} r)\,dr = \frac{\lambda_{0,\hat{n}}}{2} A_{0,\hat{n}} \left(J_1(\lambda_{0,n})\right)^2$$
$$A_{0,\hat{n}} = \frac{2}{\lambda_{0,\hat{n}} (J_1(\lambda_{0,\hat{n}}))^2} \int_0^1 (r - r^2) J_0(\lambda_{0,\hat{n}} r)\,dr$$

9.2.4 Assuming the radius of the drumhead is $r_0 > 0$ then the initial boundary value problem can be stated as

$$u_{tt} = c^2\left(u_{rr} + \frac{1}{r}u_r + \frac{1}{r^2}u_{\theta\theta}\right) \text{ for } r < r_0,\ -\infty < \theta < \infty, \text{ and } t > 0$$
$$u(r_0,\theta,t) = 0 \text{ for } -\infty < \theta < \infty \text{ and } t > 0$$
$$u(r,\theta,0) = 0 \text{ for } r < r_0 \text{ and } -\infty < \theta < \infty$$
$$u_t(r,\theta,0) = g(r,\theta) \text{ for } r < r_0 \text{ and } -\infty < \theta < \infty.$$

The formal solution to the initial boundary value problem found through separation of variables and the Principle of Superposition can be written as

$$u(r,\theta,t) = \sum_{m=0}^{\infty}\sum_{n=1}^{\infty}\left[J_m\left(\frac{\lambda_{m,n} r}{r_0}\right)(A_{m,n}\cos(m\theta)+B_{m,n}\sin(m\theta))\sin\left(\frac{c\lambda_{m,n} t}{r_0}\right)\right].$$

Assuming the infinite series can be differentiated term-by-term, then

$$g(r,\theta) = \sum_{m=0}^{\infty}\sum_{n=1}^{\infty}\left[\frac{c\lambda_{m,n}}{r_0}J_m\left(\frac{\lambda_{m,n} r}{r_0}\right)(A_{m,n}\cos(m\theta)+B_{m,n}\sin(m\theta))\right].$$

The values of the coefficients can be determined from the definite integrals below.

$$\begin{aligned}
A_{0,n} &= \frac{2r_0}{c\lambda_{0,n}\pi(J_1(\lambda_{0,n}))^2}\int_0^1 z\,J_0(\lambda_{0,n}z)\left(\int_{-\pi}^{\pi} g(r_0 z,\theta)\,d\theta\right)dz\\
A_{m,n} &= \frac{2r_0}{c\lambda_{m,n}\pi(J_{m+1}(\lambda_{m,n}))^2}\int_0^1 z\,J_m(\lambda_{m,n}z)\left(\int_{-\pi}^{\pi} g(r_0 z,\theta)\cos(m\theta)\,d\theta\right)dz\\
B_{m,n} &= \frac{2r_0}{c\lambda_{m,n}\pi(J_{m+1}(\lambda_{m,n}))^2}\int_0^1 z\,J_m(\lambda_{m,n}z)\left(\int_{-\pi}^{\pi} g(r_0 z,\theta)\sin(m\theta)\,d\theta\right)dz
\end{aligned}$$

9.2.6 Assuming the radius of the drumhead is $r_0 > 0$ then the initial boundary value problem can be stated as

$$\begin{aligned}
u_{tt} &= c^2\left(u_{rr}+\frac{1}{r}u_r+\frac{1}{r^2}u_{\theta\theta}\right) \text{ for } r<r_0,\ -\infty<\theta<\infty, \text{ and } t>0\\
u(r_0,\theta,t) &= 0 \text{ for } -\infty<\theta<\infty \text{ and } t>0\\
u(r,\theta,0) &= f(r,\theta) \text{ for } r<r_0 \text{ and } -\infty<\theta<\infty\\
u_t(r,\theta,0) &= g(r,\theta) \text{ for } r<r_0 \text{ and } -\infty<\theta<\infty.
\end{aligned}$$

Let $u(r,\theta,t) = v(r,\theta,t)+w(r,\theta,t)$ where function $v(r,\theta,t)$ solves the initial boundary value problem

$$\begin{aligned}
v_{tt} &= c^2\left(v_{rr}+\frac{1}{r}v_r+\frac{1}{r^2}v_{\theta\theta}\right) \text{ for } r<r_0,\ -\infty<\theta<\infty, \text{ and } t>0\\
v(r_0,\theta,t) &= 0 \text{ for } -\infty<\theta<\infty \text{ and } t>0\\
v(r,\theta,0) &= f(r,\theta) \text{ for } r<r_0 \text{ and } -\infty<\theta<\infty\\
v_t(r,\theta,0) &= 0 \text{ for } r<r_0 \text{ and } -\infty<\theta<\infty
\end{aligned}$$

and $w(r,\theta,t)$ solves the initial boundary value problem

$$\begin{aligned}
w_{tt} &= c^2\left(w_{rr}+\frac{1}{r}w_r+\frac{1}{r^2}w_{\theta\theta}\right) \text{ for } r<r_0,\ -\infty<\theta<\infty, \text{ and } t>0\\
w(r_0,\theta,t) &= 0 \text{ for } -\infty<\theta<\infty \text{ and } t>0\\
w(r,\theta,0) &= 0 \text{ for } r<r_0 \text{ and } -\infty<\theta<\infty\\
w_t(r,\theta,0) &= g(r,\theta) \text{ for } r<r_0 \text{ and } -\infty<\theta<\infty.
\end{aligned}$$

The two solutions can be expressed as infinite series where

$$v(r,\theta,t) = \sum_{m=0}^{\infty}\sum_{n=1}^{\infty}\left[J_m\left(\frac{\lambda_{m,n} r}{r_0}\right)(A_{m,n}\cos(m\theta)+B_{m,n}\sin(m\theta))\cos\left(\frac{c\lambda_{m,n} t}{r_0}\right)\right]$$

with

$$A_{0,n} = \frac{2}{\pi(J_1(\lambda_{0,n}))^2} \int_0^1 z\, J_0(\lambda_{0,n} z) \left(\int_{-\pi}^{\pi} f(r_0 z, \theta)\, d\theta \right) dz$$

$$A_{m,n} = \frac{2}{\pi(J_{m+1}(\lambda_{m,n}))^2} \int_0^1 z\, J_m(\lambda_{m,n} z) \left(\int_{-\pi}^{\pi} f(r_0 z, \theta) \cos(m\theta)\, d\theta \right) dz$$

$$B_{m,n} = \frac{2}{\pi(J_{m+1}(\lambda_{m,n}))^2} \int_0^1 z\, J_m(\lambda_{m,n} z) \left(\int_{-\pi}^{\pi} f(r_0 z, \theta) \sin(m\theta)\, d\theta \right) dz$$

and

$$w(r, \theta, t) = \sum_{m=0}^{\infty} \sum_{n=1}^{\infty} \left[J_m\left(\frac{\lambda_{m,n} r}{r_0} \right) (\hat{A}_{m,n} \cos(m\theta) + \hat{B}_{m,n} \sin(m\theta)) \sin\left(\frac{c\lambda_{m,n} t}{r_0} \right) \right]$$

with

$$\hat{A}_{0,n} = \frac{2r_0}{c\lambda_{0,n}\pi(J_1(\lambda_{0,n}))^2} \int_0^1 z\, J_0(\lambda_{0,n} z) \left(\int_{-\pi}^{\pi} g(r_0 z, \theta)\, d\theta \right) dz$$

$$\hat{A}_{m,n} = \frac{2r_0}{c\lambda_{m,n}\pi(J_{m+1}(\lambda_{m,n}))^2} \int_0^1 z\, J_m(\lambda_{m,n} z) \left(\int_{-\pi}^{\pi} g(r_0 z, \theta) \cos(m\theta)\, d\theta \right) dz$$

$$\hat{B}_{m,n} = \frac{2r_0}{c\lambda_{m,n}\pi(J_{m+1}(\lambda_{m,n}))^2} \int_0^1 z\, J_m(\lambda_{m,n} z) \left(\int_{-\pi}^{\pi} g(r_0 z, \theta) \sin(m\theta)\, d\theta \right) dz.$$

9.3.2 The formal solution may be expressed as an infinite series of the form

$$u(\rho, \varphi) = \sum_{n=0}^{\infty} a_n \rho^n P_n(\cos\varphi).$$

On the boundary,

$$\sum_{n=0}^{\infty} a_n P_n(\cos\varphi) = \cos^2\varphi$$
$$= \frac{2}{3} P_2(\cos\varphi) + \frac{1}{3} P_0(\varphi).$$

Therefore the solution can be expressed as the finite sum,

$$u(\rho, \varphi) = \frac{1}{3} + \frac{2}{3}\rho^2 P_2(\cos\varphi).$$

9.3.4 The formal solution may be expressed as an infinite series of the form

$$u(\rho, \varphi) = \sum_{n=0}^{\infty} \frac{b_n}{\rho^{n+1}} P_n(\cos\varphi).$$

On the boundary,

$$\sum_{n=0}^{\infty} b_n P_n(\cos\varphi) = \cos^2\varphi$$
$$= \frac{1}{3} P_0(\cos\varphi) + \frac{2}{3} P_2(\cos\varphi)$$

Therefore the solution can be expressed as the finite sum,

$$u(\rho, \varphi) = \frac{1}{3\rho} P_0(\cos\varphi) + \frac{2}{3\rho^3} P_2(\cos\varphi).$$

9.3.6 Since the mass density is constant and the potential on the surface of the sphere is independent of the angular coordinates Eq. (8.80) allows Poisson's equation to be written as

$$\begin{aligned} u_{\rho\rho} + \frac{2}{\rho} u_\rho &= 4\pi G\sigma_0 \\ \rho^2 u_{\rho\rho} + 2\rho u_\rho &= 4\pi G\sigma_0 \rho^2 \\ [\rho^2 u_\rho]_\rho &= 4\pi G\sigma_0 \rho^2 \\ \rho^2 u_\rho &= \frac{4\pi G\sigma_0}{3}\rho^3 + A \\ u_\rho &= \frac{4\pi G\sigma_0}{3}\rho + \frac{A}{\rho^2} \\ u(\rho,\varphi,\theta) &= \frac{2\pi G\sigma_0}{3}\rho^2 - \frac{A}{\rho} + B. \end{aligned}$$

In order for the solution to be bounded as $\rho \to 0^+$ the constant $A = 0$. Therefore,

$$u(\rho,\varphi,\theta) = \frac{2\pi G\sigma_0}{3}\rho^2 + B$$

for $0 \le \rho \le R_0$ where B is a constant.

9.3.8 If the solution is smooth at $\rho = R_0$ then the partial derivative with respect to ρ is continuous at $\rho = R_0$. This implies

$$\frac{4}{3}\pi G\sigma_0 R_0 = \frac{A}{R_0^2} \iff A = \frac{4}{3}\pi G\sigma_0 R_0^3.$$

Hence the $u(\rho,\varphi,\theta) = -\frac{4\pi G\sigma_0 R_0^3}{3\rho}$ for $\rho > R_0$. The solution must also be a continuous function of ρ at $\rho = R_0$. Therefore,

$$\frac{2}{3}\pi G\sigma_0 R_0^2 + B = -\frac{4}{3}\pi G\sigma_0 R_0^2 \iff B = -2\pi G\sigma_0 R_0^2$$

which implies for $0 \le \rho \le R_0$

$$\begin{aligned} u(\rho,\varphi,\theta) &= \frac{2}{3}\pi G\sigma_0\rho^2 - 2\pi G\sigma_0 R_0^2 \\ &= \frac{2}{3}\pi G\sigma_0(\rho^2 - 3R_0^2). \end{aligned}$$

The gravitational potential can be expressed as the piecewise-defined function,

$$u(\rho,\varphi,\theta) = \begin{cases} \frac{2}{3}\pi G\sigma_0(\rho^2 - 3R_0^2) & \text{for } 0 \le \rho \le R_0, \\ -\frac{4\pi G\sigma_0 R_0^3}{3\rho} & \text{for } \rho > R_0. \end{cases}$$

9.4.1

$$\begin{aligned} Y_n^{-m}(\varphi,\theta) &= \sqrt{\frac{2n+1}{4\pi}\frac{(n-(-m))!}{(n+(-m))!}} e^{i(-m)\theta} P_n^{-m}(\cos\varphi) \\ &= \sqrt{\frac{2n+1}{4\pi}\frac{(n+m)!}{(n-m)!}} e^{i(-m)\theta} (-1)^m \frac{(n-m)!}{(n+m)!} P_n^m(\cos\varphi) \text{ (by Eq. (8.66))} \\ &= (-1)^m \sqrt{\frac{2n+1}{4\pi}\frac{(n-m)!}{(n+m)!}} e^{im(-\theta)} P_n^m(\cos\varphi) \\ &= (-1)^m Y_n^m(\varphi,-\theta) \end{aligned}$$

9.4.3

$$\begin{aligned}\int_{-\pi}^{\pi} e^{\pm 2i\theta}\sin(2\theta)\,d\theta &= \int_{-\pi}^{\pi}(\cos(\pm 2\theta)+i\sin(\pm 2\theta))\sin(2\theta)\,d\theta\\ &= \int_{-\pi}^{\pi}\cos(2\theta)\sin(2\theta)\,d\theta \pm i\int_{-\pi}^{\pi}\sin^2(2\theta)\,d\theta\\ &= \pm\frac{i}{2}\int_{-\pi}^{\pi}(1-\cos(4\theta))\,d\theta\\ &= \pm\frac{i}{2}(2\pi) = \pm\pi\, i\end{aligned}$$

9.4.5 The displacement can be written formally as an infinite series of the form

$$u(\varphi,\theta,t) = \sum_{n=0}^{\infty}\sum_{m=-n}^{n} Y_n^m(\varphi,\theta)(a_{n,m}\cos(\sqrt{n(n+1)}t) + b_{n,m}\sin(\sqrt{n(n+1)}t)$$

where $b_{n,m}=0$ and

$$a_{n,m} = \sqrt{\frac{2n+1}{4\pi}\frac{(n-m)!}{(n+m)!}}\int_{-\pi}^{\pi}\int_0^{\pi}\frac{1}{16}\sin(2\varphi)\sin(3\theta)e^{-im\theta}P_n^m(\cos\varphi)\sin\varphi\,d\varphi\,d\theta.$$

Note that $a_{n,m}=0$ except when $m=\pm 3$ and $a_{n,-3}=a_{n,3}$.

$$\begin{aligned}a_{n,3} &= -\frac{i}{16}\pi\sqrt{\frac{2n+1}{4\pi}\frac{(n-3)!}{(n+3)!}}\int_0^{\pi}\sin(2\varphi)P_n^3(\cos\varphi)\sin\varphi\,d\varphi\\ &= -\frac{i}{16}\sqrt{(2n+1)\pi\frac{(n-3)!}{(n+3)!}}\int_0^{\pi}\cos\varphi\sin^2(\varphi)P_n^3(\cos\varphi)\,d\varphi\\ &= -\frac{i}{16}\sqrt{(2n+1)\pi\frac{(n-3)!}{(n+3)!}}\int_{-1}^{1}x(1-x^2)^{1/2}P_n^3(x)\,dx\end{aligned}$$

If n is odd then $n+3$ is even and therefore the integrand is an odd function and hence $a_{n,3}=0$ when n is odd. Suppose $n=2k$ is even.

$$\begin{aligned}a_{2k,3} &= -\frac{i}{16}\sqrt{(4k+1)\pi\frac{(2k-3)!}{(2k+3)!}}\int_{-1}^{1}x(1-x^2)^{1/2}P_{2k}^3(x)\,dx\\ &= \frac{i}{16}\sqrt{(4k+1)\pi\frac{(2k-3)!}{(2k+3)!}}\int_{-1}^{1}x(1-x^2)^2\frac{d^3}{dx^3}\left[P_{2k}(x)\right]\,dx\\ &= -\frac{i}{16}\sqrt{(4k+1)\pi\frac{(2k-3)!}{(2k+3)!}}\int_{-1}^{1}(1-x^2)(1-5x^2)\frac{d^2}{dx^2}\left[P_{2k}(x)\right]\,dx\\ &= \frac{i}{4}\sqrt{(4k+1)\pi\frac{(2k-3)!}{(2k+3)!}}\int_{-1}^{1}x(5x^2-3)\frac{d}{dx}\left[P_{2k}(x)\right]\,dx\\ &= \frac{i}{4}\sqrt{(4k+1)\pi\frac{(2k-3)!}{(2k+3)!}}\left(\left[x(5x^2-3)P_{2k}(x)\right]_{x=-1}^{x=1} - \int_{-1}^{1}(15x^2-3)P_{2k}(x)\,dx\right)\\ &= i\sqrt{(4k+1)\pi\frac{(2k-3)!}{(2k+3)!}}\end{aligned}$$

Note that k must be greater than or equal to 2 for otherwise $P_{2k}^3(x)$ is not defined. Thus the displacement of the spherical membrane is

$$\begin{aligned}
u(\varphi,\theta,t) &= \sum_{k=2}^{\infty}\left[i\sqrt{(4k+1)\pi\frac{(2k-3)!}{(2k+3)!}}\cos(\sqrt{2k(2k+1)}t)\left(Y_{2k}^{-3}(\varphi,\theta)+Y_{2k}^{3}(\varphi,\theta)\right)\right] \\
&= \sum_{k=2}^{\infty}\left[i\sqrt{(4k+1)\pi\frac{(2k-3)!}{(2k+3)!}}\cos(\sqrt{2k(2k+1)}t)\left(Y_{2k}^{3}(\varphi,\theta)-Y_{2k}^{3}(\varphi,-\theta)\right)\right] \\
&= \sum_{k=2}^{\infty}\left[i\sqrt{(4k+1)\pi\frac{(2k-3)!}{(2k+3)!}}\sqrt{\frac{4k+1}{4\pi}\frac{(2k-3)!}{(2k+3)!}}\cos(\sqrt{2k(2k+1)}t)P_{2k}^{3}(\cos\varphi)\left(e^{i3\theta}-e^{-i3\theta}\right)\right] \\
&= -\sin(3\theta)\sum_{k=2}^{\infty}\left[(4k+1)\frac{(2k-3)!}{(2k+3)!}\cos(\sqrt{2k(2k+1)}t)P_{2k}^{3}(\cos\varphi)\right].
\end{aligned}$$

9.5.1 The formal solution may be expressed as an infinite series of the form

$$u(\rho,\varphi,\theta,t) = \sum_{n=0}^{\infty}\sum_{q=1}^{\infty}\sum_{m=-n}^{n} A_{n,m,q}j_n(\lambda_{n+\frac{1}{2},q}\rho)Y_n^m(\varphi,\theta)e^{-\lambda_{n+\frac{1}{2},q}^2 t}.$$

Since the initial condition is independent of φ and θ the solution to the heat equation will be as well. This simplifies the series solution to

$$u(\rho,t) = \sum_{q=1}^{\infty} A_{0,0,q}j_0(\lambda_{\frac{1}{2},q}\rho)e^{-\lambda_{\frac{1}{2},q}^2 t}.$$

At time $t=0$,

$$\begin{aligned}
u(\rho,0) &= \sin(\pi\rho) \\
\sum_{q=1}^{\infty} A_{0,0,q}j_0(\lambda_{\frac{1}{2},q}\rho) &= \pi\rho j_0(\pi\rho) \\
&= \lambda_{\frac{1}{2},1}\rho j_0(\lambda_{\frac{1}{2},1}\rho).
\end{aligned}$$

Multiply both sides of this equation by $j_0(\lambda_{\frac{1}{2},\hat{q}}\rho)\rho^2$ and integrate with respect to ρ over the interval $[0,1]$.

$$\begin{aligned}
\sum_{q=1}^{\infty} A_{0,0,q}\int_0^1 j_0(\lambda_{\frac{1}{2},q}\rho)j_0(\lambda_{\frac{1}{2},\hat{q}}\rho)\rho^2\,d\rho &= \int_0^1 \lambda_{\frac{1}{2},1}\rho j_0(\lambda_{\frac{1}{2},1}\rho)j_0(\lambda_{\frac{1}{2},\hat{q}}\rho)\rho^2\,d\rho \\
\frac{1}{2}A_{0,0,\hat{q}}(j_1(\lambda_{\frac{1}{2},\hat{q}}))^2 &= \frac{1}{\hat{q}\pi}\int_0^1 \rho\sin(\pi\rho)\sin(\hat{q}\pi\rho)\,d\rho \\
A_{0,0,\hat{q}}(j_1(\hat{q}\pi))^2 &= \frac{1}{\hat{q}\pi}\int_0^1 \rho(\cos(\pi(1-\hat{q})\rho)-\cos(\pi(1+\hat{q})\rho))\,d\rho \\
A_{0,0,\hat{q}}\frac{1}{(\hat{q}\pi)^2} &= \frac{-4(1+(-1)^{\hat{q}})}{\pi^3(\hat{q}^2-1)^2} \\
A_{0,0,\hat{q}} &= \frac{-4\hat{q}^2(1+(-1)^{\hat{q}})}{\pi(\hat{q}^2-1)^2},
\end{aligned}$$

if $\hat{q} > 1$. For the case when $q = 1$,

$$\begin{aligned}
A_{0,0,1}(j_1(\pi))^2 &= \frac{1}{\pi}\int_0^1 \rho(1-\cos(2\pi\rho))\,d\rho \\
A_{0,0,1}\frac{1}{\pi^2} &= \frac{1}{2\pi} \\
A_{0,0,1} &= \frac{\pi}{2}.
\end{aligned}$$

Formally the solution can be expressed as the infinite series,

$$u(\rho,t) = \frac{\pi}{2} j_0(\pi\rho)e^{-\pi^2 t} - \frac{8}{\pi}\sum_{q=1}^{\infty}\frac{4q^2}{(4q^2-1)^2} j_0(2q\pi\rho)e^{-4q^2\pi^2 t}.$$

9.5.3 The formal solution may be expressed as an infinite series of the form

$$u(\rho,\varphi,\theta,t) = \sum_{n=0}^{\infty}\sum_{q=1}^{\infty}\sum_{m=-n}^{n} A_{n,m,q} j_n(\lambda_{n+\frac{1}{2},q}\rho)Y_n^m(\varphi,\theta)e^{-\lambda^2_{n+\frac{1}{2},q}t}.$$

Since the initial condition is independent of φ and θ the solution to the heat equation will be as well. This simplifies the series solution to

$$u(\rho,t) = \sum_{q=1}^{\infty} A_{0,0,q} j_0(\lambda_{\frac{1}{2},q}\rho)e^{-\lambda^2_{\frac{1}{2},q}t}.$$

At time $t = 0$,

$$u(\rho,0) = \sum_{q=1}^{\infty} A_{0,0,q} j_0(\lambda_{\frac{1}{2},q}\rho) = \rho - \rho^2.$$

Multiply both sides of this equation by $j_0(\lambda_{\frac{1}{2},\hat{q}}\rho)\rho^2$ and integrate with respect to ρ over the interval $[0,1]$.

$$\begin{aligned}
\sum_{q=1}^{\infty} A_{0,0,q}\int_0^1 j_0(\lambda_{\frac{1}{2},q}\rho)j_0(\lambda_{\frac{1}{2},\hat{q}}\rho)\rho^2\,d\rho &= \int_0^1 (\rho-\rho^2)j_0(\lambda_{\frac{1}{2},\hat{q}}\rho)\rho^2\,d\rho \\
\frac{1}{2}A_{0,0,\hat{q}}(j_1(\lambda_{\frac{1}{2},\hat{q}}))^2 &= \int_0^1 (\rho^3-\rho^4)j_0(\lambda_{\frac{1}{2},\hat{q}}\rho)\,d\rho \\
\frac{1}{2}A_{0,0,\hat{q}}\frac{1}{(\hat{q}\pi)^2} &= \frac{1}{\hat{q}\pi}\int_0^1 (\rho^2-\rho^3)\sin(\hat{q}\pi\rho)\,d\rho \\
&= \frac{-2(1+2(-1)^{\hat{q}})}{(\hat{q}\pi)^4} \\
A_{0,0,\hat{q}} &= \frac{-4(1+2(-1)^{\hat{q}})}{(\hat{q}\pi)^2}.
\end{aligned}$$

Formally the solution can be expressed as the infinite series,

$$u(\rho,t) = \frac{-4}{\pi^2}\sum_{q=1}^{\infty}\frac{(1+2(-1)^q)}{q^2} j_0(q\pi\rho)e^{-q^2\pi^2 t}.$$

9.5.5 The formal solution may be expressed as an infinite series of the form

$$u(\rho,\varphi,\theta,t) = \sum_{n=0}^{\infty}\sum_{q=1}^{\infty}\sum_{m=-n}^{n} A_{n,m,q} j_n(\lambda_{n+\frac{1}{2},q}\rho)Y_n^m(\varphi,\theta)e^{-\lambda^2_{n+\frac{1}{2},q}t}.$$

Since the initial condition is independent of θ the solution to the heat equation will be as well. This simplifies the series solution to

$$u(\rho, \varphi, t) = \sum_{n=0}^{\infty} \sum_{q=1}^{\infty} A_{n,0,q} j_n(\lambda_{n+\frac{1}{2},q}\rho) P_n(\cos\varphi) e^{-\lambda^2_{n+\frac{1}{2},q} t}.$$

At time $t = 0$,

$$\begin{aligned}
u(\rho, \varphi, 0) &= \sum_{n=0}^{\infty} \sum_{q=1}^{\infty} A_{n,0,q} j_n(\lambda_{n+\frac{1}{2},q}\rho) P_n(\cos\varphi) \\
&= \begin{cases} (1-\rho)\cos\varphi & \text{for } 0 < \rho < 1,\ 0 < \varphi < \frac{\pi}{2}, \\ 0 & \text{for } 0 < \rho < 1,\ \frac{\pi}{2} < \varphi < \pi. \end{cases}
\end{aligned}$$

Multiply both sides of the last equation by $j_{\hat{n}}(\lambda_{\hat{n}+\frac{1}{2},\hat{q}}\rho) P_{\hat{n}}(\cos\varphi)\rho^2 \sin\varphi$ and integrate over the unit sphere.

$$\begin{aligned}
&\sum_{n=0}^{\infty} \sum_{q=1}^{\infty} A_{n,0,q} \int_0^1 \int_0^{\pi} \int_{-\pi}^{\pi} j_n(\lambda_{n+\frac{1}{2},q}\rho) P_n(\cos\varphi) j_{\hat{n}}(\lambda_{\hat{n}+\frac{1}{2},\hat{q}}\rho) P_{\hat{n}}(\cos\varphi)\rho^2 \sin\varphi \, d\varphi \, d\theta \, d\rho \\
&= \int_0^1 \int_0^{\frac{\pi}{2}} \int_{-\pi}^{\pi} (1-\rho)\cos\varphi j_{\hat{n}}(\lambda_{\hat{n}+\frac{1}{2},\hat{q}}\rho) P_{\hat{n}}(\cos\varphi)\rho^2 \sin\varphi \, d\varphi \, d\theta \, d\rho
\end{aligned}$$

$$\begin{aligned}
&\sum_{n=0}^{\infty} \sum_{q=1}^{\infty} A_{n,0,q} \int_0^1 \int_0^{\pi} j_n(\lambda_{n+\frac{1}{2},q}\rho) j_{\hat{n}}(\lambda_{\hat{n}+\frac{1}{2},\hat{q}}\rho)\rho^2 P_n(\cos\varphi) P_{\hat{n}}(\cos\varphi) \sin\varphi \, d\varphi \, d\rho \\
&= \int_0^1 \int_0^{\frac{\pi}{2}} (\rho^2 - \rho^3) j_{\hat{n}}(\lambda_{\hat{n}+\frac{1}{2},\hat{q}}\rho) P_{\hat{n}}(\cos\varphi) \cos\varphi \sin\varphi \, d\varphi \, d\rho
\end{aligned}$$

If $n \neq \hat{n}$ or if $q \neq \hat{q}$ the left-hand side of the equation is zero. When $n = \hat{n}$, and $q = \hat{q}$,

$$\frac{A_{\hat{n},0,\hat{q}}}{(2\hat{n}+1)(j_{\hat{n}+1}(\lambda_{\hat{n}+\frac{1}{2},\hat{q}}))^2} = \int_0^1 (\rho^2 - \rho^3) j_{\hat{n}}(\lambda_{\hat{n}+\frac{1}{2},\hat{q}}\rho) \, d\rho \int_0^{\frac{\pi}{2}} P_{\hat{n}}(\cos\varphi) \cos\varphi \sin\varphi \, d\varphi$$

Consider the definite integral with respect to the zenith angle φ. For the case of $\hat{n} = 0$,

$$\int_0^{\frac{\pi}{2}} P_0(\cos\varphi) \cos\varphi \sin\varphi \, d\varphi = \int_0^1 x \, dx = \frac{1}{2}.$$

If $\hat{n} = 1$,

$$\int_0^{\frac{\pi}{2}} P_1(\cos\varphi) \cos\varphi \sin\varphi \, d\varphi = \int_0^1 x^2 \, dx = \frac{1}{3}.$$

From Eq. (8.76) if $\hat{n} \neq 1$ and $t = 0$, then

$$\begin{aligned}
\int_0^{\frac{\pi}{2}} P_{\hat{n}}(\cos\varphi) \cos\varphi \sin\varphi \, d\varphi &= \int_0^1 P_1(x) P_{\hat{n}}(x) \, dx \\
&= \frac{P_1'(0) P_n(0) - P_1(0) P_n'(0)}{(1)(2) - \hat{n}(\hat{n}+1)} \\
&= \frac{P_{\hat{n}}(0)}{2 - \hat{n}(\hat{n}+1)} \\
&= \begin{cases} 0 & \text{if } \hat{n} \text{ is odd and greater than 1}, \\ (-1)^k \dfrac{(2k)!}{2^{2k}(k!)^2} & \text{if } \hat{n} = 2k \text{ with } k \in \mathbb{N}, \text{ Eq. (8.51)}. \end{cases}
\end{aligned}$$

Therefore if $\hat{n} = 2k$ (an even integer)

$$\frac{A_{2k,0,\hat{q}}}{(4k+1)(j_{2k+1}(\lambda_{2k+\frac{1}{2},\hat{q}}))^2} = (-1)^k \frac{(2k)!}{2^{2k}(k!)^2} \int_0^1 (\rho^2 - \rho^3) j_{2k}(\lambda_{2k+\frac{1}{2},\hat{q}}\rho)\, d\rho$$

$$A_{2k,0,\hat{q}} = (-1)^k \frac{(4k+1)(j_{2k+1}(\lambda_{2k+\frac{1}{2},\hat{q}}))^2(2k)!}{2^{2k}(k!)^2} \int_0^1 (\rho^2 - \rho^3) j_{2k}(\lambda_{2k+\frac{1}{2},\hat{q}}\rho)\, d\rho.$$

If $\hat{n}$ is an odd integer great than 1, then $A_{\hat{n},0,q} = 0$. Finally if $\hat{n} = 1$,

$$\frac{A_{1,0,\hat{q}}}{3(j_2(\lambda_{\frac{3}{2},\hat{q}}))^2} = \frac{1}{3} \int_0^1 (\rho^2 - \rho^3) j_1(\lambda_{\frac{3}{2},\hat{q}}\rho)\, d\rho$$

$$A_{1,0,\hat{q}} = (j_2(\lambda_{\frac{3}{2},\hat{q}}))^2 \int_0^1 (\rho^2 - \rho^3) j_1(\lambda_{\frac{3}{2},\hat{q}}\rho)\, d\rho.$$

9.5.7 The initial, boundary value problem describing this situation is as follows.

$$u_t = \Delta u \text{ for } 0 < \rho < 1,\ 0 < \varphi < \pi,\ 0 \leq \theta < 2\pi, \text{ and } t > 0$$

$$u(1, \theta, \varphi, t) = 0 \text{ for } 0 < \varphi < \pi,\ 0 \leq \theta < 2\pi, \text{ and } t > 0$$

$$u(\rho, \theta, \varphi, 0) = \frac{\sin(\pi\rho)\cos\varphi}{\rho} \text{ for } 0 < \rho < 1,\ 0 < \varphi < \pi, \text{ and } 0 \leq \theta < 2\pi$$

The formal series solution has the form,

$$u(\rho, \varphi, \theta, t) = \sum_{n=0}^{\infty} \sum_{q=1}^{\infty} \sum_{m=-n}^{n} A_{n,m,q} j_n\left(\lambda_{n+1/2,q}\rho\right) Y_n^m(\varphi, \theta) e^{-\kappa\lambda_{n+1/2,q}^2 t}.$$

Note that the initial condition

$$\frac{\sin(\pi\rho)\cos\varphi}{\rho} = \pi j_0(\lambda_{\frac{1}{2},1}\rho) P_1^0(\cos\varphi)$$

Using the orthogonality of the eigenfunctions, $A_{n,m,q} = 0$ except when $n = 1$ and $m = 0$. The Fourier series expansion of the initial condition can be simplified to

$$u(\rho, \varphi, \theta, 0) = \sum_{q=1}^{\infty} A_{1,0,q} j_1\left(\lambda_{\frac{3}{2},q}\rho\right) Y_1^0(\varphi, \theta)$$

$$\pi j_0(\lambda_{\frac{1}{2},1}\rho)\cos\varphi = \frac{1}{2}\sqrt{\frac{3}{\pi}} \sum_{q=1}^{\infty} A_{1,0,q} j_1\left(\lambda_{\frac{3}{2},q}\rho\right)\cos\varphi$$

$$2\pi\sqrt{\frac{\pi}{3}} j_0(\lambda_{\frac{1}{2},1}\rho) = \sum_{q=1}^{\infty} A_{1,0,q} j_1\left(\lambda_{\frac{3}{2},q}\rho\right).$$

Multiply both sides of the last equation by $j_1(\lambda_{\frac{3}{2},\hat{q}}\rho)\rho^2$ and integrate with respect to ρ over the interval $[0, 1]$. According to Cor. 8.2,

$$2\pi\sqrt{\frac{\pi}{3}} \int_0^1 j_0(\lambda_{\frac{1}{2},1}\rho) j_1(\lambda_{\frac{3}{2},\hat{q}}\rho)\rho^2\, d\rho = \sum_{q=1}^{\infty} A_{1,0,q} \int_0^1 j_1\left(\lambda_{\frac{3}{2},q}\rho\right) j_1(\lambda_{\frac{3}{2},\hat{q}}\rho)\rho^2\, d\rho$$

$$= \frac{1}{2} A_{1,0,\hat{q}} \left(j_2(\lambda_{\frac{3}{2},\hat{q}})\right)^2$$

$$\frac{4\pi\sqrt{\frac{\pi}{3}}}{\left(j_2(\lambda_{\frac{3}{2},\hat{q}})\right)^2} \int_0^1 j_0(\lambda_{\frac{1}{2},1}\rho) j_1(\lambda_{\frac{3}{2},\hat{q}}\rho)\rho^2\, d\rho = A_{1,0,\hat{q}}.$$

9.6.2 Note that $0 < P(u)$ for $-A < u < A$.

$$\int_{-A}^{A} \frac{1}{\pi\sqrt{A^2 - u^2}}\, du = \frac{1}{\pi} \int_{-\pi/2}^{\pi/2} \frac{A\cos t}{\sqrt{A^2 - A^2\sin^2 t}}\, dt = \frac{1}{\pi} \int_{-\pi/2}^{\pi/2} 1\, dt = 1$$

Thus $P(u)$ is a valid probability density function.
9.6.4

(a) Replace $n+1$ with $2n$ in Eq. (8.111).

$$\begin{aligned} H_{2n}(x) &= 2xH_{2n-1}(x) - 2(2n-1)H_{2n-2}(x) \\ e^{\frac{-3x^2}{4}} H_{2n}(x) &= 2xe^{\frac{-3x^2}{4}} H_{2n-1}(x) - 2(2n-1)e^{\frac{-3x^2}{4}} H_{2n-2}(x) \\ \int_{-\infty}^{\infty} e^{\frac{-3x^2}{4}} H_{2n}(x)\, dx &= 2\int_{-\infty}^{\infty} xe^{\frac{-3x^2}{4}} H_{2n-1}(x)\, dx - 2(2n-1)\int_{-\infty}^{\infty} e^{\frac{-3x^2}{4}} H_{2n-2}(x)\, dx \end{aligned}$$

(b) Using integration by parts and the identity in the hint,

$$\begin{aligned} 2\int_{-\infty}^{\infty} xe^{-3x^2/4} H_{2n-1}(x)\, dx &= \left[-\frac{4}{3} e^{-3x^2/4} H_{2n-1}(x)\right]_{x\to-\infty}^{x\to\infty} + \frac{4}{3}\int_{-\infty}^{\infty} e^{-3x^2/4} H'_{2n-1}(x)\, dx \\ &= \frac{8}{3}(2n-1)\int_{-\infty}^{\infty} e^{-3x^2/4} H_{2n-2}(x)\, dx. \end{aligned}$$

(c) Combining the results of the two previous parts of the exercise yields,

$$\begin{aligned} \int_{-\infty}^{\infty} e^{-3x^2/4} H_{2n}(x)\, dx &= \frac{8}{3}(2n-1)\int_{-\infty}^{\infty} e^{-3x^2/4} H_{2n-2}(x)\, dx - 2(2n-1)\int_{-\infty}^{\infty} e^{-3x^2/4} H_{2n-2}(x)\, dx \\ &= \frac{2}{3}(2n-1)\int_{-\infty}^{\infty} e^{-3x^2/4} H_{2n-2}(x)\, dx. \end{aligned}$$

(d) When $n = 0$,

$$\begin{aligned} \int_{-\infty}^{\infty} e^{-3x^2/4} H_0(x)\, dx &= \int_{-\infty}^{\infty} e^{-3x^2/4}\, dx = \sqrt{\frac{2}{3}} \int_{-\infty}^{\infty} e^{-z^2/2}\, dz \\ &= 2\sqrt{\frac{\pi}{3}} = 2\sqrt{\frac{\pi}{3}}\, \frac{(0)!}{3^0(0!)}. \end{aligned}$$

Now suppose the result holds for $n = k$, then

$$\begin{aligned} \int_{-\infty}^{\infty} e^{-3x^2/4} H_{2(k+1)}(x)\, dx &= \frac{2}{3}(2k+1)\int_{-\infty}^{\infty} e^{-3x^2/4} H_{2k}(x)\, dx = \frac{2}{3}(2k+1)2\sqrt{\frac{\pi}{3}}\, \frac{(2k)!}{3^k(k!)} \\ &= 2\sqrt{\frac{\pi}{3}}\, \frac{(2(k+1))!}{3^{k+1}((k+1)!)} \end{aligned}$$

and the formula holds for $n = k+1$. Thus by the principle of mathematical induction, the formula holds for all nonnegative integers n.

9.6.6 The energy and amplitude for the classical harmonic oscillator are related through the equation

$A = \sqrt{2E/k}$.

$$\begin{aligned} A_n &= \sqrt{2\left(n+\frac{1}{2}\right)\frac{\hbar\omega}{k}} \\ &= \sqrt{(2n+1)\frac{\hbar\omega}{\mu\omega^2}} \\ &= \sqrt{(2n+1)\frac{\hbar}{\mu\omega}} \end{aligned}$$

9.6.8 Integrate the probability density function for the quantum harmonic oscillator over the interval $[-1, 1]$.

$$\int_{-1}^{1} \frac{1}{\sqrt{\pi}} e^{-x^2}\,dx = \frac{2}{\sqrt{\pi}} \int_0^1 e^{-x^2}\,dx = \text{erf}(1) \approx 0.8427$$

Thus the quantum harmonic oscillator will be found outside of the interval $[-1, 1]$ with probability approximately $1 - 0.8427 = 0.1583$.
9.7.2 Replacing $e_0^2\sqrt{-\mu/(32\pi^2\epsilon_0^2 E\hbar^2)}$ with n in Eq. (9.47) results in the equation,

$$xy''(x) + ((2l+1)_1 - x)y'(x) + (n - l - 1)y(x) = 0.$$

This is the ordinary differential equation satisfied by the generalized Laguerre polynomial when $\lambda = n-l-1$ is a nonnegative integer. The solution can be expressed as $L_{n-l-1}^{2l+1}(x)$.
9.7.4 By the result of Exercise 8.5.10 of Chap. 8,

$$0 = (n+1)L_{n+1}^{\alpha}(x) - (2n+\alpha+1-x)L_n^{\alpha}(x) + (n+\alpha)L_{n-1}^{\alpha}(x).$$

Multiplying both sides of this equation by $L_n^{\alpha}(x)x^{\alpha}e^{-x}$ and integrating over the interval $[0, \infty)$ produces

$$\begin{aligned} 0 = (n+1)\int_0^{\infty} L_{n+1}^{\alpha}(x)L_n^{\alpha}(x)x^{\alpha}e^{-x}\,dx - (2n+\alpha+1)\int_0^{\infty} \left(L_n^{\alpha}(x)\right)^2 x^{\alpha}e^{-x}\,dx \\ + \int_0^{\infty} \left(L_n^{\alpha}(x)\right)^2 x^{\alpha+1}e^{-x}\,dx + (n+\alpha)\int_0^{\infty} L_{n-1}^{\alpha}(x)L_n^{\alpha}(x)x^{\alpha}e^{-x}\,dx \end{aligned}$$

$$\int_0^{\infty} \left(L_n^{\alpha}(x)\right)^2 x^{\alpha+1}e^{-x}\,dx = (2n+\alpha+1)\frac{\Gamma(n+\alpha+1)}{n!}$$

by Thm. 8.14. Now replace n with $n-l-1$ and replace α with $2l+1$.
9.7.6 From Eq. (9.49)

$$\begin{aligned} E_1 &= -\frac{\frac{(1.67262\times10^{-27})(9.10938\times10^{-31})}{(1.67262\times10^{-27})+(9.10938\times10^{-31})}(1.60218\times10^{-19})^4}{32\pi^2(8.854\times10^{-12})^2(1.05457\times10^{-34})^2} \\ &= -2.17869\times10^{-18} \end{aligned}$$

The eigenfunction associated with this energy is $\Psi_{100}(\rho,\varphi,\theta,t)$ where

$$\begin{aligned}
\Psi_{100}(\rho,\varphi,\theta,t) &= \sqrt{\frac{\alpha^3(1-0-1)!}{2\Gamma(1+0+1)}}e^{-\frac{iE_1}{\hbar}t}e^{-\frac{\alpha\rho}{2}}(\alpha\rho)^0 L_{1-0-1}^{2(0)+1}(\alpha\rho)Y_0^0(\varphi,\theta) \\
&= \sqrt{\frac{\alpha^3}{2}}e^{-\frac{iE_1}{\hbar}t}e^{-\frac{\alpha\rho}{2}}\frac{1}{2\sqrt{\pi}} \\
&= \sqrt{\frac{\alpha^3}{8\pi}}e^{-\frac{iE_1}{\hbar}t}e^{-\frac{\alpha\rho}{2}} \\
&= \sqrt{\frac{\left(\frac{2}{a_0}\right)^3}{8\pi}}e^{-\frac{iE_1}{\hbar}t}e^{-\frac{\frac{2}{a_0}\rho}{2}} \\
&= \sqrt{\frac{1}{a_0^3\pi}}e^{-\frac{iE_1}{\hbar}t}e^{-\frac{\rho}{a_0}}.
\end{aligned}$$

9.7.8 Let $P(\rho,\varphi,\theta) = \Psi_{100}(\rho,\varphi,\theta,t)\overline{\Psi_{100}(\rho,\varphi,\theta,t)}$, then

$$P(\rho,\varphi,\theta) = \frac{1}{a_0^3\pi}e^{-\frac{2\rho}{a_0}}$$

according to Eq. (9.52) and the result of Exercise 9.7.5. The probability of the electron having $0<\rho<a_0$ is then

$$\int_0^{a_0}\int_0^{\pi}\int_{-\pi}^{\pi}\frac{1}{a_0^3\pi}e^{-\frac{2\rho}{a_0}}\rho^2\sin\varphi\,d\theta\,d\varphi\,d\rho = \frac{4\pi}{a_0^3\pi}\int_0^{a_0}e^{-\frac{2\rho}{a_0}}\rho^2\,d\rho = \frac{4}{a_0^3}\frac{a_0^3(e^2-5)}{4e^2} = \frac{(e^2-5)}{e^2} \approx 0.323324.$$

10.1.1 Suppose $u(x,t) = v(x,t)\Delta t$ is a solution to Eq. (10.6) with $\Delta t > 0$ constant.

$$\begin{aligned}
u_t(x,t) &= u_t(x,t)\Delta t \\
u_{xx}(x,t) &= u_{xx}(x,t)\Delta t
\end{aligned}$$

Note that $u(0,t) = v(0,t)\Delta t = 0$ and $u(L,t) = v(L,t)\Delta t = 0$ for $t>\Delta t$. This implies $v(0,t) = v(L,t) = 0$ for $t>\Delta t$. When $t=\Delta t$ then $u(x,\Delta t) = v(x,\Delta t)\Delta t = g(x,0)\Delta t$ which implies $v(x,\Delta t) = g(x,0)$. Hence $v(x,t)$ solves the initial boundary value problem

$$\begin{aligned}
v_t &= \kappa\, v_{xx} \text{ for } 0<x<L \text{ and } t>\Delta t, \\
v(0,t) = v(L,t) &= 0 \text{ for } t>\Delta t, \\
v(x,\Delta t) &= g(x,0) \text{ for } 0\le x\le L.
\end{aligned}$$

10.1.3 Using Lemma 1.1,

$$\frac{d}{dt}\int_0^t \tan^{-1}(s^2t)(1+st)^2\,ds = \tan^{-1}(t^3)(1+t^2)^2 + \int_0^t\left[\frac{s^2(1+st)^2}{1+s^4t^2} + 2s(1+st)\tan^{-1}(s^2t)\right]ds.$$

10.1.5 Differentiate the proposed solution with respect to variable x twice.

$$\begin{aligned}
u_x(x,t) &= \int_0^t\int_{-\infty}^{\infty}\frac{1}{\sqrt{4\pi\kappa(t-s)}}e^{\frac{-(x-y)^2}{4\kappa(t-s)}}\left(\frac{-2(x-y)}{4\kappa(t-s)}\right)f(y,s)\,dy\,ds \\
u_{xx}(x,t) &= \int_0^t\int_{-\infty}^{\infty}\frac{1}{\sqrt{4\pi\kappa(t-s)}}e^{\frac{-(x-y)^2}{4\kappa(t-s)}}\left(\frac{(x-y)^2}{4\kappa^2(t-s)^2} - \frac{1}{2\kappa(t-s)}\right)f(y,s)\,dy\,ds
\end{aligned}$$

Now carefully differentiate with $u(x,t)$ with respect to variable t.

$$\begin{aligned}
u_t(x,t) &= f(x,t) + \int_0^t \int_{-\infty}^{\infty} \frac{1}{\sqrt{4\pi\kappa(t-s)}} \left[\frac{-2\pi\kappa}{4\pi\kappa(t-s)} + \frac{(x-y)^2}{4\kappa(t-s)^2}\right] e^{\frac{-(x-y)^2}{4\kappa(t-s)}} f(y,s)\,dy\,ds \\
&= f(x,t) + \int_0^t \int_{-\infty}^{\infty} \frac{1}{\sqrt{4\pi\kappa(t-s)}} e^{\frac{-(x-y)^2}{4\kappa(t-s)}} \left[\frac{(x-y)^2}{4\kappa(t-s)^2} - \frac{1}{2(t-s)}\right] f(y,s)\,dy\,ds \\
&= f(x,t) + \kappa \int_0^t \int_{-\infty}^{\infty} \frac{1}{\sqrt{4\pi\kappa(t-s)}} e^{\frac{-(x-y)^2}{4\kappa(t-s)}} \left[\frac{(x-y)^2}{4\kappa^2(t-s)^2} - \frac{1}{2\kappa(t-s)}\right] f(y,s)\,dy\,ds \\
&= \kappa u_{xx} + f(x,t)
\end{aligned}$$

10.2.1 The solution can be expressed as $u(x,t) = \int_0^t \hat{v}(x,t-s;s)\,ds$ where $\hat{v}(x,t;s)$ is the solution to the auxiliary initial boundary value problem

$$\begin{aligned}
\hat{v}_t(x,t;s) &= \hat{v}_{xx}(x,t;s) \text{ for } 0 < x < \pi \text{ and } t > 0, \\
\hat{v}(0,t;s) &= \hat{v}(\pi,t;s) = 0 \text{ for } t > 0, \\
\hat{v}(x,0;s) &= \sin(3x) \text{ for } 0 < x < \pi.
\end{aligned}$$

By direct substitution $\hat{v}(x,t;s) = e^{-9t}\sin(3x)$ and thus

$$u(x,t) = \int_0^t e^{-9(t-s)} \sin(3x)\,ds = e^{-9t}\sin(3x) \int_0^t e^{9s}\,ds = \frac{1}{9}(1-e^{-9t})\sin(3x).$$

10.2.3 The solution can be expressed as the sum of the solution to the homogeneous portion of the initial boundary value problem and the particular solution to the nonhomogeneous partial differential equation with homogeneous boundary conditions and zero initial conditions (found in Exercise 10.2.1). The solution to

$$\begin{aligned}
v_t &= v_{xx} \text{ for } 0 < x < \pi \text{ and } t > 0, \\
v(0,t) &= v(\pi,t) = 0 \text{ for } t > 0, \\
v(x,0) &= x\left(1 - \frac{x}{\pi}\right) \text{ for } 0 < x < \pi
\end{aligned}$$

can be expressed as

$$v(x,t) = \sum_{n=1}^{\infty} a_n e^{-n^2 t} \sin(nx)$$

where

$$a_n = \frac{2}{\pi} \int_0^{\pi} x\left(1 - \frac{x}{\pi}\right) \sin(nx)\,dx = \begin{cases} 0 & \text{if } n \text{ is even,} \\ 8/(n^3\pi^2) & \text{if } n \text{ is odd.} \end{cases}$$

Thus the solution to the original initial value problem is

$$u(x,t) = \frac{1}{9}(1-e^{-9t})\sin(3x) + \frac{8}{\pi^2} \sum_{n=1}^{\infty} \frac{e^{-(2n-1)^2 t}}{(2n-1)^3} \sin((2n-1)x).$$

10.2.5 If function $u_1(x,t) = (e^{-t}-1)x/\pi + 1$ and $u(x,t) = u_1(x,t) + w(x,t)$, then $w(x,t)$ solves the following initial value problem.

$$\begin{aligned}
w_t &= w_{xx} + \sin(3x) + \frac{x}{\pi}e^{-t} \text{ for } 0 < x < \pi \text{ and } t > 0, \\
w(0,t) &= w(\pi,t) = 0 \text{ for } t > 0, \\
w(x,0) &= \cos(2x) - 1 \text{ for } 0 < x < \pi
\end{aligned}$$

The solution to this initial boundary value problem can be expressed as the sum of the solution to the homogeneous portion of the initial boundary value problem and the particular solution to the nonhomogeneous partial differential equation with homogeneous boundary conditions and zero initial conditions. The solution to

$$\begin{aligned} w_t &= w_{xx} \text{ for } 0 < x < \pi \text{ and } t > 0, \\ w(0,t) &= w(\pi,t) = 0 \text{ for } t > 0, \\ w(x,0) &= \cos(2x) - 1 \text{ for } 0 < x < \pi \end{aligned}$$

can be expressed as

$$w(x,t) = \sum_{n=1}^{\infty} a_n e^{-n^2 t} \sin(nx)$$

where

$$a_n = \frac{2}{\pi} \int_0^{\pi} (\cos(2x) - 1) \sin(nx)\, dx = \begin{cases} 0 & \text{if } n \text{ is even,} \\ 16/(n(n^2-4)\pi) & \text{if } n \text{ is odd.} \end{cases}$$

Therefore define

$$u_2(x,t) = \frac{16}{\pi} \sum_{n=1}^{\infty} \frac{e^{-(2n-1)^2 t}}{(2n-1)((2n-1)^2-4)} \sin((2n-1)x).$$

The solution to the nonhomogeneous initial boundary value problem

$$\begin{aligned} w_t &= w_{xx} + \sin(3x) + \frac{x}{\pi} e^{-t} \text{ for } 0 < x < \pi \text{ and } t > 0, \\ w(0,t) &= w(\pi,t) = 0 \text{ for } t > 0, \\ w(x,0) &= 0 \text{ for } 0 < x < \pi \end{aligned}$$

can be expressed as $w(x,t) = \int_0^t \hat{v}(x, t-s; s)\, ds$ where $\hat{v}(x,t;s)$ is the solution to

$$\begin{aligned} \hat{v}_t(x,t;s) &= \hat{v}_{xx}(x,t;s) \text{ for } 0 < x < \pi \text{ and } t > 0, \\ \hat{v}(0,t;s) &= \hat{v}(\pi,t;s) = 0 \text{ for } t > 0, \\ \hat{v}(x,0;s) &= \sin(3x) + \frac{x}{\pi} e^{-s} \text{ for } 0 < x < \pi \end{aligned}$$

The function $\hat{v}(x,t;s)$ is

$$\hat{v}(x,t;s) = \sum_{n=1}^{\infty} b_n e^{-n^2 t} \sin(nx)$$

where

$$b_n = \frac{2}{\pi} \int_0^{\pi} \left(\sin(3x) + \frac{x}{\pi} e^{-s} \right) \sin(nx)\, dx = \begin{cases} 1 + 2e^{-s}/(3\pi) & \text{if } n = 3, \\ \frac{2(-1)^{n+1} e^{-s}}{n\pi} & \text{if } n \neq 3. \end{cases}$$

Consequently

$$\hat{v}(x, t-s; s) = e^{-9(t-s)} \sin(3x) - \frac{2e^{-s}}{\pi} \sum_{n=1}^{\infty} \frac{(-1)^n}{n} e^{-n^2(t-s)} \sin(nx).$$

Assuming $\hat{v}$ can be integrated term by term then the particular solution takes the form

$$u_3(x,t) = \frac{1}{9}(1 - e^{-9t}) \sin(3x) + \frac{2te^{-t}}{\pi} \sin x - \frac{2}{\pi} \sum_{n=2}^{\infty} \frac{(-1)^n}{n(n^2-1)} \left(e^{-t} - e^{-n^2 t} \right) \sin(nx).$$

Finally the solution to the original initial boundary value problem is $u(x,t) = u_1(x,t) + u_2(x,t) + u_3(x,t)$.

10.2.7 Multiply both sides of the ordinary differential equation by the integrating factor $\mu(t) = e^{4t}$. This produces

$$\begin{aligned}
(T_2'(t) + 4T_2(t))\, e^{4t} &= \frac{e^{4t}}{\pi}\sin t + e^{3t}\left(1 - \frac{1}{\pi}\right) \\
\frac{d}{dt}\left[e^{4t}T_2(t)\right] &= \frac{e^{4t}}{\pi}\sin t + e^{3t}\left(1 - \frac{1}{\pi}\right) \\
\int_0^t \frac{d}{ds}\left[e^{4s}T_2(s)\right]\, ds &= \int_0^t \frac{e^{4s}}{\pi}\sin s + e^{3s}\left(1 - \frac{1}{\pi}\right)\, ds \\
e^{4t}T_2(t) &= \frac{1}{17\pi}\left(1 - e^{4t}(\cos t - 4\sin t)\right) + \frac{\pi - 1}{3\pi}(e^{3t} - 1) \\
T_2(t) &= \frac{1}{17\pi}\left(e^{-4t} - \cos t + 4\sin t\right) + \frac{\pi - 1}{3\pi}(e^{-t} - e^{-4t}) \\
&= \frac{4}{17\pi}\sin t + \frac{(20 - 17\pi)e^{-4t}}{51\pi} + \frac{(\pi - 1)e^{-t}}{3\pi} - \frac{1}{17\pi}\cos t.
\end{aligned}$$

10.2.9 If function $u_1(x,t) = \sin t$ and $u(x,t) = u_1(x,t) + w(x,t)$, then $w(x,t)$ solves the following initial value problem.

$$\begin{aligned}
w_t &= w_{xx} + \cos(x) - \cos(t) \text{ for } 0 < x < \pi \text{ and } t > 0, \\
w_x(0,t) &= w(\pi,t) = 0 \text{ for } t > 0, \\
w(x,0) &= 0 \text{ for } 0 < x < \pi
\end{aligned}$$

The solution to this nonhomogeneous initial boundary value problem can be expressed as $w(x,t) = \int_0^t \hat{v}(x,t-s;s)\, ds$ where $\hat{v}(x,t;s)$ is the solution to

$$\begin{aligned}
\hat{v}_t(x,t;s) &= \hat{v}_{xx}(x,t;s) \text{ for } 0 < x < \pi \text{ and } t > 0, \\
\hat{v}_x(0,t;s) &= \hat{v}(\pi,t;s) = 0 \text{ for } t > 0, \\
\hat{v}(x,0;s) &= \cos(x) - \cos(s) \text{ for } 0 < x < \pi
\end{aligned}$$

The function $\hat{v}(x,t;s)$ is

$$\hat{v}(x,t;s) = \sum_{n=1}^{\infty} b_n e^{-(2n-1)^2 t/4} \cos\left(\frac{(2n-1)x}{2}\right)$$

where

$$b_n = \frac{2}{\pi}\int_0^{\pi} (\cos(x) - \cos(s))\cos\left(\frac{(2n-1)x}{2}\right)\, dx = \frac{4(-1)^n(4 + (4n^2 - 4n - 3)(1 + \cos s))}{(4n^2 - 1)(2n - 3)\pi}.$$

Consequently

$$\hat{v}(x,t-s;s) = \frac{4}{\pi}\sum_{n=1}^{\infty} \frac{(-1)^n(4 + (4n^2 - 4n - 3)(1 + \cos s))}{(4n^2 - 1)(2n - 3)} e^{-(2n-1)^2(t-s)/4} \cos\left(\frac{(2n-1)x}{2}\right).$$

Assuming $\hat{v}$ can be integrated term by term then the particular solution takes the form

$$\begin{aligned}
u_3(x,t) &= \frac{16}{\pi}\sum_{n=1}^{\infty} \frac{(-1)^n(1 - e^{-(2n-1)^2 t/4})}{(4n^2 - 1)(2n - 3)} \cos\left(\frac{(2n-1)x}{2}\right) \\
&\quad + \frac{16}{\pi}\sum_{n=1}^{\infty} \frac{(-1)^n\left(4\sin t + (2n-1)^2\left[\cos t - e^{-(2n-1)^2 t/4}\right]\right)}{(2n-1)(8n(n-1)(2n^2 - 2n + 1) + 17)} \cos\left(\frac{(2n-1)x}{2}\right).
\end{aligned}$$

Finally the solution to the original initial boundary value problem is $u(x,t) = u_1(x,t) + u_3(x,t)$.
10.2.11 The solution $u(x,t) = u_1(x,t) + u_2(x,t)$ where $u_1(x,t)$ is the solution to the following initial value problem.

$$\begin{aligned} (u_1)_t &= \kappa(u_1)_{xx} \text{ for } -\infty < x < \infty \text{ and } t > 0, \\ u_1(x,0) &= \phi(x) \text{ for } -\infty < x < \infty. \end{aligned}$$

According to Thm. 4.3

$$u_1(x,t) = \int_{-\infty}^{\infty} U(x-y,t)\phi(y)\,dy$$

for $t > 0$. The other component $u_2(x,t)$ solves

$$\begin{aligned} (u_2)_t &= \kappa(u_1)_{xx} + f(x,t) \text{ for } -\infty < x < \infty \text{ and } t > 0, \\ u_1(x,0) &= 0 \text{ for } -\infty < x < \infty. \end{aligned}$$

By Thm. 10.3

$$u_2(x,t) = \int_0^t \int_{\infty}^{\infty} U(x-y,t-s)f(y,s)\,dy\,ds.$$

10.3.1 The solution can be represented as the sum of a solution to the homogeneous initial value problem

$$\begin{aligned} v_{tt} &= v_{xx} \text{ for } -\infty < x < \infty \text{ and } t > 0, \\ v(x,0) &= 0 \text{ for } -\infty < x < \infty, \\ v_t(x,0) &= \sin x \text{ for } -\infty < x < \infty. \end{aligned}$$

which takes the form of a d'Alembert solution

$$v(x,t) = \frac{1}{2}\int_{x-t}^{x+t} \sin s\,ds = \sin x \sin t,$$

and a particular solution $w(x,t)$ to the nonhomogeneous problem

$$\begin{aligned} w_{tt} &= w_{xx} + \sin(x-t) \text{ for } -\infty < x < \infty \text{ and } t > 0, \\ w(x,0) &= 0 \text{ for } -\infty < x < \infty, \\ w_t(x,0) &= 0 \text{ for } -\infty < x < \infty. \end{aligned}$$

According to Thm. 10.4,

$$\begin{aligned} w(x,t) &= \frac{1}{2}\int_0^t \int_{x-(t-s)}^{x+(t-s)} \sin(r-s)\,dr\,ds = \frac{1}{2}\int_0^t \left(\cos(x-t) - \cos(x+t-2s)\right)\,ds \\ &= \frac{t}{2}\cos(x-t) - \frac{1}{2}\cos x \sin t. \end{aligned}$$

The solution to the original initial value problem is thus

$$u(x,t) = \sin x \sin t + \frac{t}{2}\cos(x-t) - \frac{1}{2}\cos x \sin t.$$

10.3.3 Suppose that

$$\frac{\omega^2 x}{L}\sin(\omega s) = \sum_{n=1}^{\infty} \frac{a_n(s) n\pi c}{L} \sin\frac{n\pi x}{L},$$

then multiplying both sides of the equation by $\sin(m\pi x/L)$ and integrating over $[0, L]$ produces the following.

$$\begin{aligned}
\frac{\omega^2}{L}\sin(\omega s)\int_0^L x\sin\frac{m\pi x}{L}\,dx &= \sum_{n=1}^{\infty}\frac{a_n(s)n\pi c}{L}\int_0^L \sin\frac{n\pi x}{L}\sin\frac{m\pi x}{L}\,dx \\
\frac{\omega^2}{L}\sin(\omega s)\frac{(-1)^{m+1}L^2}{m\pi} &= \frac{a_m(s)m\pi c}{L}\frac{L}{2} \\
a_m(s) &= \frac{2(-1)^{m+1}L\omega^2}{m^2\pi^2 c}\sin(\omega s).
\end{aligned}$$

This assumes that the order of integration and summation can be interchanged.

10.3.5 Let $u_1(x,t) = (-t + \sin t)x/\pi$ then if $u(x,t) = u_1(x,t) + v(x,t)$, the function $v(x,t)$ satisfies the following initial boundary value problem.

$$\begin{aligned}
v_{tt} &= v_{xx} + \sin(x - t) + \frac{x}{\pi}\sin t \text{ for } 0 < x < \pi \text{ and } t > 0, \\
v(0,t) &= v(\pi,t) = 0 \text{ for } t > 0, \\
v(x,0) &= v_t(x,0) = 0 \text{ for } 0 < x < \pi
\end{aligned}$$

Suppose the solution can be written formally as

$$v(x,t) = \sum_{n=1}^{\infty} a_n(t)\sin(nx),$$

where the functions $a_n(t)$ are as yet unknown. The partial differential equation can be written as

$$\sin(x-t) + \frac{x}{\pi}\sin t = \sum_{n=1}^{\infty}(a_n''(t) + n^2 a_n(t))\sin(nx).$$

Multiplying both sides of the equation by $\sin(mx)$ and integrating over the interval $[0, \pi]$ produces the following equation.

$$a_m''(t) + m^2 a_m(t) = \frac{2(-1)^{m+1}\sin t}{m\pi} + \begin{cases} \cos t & \text{if } m = 1, \\ 0 & \text{if } m \text{ is odd and } m > 2, \\ \frac{-4m}{(m^2-1)\pi}\sin t & \text{if } m \text{ is even.} \end{cases}$$

If $a_m(0) = a_m'(0) = 0$ then the initial conditions of the initial boundary value problem will be satisfied. Thus when $m = 1$ the following initial value problem must be solved.

$$\begin{aligned}
a_1''(t) + a_1(t) &= \frac{2}{\pi}\sin t + \cos t \\
a_1(0) &= a_1'(0) = 0
\end{aligned}$$

The reader can verify the solution is $a_1(t) = (1/\pi + t/2)\sin t - (t/\pi)\cos t$. When m is odd and greater than 1 the initial value problem to be solved takes the form

$$\begin{aligned}
a_m''(t) + m^2 a_m(t) &= \frac{2(-1)^{m+1}\sin t}{m\pi} \\
a_m(0) &= a_m'(0) = 0.
\end{aligned}$$

The solution in this case can be written as

$$a_m(t) = \frac{2(-1)^m(\sin(mt) - m\sin t)}{m^2(m^2-1)\pi}.$$

When m is even the initial value problem becomes

$$a_m''(t) + m^2 a_m(t) = \frac{2(-1)^{m+1}\sin t}{m\pi} - \frac{4m}{(m^2-1)\pi}\sin t$$
$$a_m(0) = a_m'(0) = 0$$

The solution can be verified to be

$$a_m(t) = \frac{2(-1)^m(\sin(mt) - m\sin t)}{m^2(m^2-1)\pi} + \frac{4(\sin(mt) - m\sin t)}{(m^2-1)^2\pi}$$

Thus the solution to the original initial boundary value problem can be formally written as

$$\begin{aligned} u(x,t) &= (-t+\sin t)\frac{x}{\pi} + \left(\left[\frac{1}{\pi} + \frac{t}{2}\right]\sin t - \frac{t}{\pi}\cos t\right)\sin x \\ &+ \sum_{n=2}^{\infty} \frac{(2(2n-1)\sin t - \sin((2n-1)t))}{(2n-1)^2((2n-1)^2-1)\pi}\sin((2n-1)x) \\ &+ \sum_{n=1}^{\infty}\left(\frac{2(\sin(2nt) - 2n\sin t)}{4n^2(4n^2-1)\pi} + \frac{4(\sin(2nt)-2n\sin t)}{(4n^2-1)^2\pi}\right)\sin(2nx). \end{aligned}$$

11.1.1

$$\begin{aligned} \hat{f}(\omega) &= \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} f(x)e^{-i\omega x}\,dx \\ &= \frac{1}{\sqrt{2\pi}}\int_0^{\infty} e^{-ax}e^{-i\omega x}\,dx \\ &= \frac{1}{\sqrt{2\pi}}\lim_{M\to\infty}\int_0^M e^{-(a+i\omega)x}\,dx \\ &= \frac{-1}{(a+i\omega)\sqrt{2\pi}}\lim_{M\to\infty}\left[e^{-(a+i\omega)x}\right]_{x=0}^{x=M} \\ &= \frac{-1}{(a+i\omega)\sqrt{2\pi}}\lim_{M\to\infty}\left(e^{-(a+i\omega)M} - 1\right) \end{aligned}$$

Since $0 \leq |e^{-i\omega M}|e^{-aM} \leq e^{-aM}$ then by the Squeeze Theorem (Thm. 2.7 of [**?**]),

$$\lim_{M\to\infty}\left(e^{-(a+i\omega)M} - 1\right) = -1 \implies \hat{f}(\omega) = \frac{1}{(a+i\omega)\sqrt{2\pi}}.$$

11.1.3

(a) Using the definition of the Fourier transform, inverse Fourier transform, and a "dummy" variable of integration yield

$$\begin{aligned} \mathcal{F}^{-1}[\hat{f}](x) &= \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}\left(\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} f(y)e^{-i\omega y}\,dy\right)e^{i\omega x}\,d\omega \\ &= \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\frac{1}{\sqrt{2\pi}}f(y)e^{i\omega(x-y)}\,dy\,d\omega \end{aligned}$$

(b) Make the substitution $u = x - y$.

$$\begin{aligned} \mathcal{F}^{-1}[\hat{f}](x) &= \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\frac{1}{\sqrt{2\pi}}f(x-u)e^{i\omega u}\,du\,d\omega \\ &= \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} f(x-u)\int_{-\infty}^{\infty}\frac{1}{\sqrt{2\pi}}e^{i\omega u}\,d\omega\,du \end{aligned}$$

(c) Since the Fourier transform of the Dirac delta function is $1/\sqrt{2\pi}$ then

$$
\begin{aligned}
\mathcal{F}^{-1}[\hat{f}](x) &= \int_{-\infty}^{\infty} f(x-u)\left(\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}\frac{1}{\sqrt{2\pi}}e^{i\omega u}\,d\omega\right)du \\
&= \int_{-\infty}^{\infty} f(x-u)\delta(u)\,du \\
&= f(x).
\end{aligned}
$$

11.1.5

$$
\begin{aligned}
\hat{f}(\omega) &= \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} xe^{-x^2}e^{-i\omega x}\,dx \\
&= \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} xe^{-(x^2+i\omega x)}\,dx \\
&= \frac{e^{-\omega^2/4}}{\sqrt{2\pi}}\int_{-\infty}^{\infty} xe^{-(x+i\omega/2)^2}\,dx \\
&= \frac{e^{-\omega^2/4}}{\sqrt{2\pi}}\int_{-\infty}^{\infty}\left(x+\frac{i\omega}{2}\right)e^{-(x+i\omega/2)^2}\,dx - \frac{e^{-\omega^2/4}}{\sqrt{2\pi}}\int_{-\infty}^{\infty}\frac{i\omega}{2}e^{-(x+i\omega/2)^2}\,dx \\
&= \frac{e^{-\omega^2/4}}{\sqrt{2\pi}}\int_{-\infty}^{\infty} ze^{-z^2}\,dz - \frac{i\omega}{2}\frac{e^{-\omega^2/4}}{\sqrt{2\pi}}\int_{-\infty}^{\infty} e^{-z^2}\,dz \\
&= -\frac{i\omega}{2}e^{-\omega^2/4}\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} e^{-z^2}\,dz \\
&= -\frac{i\omega}{2\sqrt{2}}e^{-w^2/4}
\end{aligned}
$$

11.1.7

$$
\begin{aligned}
f(x) &= \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}\hat{f}(\omega)e^{i\omega x}\,d\omega \\
&= \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} e^{-a\omega^2}e^{i\omega x}\,d\omega \\
&= \frac{1}{\sqrt{2\pi}}e^{-x^2/4a}\int_{-\infty}^{\infty} e^{-a\left(\omega-\frac{ix}{2a}\right)^2}\,d\omega \\
&= \frac{1}{\sqrt{2\pi}}e^{-x^2/(4a)}\int_{-\infty}^{\infty} e^{-az^2}\,dz \\
&= \frac{1}{\sqrt{2a}}e^{-x^2/(4a)}
\end{aligned}
$$

The last step follows since

$$\int_{-\infty}^{\infty} e^{-az^2}\,dz = \sqrt{\frac{\pi}{a}}.$$

11.1.9 Use the complex exponential form of $\sin x$ and the technique of completing the square in the exponents to evaluate the integral.

$$\begin{aligned}
\mathcal{F}[e^{-x^2}\sin x](\omega) &= \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} e^{-x^2}\frac{e^{ix}-e^{-ix}}{2i}e^{-i\omega x}\,dx \\
&= \frac{1}{2i\sqrt{2\pi}}e^{\frac{-(\omega-1)^2}{4}}\int_{-\infty}^{\infty} e^{-\left(x+\frac{i(\omega-1)}{2}\right)^2}\,dx - \frac{1}{2i\sqrt{2\pi}}e^{\frac{-(\omega+1)^2}{4}}\int_{-\infty}^{\infty} e^{-\left(x+\frac{i(\omega+1)}{2}\right)^2}\,dx \\
&= \frac{1}{2i\sqrt{2}}e^{-\frac{(\omega-1)^2}{4}} - \frac{1}{2i\sqrt{2}}e^{-\frac{(\omega+1)^2}{4}} \\
&= \frac{i}{2\sqrt{2}}\left(e^{-\frac{(\omega+1)^2}{4}} - e^{-\frac{(\omega-1)^2}{4}}\right)
\end{aligned}$$

11.1.11 Using the definition of the Fourier transform given in Eq. (11.1) and the complex exponential formula for $\sin x$ produce,

$$\begin{aligned}
\mathcal{F}[f](\omega) &= \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} f(x)e^{-i\omega x}\,dx = \frac{1}{\sqrt{2\pi}}\int_{0}^{\infty} e^{-x}\frac{e^{ix}-e^{-ix}}{2i}e^{-i\omega x}\,dx \\
&= \frac{1}{2i\sqrt{2\pi}}\int_{0}^{\infty}\left(e^{(i(1-\omega)-1)x} - e^{-(i(1+\omega)+1)x}\right)dx \\
&= \lim_{M\to\infty}\left[\frac{1}{2i\sqrt{2\pi}}\left(\frac{e^{(i(1-\omega)-1)x}}{i(1-\omega)-1} + \frac{e^{-(i(1+\omega)+1)x}}{i(1+\omega)+1}\right)\right]_{x=0}^{x=M} \\
&= \frac{-1}{2i\sqrt{2\pi}}\left(\frac{1}{i(1-\omega)-1} + \frac{1}{i(1+\omega)+1}\right) \\
&= \frac{-1}{((\omega-i)^2-1)\sqrt{2\pi}}.
\end{aligned}$$

11.2.1 Making use of Thm. 11.3,

$$\begin{aligned}
\mathcal{F}[f](\omega) &= i\frac{d}{d\omega}\left[\mathcal{F}[\mathrm{erf}](\omega)\right] \\
&= i\frac{d}{d\omega}\left[\frac{\sqrt{2}}{i\omega\sqrt{\pi}}e^{-\frac{\omega^2}{4}}\right] \quad \text{(using Exercise 11.1.12)} \\
&= \frac{\sqrt{2}}{\sqrt{\pi}}\frac{d}{d\omega}\left[\frac{1}{\omega}e^{-\frac{\omega^2}{4}}\right] \\
&= \frac{\sqrt{2}}{\sqrt{\pi}}\left(\frac{-1}{\omega^2}e^{-\frac{\omega^2}{4}} - \frac{1}{2}e^{-\frac{\omega^2}{4}}\right) \\
&= -\frac{(2+\omega^2)}{\omega^2\sqrt{2\pi}}e^{-\frac{\omega^2}{4}}.
\end{aligned}$$

11.2.3 The transform $G(\omega)$ can be re-written as

$$\begin{aligned}
G(\omega) &= \sqrt{\frac{2}{\pi}}\left(\sqrt{\frac{\pi}{2}}\delta(\omega-\omega_0) - \sqrt{\frac{\pi}{2}}\delta(\omega+\omega_0)\right) \\
&= \sqrt{\frac{2}{\pi}}\left(\sqrt{\frac{\pi}{2}}\delta(\omega-\omega_0) - \frac{i}{(\omega-\omega_0)\sqrt{2\pi}} - \sqrt{\frac{\pi}{2}}\delta(\omega+\omega_0) + \frac{i}{(\omega+\omega_0)\sqrt{2\pi}}\right).
\end{aligned}$$

Using the linearity property, the shifting property, and the result of Example 11.5,

$$\begin{aligned}\mathcal{F}^{-1}[G](x) &= \sqrt{\frac{2}{\pi}}\left(e^{i\omega_0 x}H(x) - e^{-i\omega_0 x}H(x)\right)\\ &= \sqrt{\frac{2}{\pi}}\left(e^{i\omega_0 x} - e^{-i\omega_0 x}\right)H(x)\\ &= 2\sqrt{\frac{2}{\pi}} i\sin(\omega_0 x)H(x).\end{aligned}$$

11.2.5 The piecewise-defined function $f(x)$ from Exercise 11.1.11 can be written using the Heaviside function $H(x)$ as

$$f(x) = H(x)e^{-x}\sin x.$$

Thus from Exercise 11.1.11 and Thm. 11.4,

$$\begin{aligned}\mathcal{F}\left[f\left(x+\frac{\pi}{2}\right)\right](\omega) &= \frac{-e^{i\omega\pi/2}}{((\omega-i)^2-1)\sqrt{2\pi}}\\ \mathcal{F}\left[H\left(x+\frac{\pi}{2}\right)e^{-x-\frac{\pi}{2}}\sin\left(x+\frac{\pi}{2}\right)\right](\omega) &= \frac{-e^{i\omega\pi/2}}{((\omega-i)^2-1)\sqrt{2\pi}}\\ e^{-\frac{\pi}{2}}\mathcal{F}\left[H\left(x+\frac{\pi}{2}\right)e^{-x}\cos x\right](\omega) &= \frac{-e^{i\omega\pi/2}}{((\omega-i)^2-1)\sqrt{2\pi}}\\ \mathcal{F}\left[H\left(x+\frac{\pi}{2}\right)e^{-x}\cos x\right](\omega) &= \frac{-e^{(i\omega+1)\pi/2}}{((\omega-i)^2-1)\sqrt{2\pi}}\\ \frac{1}{\sqrt{2\pi}}\int_{-\pi/2}^{0} e^{-x}\cos x e^{-i\omega x}\,dx + \mathcal{F}\left[H(x)e^{-x}\cos x\right](\omega) &= \frac{-e^{(i\omega+1)\pi/2}}{((\omega-i)^2-1)\sqrt{2\pi}}\\ \frac{1}{\sqrt{2\pi}}\left(\frac{1+i\omega}{(\omega-i)^2-1}\right) - \frac{1}{\sqrt{2\pi}}\left(\frac{e^{(i\omega+1)\pi/2}}{(\omega-i)^2-1}\right) + \mathcal{F}[g](\omega) &= \frac{-e^{(i\omega+1)\pi/2}}{((\omega-i)^2-1)\sqrt{2\pi}}\\ \mathcal{F}[g](\omega) &= \frac{-1-i\omega}{((\omega-i)^2-1)\sqrt{2\pi}}\end{aligned}$$

11.2.7

$$\begin{aligned}(f*g)(x) &= \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} ze^{-(x-z)^2}\,dz\\ &= \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}(x-u)e^{-u^2}\,du\\ &= \frac{x}{\sqrt{2\pi}}\int_{-\infty}^{\infty}e^{-u^2}\,du - \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}ue^{-u^2}\,du\\ &= \frac{x\sqrt{\pi}}{\sqrt{2\pi}} - 0 = \frac{x}{\sqrt{2}}\end{aligned}$$

11.2.9

$$\begin{aligned}(f*g)'(x) &= \frac{d}{dx}\left[\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} f(x-z)g(z)\,dz\right] = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}\frac{d}{dx}\left[f(x-z)\right]g(z)\,dz\\ &= \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} f'(x-z)g(z)\,dz = (f'*g)(x)\end{aligned}$$

11.3.1 Make use of completing the square in the exponents.

$$
\begin{aligned}
u(x,t) &= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \frac{i}{2\sqrt{2}} \left(e^{-\frac{(\omega+1)^2}{4}} - e^{-\frac{(\omega-1)^2}{4}} \right) e^{-\kappa\omega^2 t} e^{i\omega x}\, d\omega \\
&= \frac{i}{4\sqrt{\pi}} \int_{-\infty}^{\infty} \left(e^{-\frac{(1+4\kappa t)}{4}\left(\omega^2 + \frac{2(1-2ix)\omega}{1+4\kappa t} + \frac{1}{1+4\kappa t}\right)} - e^{-\frac{(1+4\kappa t)}{4}\left(\omega^2 - \frac{2(1+2ix)\omega}{1+4\kappa t} + \frac{1}{1+4\kappa t}\right)} \right) d\omega \\
&= \frac{i}{4\sqrt{\pi}} e^{-\frac{(1+4\kappa t)}{4}\left(-\frac{(1-2ix)^2}{(1+4\kappa t)^2} + \frac{1}{1+4\kappa t}\right)} \int_{-\infty}^{\infty} e^{-\frac{(1+4\kappa t)}{4}\left(\omega + \frac{1-2ix}{1+4\kappa t}\right)^2} d\omega \\
&\quad - \frac{i}{4\sqrt{\pi}} e^{-\frac{(1+4\kappa t)}{4}\left(-\frac{(1+2ix)^2}{(1+4\kappa t)^2} + \frac{1}{1+4\kappa t}\right)} \int_{-\infty}^{\infty} e^{-\frac{(1+4\kappa t)}{4}\left(\omega - \frac{1+2ix}{1+4\kappa t}\right)^2} d\omega \\
&= \frac{i}{4\sqrt{\pi}} e^{\frac{(1-2ix)^2-(1+4\kappa t)}{4(1+4\kappa t)}} \frac{2}{\sqrt{1+4\kappa t}} \int_{-\infty}^{\infty} e^{-u^2}\, du - \frac{i}{4\sqrt{\pi}} e^{\frac{(1+2ix)^2-(1+4\kappa t)}{4(1+4\kappa t)}} \frac{2}{\sqrt{1+4\kappa t}} \int_{-\infty}^{\infty} e^{-u^2}\, du \\
&= \frac{i}{2\sqrt{1+4\kappa t}} e^{-\frac{x^2+\kappa t}{1+4\kappa t}} \left(e^{-\frac{ix}{1+4\kappa t}} - e^{\frac{ix}{1+4\kappa t}} \right) \\
&= \frac{1}{\sqrt{1+4\kappa t}} e^{-\frac{x^2+\kappa t}{1+4\kappa t}} \sin\left(\frac{x}{1+4\kappa t} \right)
\end{aligned}
$$

11.3.3 Fourier transforming both sides of the PDE and solving the resulting ODE produce $\hat{u}(\omega,t) = \hat{u}(\omega,0)e^{-\kappa\omega^2 t}$, where $\hat{u}(\omega,0) = \mathcal{F}[\delta](\omega) = \frac{1}{\sqrt{2\pi}}$ according to Exercise 11.1.2. Now find the inverse Fourier Transform of $\hat{u}(\omega,t)$.

$$
\begin{aligned}
u(x,t) &= \mathcal{F}^{-1}[\frac{1}{\sqrt{2\pi}} e^{-\kappa\omega^2 t}](x) \\
&= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi}} e^{-\kappa\omega^2 t + i\omega x}\, d\omega \\
&= \frac{1}{2\pi} e^{\frac{-x^2}{4\kappa t}} \int_{-\infty}^{\infty} e^{-\kappa t\left(\omega + \frac{ix}{2\kappa t}\right)^2} d\omega \\
&= \frac{1}{\sqrt{4\kappa\pi t}} e^{\frac{-x^2}{4\kappa t}}
\end{aligned}
$$

11.3.5 Start by taking the Fourier transform of both sides of the partial differential equation. Integrate with respect to x since the boundary condition is a function of x.

$$
\begin{aligned}
\mathcal{F}[u_{xx} + u_{yy}](\omega) &= \mathcal{F}[0](\omega) \\
-\omega^2 \hat{u}(\omega,y) + \hat{u}_{yy}(\omega,y) &= 0 \\
\hat{u}(\omega,y) &= A(\omega)e^{\omega y} + B(\omega)e^{-\omega y}
\end{aligned}
$$

If we assume $\lim_{y\to\infty} u(x,y) = 0$ then $\lim_{y\to\infty} \hat{u}(\omega,y) = 0$ which implies the function $\hat{u}(\omega,y)$ may be written as

$$\hat{u}(\omega,y) = C(\omega)e^{-|\omega|y}.$$

The Fourier transform of the boundary condition is

$$\hat{f}(\omega) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(s)e^{-i\omega s}\, ds.$$

Since $\hat{u}(\omega,0) = C(\omega) = \hat{f}(\omega)$ then $\hat{u}(\omega,y) = \hat{f}(\omega)e^{-|\omega|y}$. Now the solution to the boundary value problem is found by inverse Fourier transforming $\hat{u}(\omega, y)$.

$$\begin{aligned}
u(x,y) &= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \hat{f}(\omega) e^{-|\omega| y} e^{i\omega x}\, d\omega \\
&= \frac{1}{2\pi} \int_{-\infty}^{\infty} \left(\int_{-\infty}^{\infty} f(s) e^{-i\omega s}\, ds \right) e^{i\omega x - |\omega| y}\, d\omega \\
&= \frac{1}{2\pi} \int_{-\infty}^{\infty} f(s) \int_{-\infty}^{\infty} e^{i\omega(x-s) - |\omega| y}\, d\omega\, ds \\
&= \frac{1}{2\pi} \int_{-\infty}^{\infty} f(s) \left(\int_{-\infty}^{0} e^{i\omega(x-s)+\omega y}\, d\omega + \int_{0}^{\infty} e^{i\omega(x-s)-\omega y}\, d\omega \right) ds \\
&= \frac{1}{2\pi} \int_{-\infty}^{\infty} f(s) \left(\left[\frac{e^{i\omega(x-s)+\omega y}}{i(x-s)+y} \right]_{\omega \to -\infty}^{\omega=0} + \left[\frac{e^{i\omega(x-s)-\omega y}}{i(x-s)-y} \right]_{\omega=0}^{\omega \to \infty} \right) ds \\
&= \frac{1}{2\pi} \int_{-\infty}^{\infty} f(s) \left(\frac{1}{i(x-s)+y} - 0 + 0 - \frac{1}{i(x-s)-y} \right) ds \\
&= \frac{1}{2\pi} \int_{-\infty}^{\infty} \frac{-2y f(s)}{-(x-s)^2 - y^2}\, ds \\
&= \frac{y}{\pi} \int_{-\infty}^{\infty} \frac{f(s)}{(x-s)^2 + y^2}\, ds
\end{aligned}$$

This integral form of the solution holds for $y > 0$.

11.3.7 Note that $\hat{u}(\omega, y)$ given in Eq. (11.13) can be factored as

$$\frac{\sqrt{\pi} e^{-(1+y)|\omega|}}{\sqrt{2}} = \left(\frac{\sqrt{\pi}}{\sqrt{2}} e^{-|\omega|} \right) \left(e^{-y|\omega|} \right)$$

where

$$\mathcal{F}^{-1} \left[\frac{\sqrt{\pi}}{\sqrt{2}} e^{-|\omega|} \right](x) = \frac{1}{1+x^2} \text{ and } \mathcal{F}^{-1} \left[e^{-y|\omega|} \right](x) = \frac{\sqrt{2}}{\sqrt{\pi}} \frac{y}{x^2+y^2}.$$

Hence by Thm. 11.5,

$$\frac{\sqrt{\pi} e^{-(1+y)|\omega|}}{\sqrt{2}} = \mathcal{F} \left[\frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \frac{1}{1+(x-z)^2} \cdot \frac{\sqrt{2}}{\sqrt{\pi}} \frac{y}{z^2+y^2}\, dz \right](\omega)$$

which implies the solution to Laplace's equation can be written as the convolution,

$$u(x,y) = \frac{1}{\pi} \int_{-\infty}^{\infty} \frac{1}{1+(x-z)^2} \cdot \frac{y}{z^2+y^2}\, dz.$$

11.4.1 Since the cosine is an even function,

$$\hat{f}^c(-\omega) = \sqrt{\frac{2}{\pi}} \int_0^{\infty} f(x) \cos(-\omega x)\, dx = \sqrt{\frac{2}{\pi}} \int_0^{\infty} f(x) \cos(\omega x)\, dx = \hat{f}^c(\omega).$$

Thus $\hat{f}^c(\omega)$ is an even function of ω.

Since the sine is an odd function,

$$\hat{f}^s(-\omega) = \sqrt{\frac{2}{\pi}} \int_0^{\infty} f(x) \sin(-\omega x)\, dx = -\sqrt{\frac{2}{\pi}} \int_0^{\infty} f(x) \sin(\omega x)\, dx = -\hat{f}^s(\omega).$$

Thus $\hat{f}^s(\omega)$ is an odd function of ω.

11.4.3

(a) Suppose f is an even function, then $\hat{f}^c(\omega)$ is an even function.

$$\begin{aligned}
\mathcal{F}[f](\omega) &= \mathcal{F}_c[f](\omega) \\
\mathcal{F}^{-1}[\hat{f}](x) &= \mathcal{F}^{-1}[\hat{f}^c](x) \\
f(x) &= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \hat{f}^c(\omega) e^{i\omega x}\, d\omega \\
&= \sqrt{\frac{2}{\pi}} \int_0^{\infty} \hat{f}^c(\omega) \cos(\omega x)\, d\omega \\
&= \mathcal{F}_c^{-1}[\hat{f}^c](x)
\end{aligned}$$

(b) Suppose f is an odd function, then $\hat{f}^s(\omega)$ is an odd function.

$$\begin{aligned}
\mathcal{F}[f](\omega) &= -i\mathcal{F}_s[f](\omega) \\
\mathcal{F}^{-1}[\hat{f}](x) &= \mathcal{F}^{-1}[-i\hat{f}^s(\omega)](x) \\
f(x) &= \frac{-i}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \hat{f}^s(\omega) e^{i\omega x}\, d\omega \\
&= \sqrt{\frac{2}{\pi}} \int_0^{\infty} \hat{f}^s(\omega) \sin(\omega x)\, d\omega \\
&= \mathcal{F}_s^{-1}[\hat{f}^s](x)
\end{aligned}$$

11.4.5 Follow a line of reasoning similar to that used to prove Thm. 11.3.

$$\begin{aligned}
\mathcal{F}_s[x^2 f(x)](\omega) &= \sqrt{\frac{2}{\pi}} \int_0^{\infty} x^2 f(x) \sin(\omega x)\, dx \\
&= \sqrt{\frac{2}{\pi}} \int_0^{\infty} f(x)(x^2 \sin(\omega x))\, dx \\
&= -\sqrt{\frac{2}{\pi}} \int_0^{\infty} f(x) \frac{d^2}{d\omega^2}\left[\sin(\omega x)\right]\, dx \\
&= -\frac{d^2}{d\omega^2}\left[\sqrt{\frac{2}{\pi}} \int_0^{\infty} f(x) \sin(\omega x)\, dx\right] \\
&= -\frac{d^2}{d\omega^2}\mathcal{F}_s[f](\omega)
\end{aligned}$$

A similar calculation shows,

$$\mathcal{F}_s[x^2 f(x)](\omega) = -\frac{d^2}{d\omega^2}\mathcal{F}_c[f](\omega).$$

11.4.7 By definition,

$$\mathcal{F}_s[f](\omega) = \sqrt{\frac{2}{\pi}} \int_0^{\infty} e^{-\alpha x} \sin(\beta x) \sin(\omega x)\, dx.$$

Using the product to sum identity in Eq. (3.10) this integral can be re-written as

$$\begin{aligned}
\hat{f}^s(\omega) &= \frac{1}{2}\sqrt{\frac{2}{\pi}} \int_0^{\infty} e^{-\alpha x}(\cos([\omega-\beta]x) - \cos([\omega+\beta]x))\, dx \\
&= \frac{1}{2}\sqrt{\frac{2}{\pi}} \int_0^{\infty} e^{-\alpha x}\cos([\omega-\beta]x)\, dx - \frac{1}{2}\sqrt{\frac{2}{\pi}} \int_0^{\infty} e^{-\alpha x}\cos([\omega+\beta]x))\, dx \\
&= \frac{1}{2}\sqrt{\frac{2}{\pi}} \left(\frac{\alpha}{\alpha^2+(\omega-\beta)^2} - \frac{\alpha}{\alpha^2+(\omega+\beta)^2}\right).
\end{aligned}$$

11.4.9 By definition,

$$\mathcal{F}_s[e^{-\alpha x^2}\sin(\beta x)](\omega) = \sqrt{\frac{2}{\pi}}\int_0^\infty e^{-\alpha x^2}\sin(\beta x)\sin(\omega x)\,dx.$$

Using the product to sum identity in Eq. (3.10) this integral can be re-written as

$$\begin{aligned}
\hat{f}^s(\omega) &= \frac{1}{2}\sqrt{\frac{2}{\pi}}\int_0^\infty e^{-\alpha x^2}(\cos([\omega-\beta]x) - \cos([\omega+\beta]x))\,dx \\
&= \frac{1}{4}\sqrt{\frac{2}{\pi}}\int_0^\infty e^{-\alpha x^2}\left(e^{i(\omega-\beta)x} + e^{-i(\omega-\beta)x} - e^{i(\omega+\beta)x} - e^{-i(\omega+\beta)x}\right)dx \\
&= \frac{1}{4}\sqrt{\frac{2}{\pi}}e^{-\frac{(\omega-\beta)^2}{4\alpha}}\int_0^\infty e^{-\alpha\left(x-\frac{i(\omega-\beta)}{2\alpha}\right)^2}dx + \frac{1}{4}\sqrt{\frac{2}{\pi}}e^{-\frac{(\omega-\beta)^2}{4\alpha}}\int_0^\infty e^{-\alpha\left(x+\frac{i(\omega-\beta)}{2\alpha}\right)^2}dx \\
&\quad - \frac{1}{4}\sqrt{\frac{2}{\pi}}e^{-\frac{(\omega+\beta)^2}{4\alpha}}\int_0^\infty e^{-\alpha\left(x-\frac{i(\omega+\beta)}{2\alpha}\right)^2}dx - \frac{1}{4}\sqrt{\frac{2}{\pi}}e^{-\frac{(\omega+\beta)^2}{4\alpha}}\int_0^\infty e^{-\alpha\left(x+\frac{i(\omega+\beta)}{2\alpha}\right)^2}dx \\
&= \frac{\sqrt{2}}{4\sqrt{\pi}}e^{-\frac{(\omega-\beta)^2}{4\alpha}}\left(\frac{\sqrt{\pi}}{2\sqrt{\alpha}} - \frac{1}{2\sqrt{\alpha}}\operatorname{erf}\left(\frac{i(\beta-\omega)}{2\sqrt{\alpha}}\right)\right) + \frac{\sqrt{2}}{4\sqrt{\pi}}e^{-\frac{(\omega-\beta)^2}{4\alpha}}\left(\frac{\sqrt{\pi}}{2\sqrt{\alpha}} + \frac{1}{2\sqrt{\alpha}}\operatorname{erf}\left(\frac{i(\beta-\omega)}{2\sqrt{\alpha}}\right)\right) \\
&\quad - \frac{\sqrt{2}}{4\sqrt{\pi}}e^{-\frac{(\omega+\beta)^2}{4\alpha}}\left(\frac{\sqrt{\pi}}{2\sqrt{\alpha}} + \frac{1}{2\sqrt{\alpha}}\operatorname{erf}\left(\frac{i(\beta+\omega)}{2\sqrt{\alpha}}\right)\right) - \frac{\sqrt{2}}{4\sqrt{\pi}}e^{-\frac{(\omega+\beta)^2}{4\alpha}}\left(\frac{\sqrt{\pi}}{2\sqrt{\alpha}} - \frac{1}{2\sqrt{\alpha}}\operatorname{erf}\left(\frac{i(\beta+\omega)}{2\sqrt{\alpha}}\right)\right) \\
&= \frac{1}{2\sqrt{2\alpha}}e^{-\frac{(\omega-\beta)^2}{4\alpha}} - \frac{1}{2\sqrt{2\alpha}}e^{-\frac{(\omega+\beta)^2}{4\alpha}} \\
&= \frac{1}{\sqrt{2\alpha}}e^{-\frac{\omega^2+\beta^2}{4\alpha}}\sinh\left(\frac{\beta\omega}{2\alpha}\right).
\end{aligned}$$

11.5.1 Using the definition of the multidimensional Fourier transform in Eq. (11.21),

$$\begin{aligned}
\mathcal{F}[af+bg](\boldsymbol{\omega}) &= \left(\frac{1}{\sqrt{2\pi}}\right)^n\int_{-\infty}^\infty\cdots\int_{-\infty}^\infty[af(x_1,\dots,x_n)+bg(x_1,\dots,x_n)]e^{-i\omega_1x_1-\cdots-i\omega_nx_n}\,dx_1\cdots dx_n \\
&= a\left(\frac{1}{\sqrt{2\pi}}\right)^n\int_{-\infty}^\infty\cdots\int_{-\infty}^\infty f(x_1,\dots,x_n)e^{-i\omega_1x_1-\cdots-i\omega_nx_n}\,dx_1\cdots dx_n \\
&\quad + b\left(\frac{1}{\sqrt{2\pi}}\right)^n\int_{-\infty}^\infty\cdots\int_{-\infty}^\infty g(x_1,\dots,x_n)e^{-i\omega_1x_1-\cdots-i\omega_nx_n}\,dx_1\cdots dx_n \\
&= a\mathcal{F}[f](\boldsymbol{\omega}) + b\mathcal{F}[g](\boldsymbol{\omega}).
\end{aligned}$$

11.5.3

$$\begin{aligned}
\mathcal{F}^{-1}[\delta(\omega_1-\omega_{1,0},\omega_2-\omega_{2,0})](x_1,x_2) &= \frac{1}{2\pi}\int_{-\infty}^\infty\int_{-\infty}^\infty\delta(\omega_1-\omega_{1,0},\omega_2-\omega_{2,0})e^{i\omega_1x_1+i\omega_2x_2}\,d\omega_1\,d\omega_2 \\
&= \frac{1}{2\pi}e^{i\omega_{1,0}x_1+i\omega_{2,0}x_2}
\end{aligned}$$

11.5.5 By definition

$$\mathcal{F}[f(a_1x_1,a_2x_2)](\omega_1,\omega_2) = \frac{1}{2\pi}\int_{-\infty}^\infty\int_{-\infty}^\infty f(a_1x_1,a_2x_2)e^{-i\omega_1x_1-i\omega_2x_2}\,dx_1\,dx_2.$$

Make the substitutions $u_1 = a_1x_1$ and $u_2 = a_2x_2$.

$$\begin{aligned}
\mathcal{F}[f(a_1x_1,a_2x_2)](\omega_1,\omega_2) &= \frac{1}{2a_1a_2\pi}\int_{-\infty}^\infty\int_{-\infty}^\infty f(u_1,u_2)e^{-\frac{i\omega_1u_1}{a_1}-\frac{i\omega_2u_2}{a_2}}\,du_1\,du_2 \\
&= \frac{1}{a_1a_2}\hat{f}\left(\frac{\omega_1}{a_1},\frac{\omega_2}{a_2}\right)
\end{aligned}$$

11.5.7 Suppose $t > 0$. According to the chain rule and product rule for derivatives,

$$\begin{aligned}
\frac{\partial}{\partial t}[u(\boldsymbol{x},t)] &= u(\boldsymbol{x},t)\left(\frac{\boldsymbol{x}\cdot\boldsymbol{x}}{4\kappa t^2} - \frac{3}{2t}\right)\\
\frac{\partial^2}{\partial x_1^2}[u(\boldsymbol{x},t)] &= \frac{1}{2\kappa t}u(\boldsymbol{x},t)\left(\frac{x_1^2}{2\kappa t} - 1\right)\\
\frac{\partial^2}{\partial x_2^2}[u(\boldsymbol{x},t)] &= \frac{1}{2\kappa t}u(\boldsymbol{x},t)\left(\frac{x_2^2}{2\kappa t} - 1\right)\\
\frac{\partial^2}{\partial x_3^2}[u(\boldsymbol{x},t)] &= \frac{1}{2\kappa t}u(\boldsymbol{x},t)\left(\frac{x_3^2}{2\kappa t} - 1\right).
\end{aligned}$$

Substituting these partial derivatives into the expression,

$$u_t - \kappa\Delta u = u(\boldsymbol{x},t)\left(\frac{\boldsymbol{x}\cdot\boldsymbol{x}}{4\kappa t^2} - \frac{3}{2t}\right) - \frac{1}{2t}u(\boldsymbol{x},t)\left(\frac{\boldsymbol{x}\cdot\boldsymbol{x}}{2\kappa t} - 3\right) = 0$$

which demonstrates the function given in Eq. (11.30) solves the heat equation.
11.5.9 Compute the two-dimensional Fourier transform of both sides of the Schrödinger equation with respect to $\boldsymbol{x}$.

$$\begin{aligned}
\mathcal{F}[i\hbar\Psi_t(\boldsymbol{x},t)](\boldsymbol{\omega}) &= \mathcal{F}[-\frac{\hbar^2}{2m}\Delta\Psi(\boldsymbol{x},t)](\boldsymbol{\omega})\\
i\hbar\frac{d}{dt}[\hat{\Psi}(\boldsymbol{\omega},t)] &= -\frac{\hbar^2}{2m}(-\omega_1^2 - \omega_2^2)\hat{\Psi}(\boldsymbol{\omega},t)\\
\frac{d}{dt}[\hat{\Psi}(\boldsymbol{\omega},t)] &= -\frac{i\hbar}{2m}(\boldsymbol{\omega}\cdot\boldsymbol{\omega})\hat{\Psi}(\boldsymbol{\omega},t)\\
\hat{\Psi}(\boldsymbol{\omega},t) &= \hat{\Psi}(\boldsymbol{\omega},0)e^{-\frac{i\hbar}{2m}(\boldsymbol{\omega}\cdot\boldsymbol{\omega})t}
\end{aligned}$$

The function $\hat{\Psi}(\boldsymbol{\omega},0)$ is the Fourier transform of the initial condition of the wave function (not explicitly stated in the problem). The wave function is the inverse Fourier transform of $\hat{\Psi}(\boldsymbol{\omega},t)$.

$$\Psi(\boldsymbol{x},t) = \frac{1}{2\pi}\int_{\mathbb{R}^2}\hat{\Psi}(\boldsymbol{\omega},0)e^{-\frac{i\hbar}{2m}(\boldsymbol{\omega}\cdot\boldsymbol{\omega})t}e^{i\boldsymbol{\omega}\cdot\boldsymbol{x}}\,d\boldsymbol{\omega} = \frac{1}{2\pi}\int_{\mathbb{R}^2}\hat{\Psi}(\boldsymbol{\omega},0)e^{i\boldsymbol{\omega}\cdot\left(\boldsymbol{x}-\frac{\hbar t}{2m}\boldsymbol{\omega}\right)}\,d\boldsymbol{\omega}$$

12.1.1 Using the derivative, for $a < x < b$

$$\begin{aligned}
\frac{d}{dx}[p(x)W[u_1,u_2](x)] &= p'(x)(u_1(x)u_2'(x) - u_1'(x)u_2(x)) + p(x)(u_1'(x)u_2'(x) + u_1(x)u_2''(x) - u_1''(x)u_2(x) - u_1'(x)u_2'(x))\\
&= u_1(x)[p(x)u_2'(x)]' - u_2(x)[p(x)u_1'(x)]'\\
&= u_1(x)(-q(x)u_2(x)) - u_2(x)(-q(x)u_1(x))\\
&= 0.
\end{aligned}$$

Hence by the Mean Value Theorem $p(x)W[u_1,u_2](x)$ is constant on (a,b).
12.1.3

$$\begin{aligned}
u'(x) = & -\int_{z_1}^{x}\frac{f(y)u_1'(x)u_2(y)}{p(y)W[u_1,u_2](y)}\,dy - \frac{f(x)u_1(x)u_2(x)}{p(x)W[u_1,u_2](x)}\\
&+\int_{z_2}^{x}\frac{f(y)u_1(y)u_2'(x)}{p(y)W[u_1,u_2](y)}\,dy + \frac{f(x)u_1(x)u_2(x)}{p(x)W[u_1,u_2](x)}\\
= & -\int_{z_1}^{x}\frac{f(y)u_1'(x)u_2(y)}{p(y)W[u_1,u_2](y)}\,dy + \int_{z_2}^{x}\frac{f(y)u_1(y)u_2'(x)}{p(y)W[u_1,u_2](y)}\,dy
\end{aligned}$$

12.1.5 By use of the inner product,

$$
\begin{aligned}
\int_a^b L[u](x)v(x)\,dx &= \int_a^b ([p(x)u'(x)]'v(x) + q(x)u(x)v(x))\,dx \\
&= [p(x)u'(x)v(x)]_{x=a}^{x=b} - \int_a^b (p(x)u'(x)v'(x) + q(x)u(x)v(x))\,dx \\
&= [p(x)(u'(x)v(x) - u(x)v'(x))]_{x=a}^{x=b} + \int_a^b ([p(x)v'(x)]' + q(x)v(x))u(x)\,dx \\
&= [p(x)(u'(x)v(x) - u(x)v'(x))]_{x=a}^{x=b} + \int_a^b L[v](x)u(x)\,dx
\end{aligned}
$$

Hence the linear differential operator of Sturm-Liouville type is formally self-adjoint. The surface terms vanish since if $\beta_1 \neq 0$ and $\beta_2 \neq 0$ (the other cases are similar)

$$
\begin{aligned}
&p(b)(u'(b)v(b) - u(b)v'(b)) - p(a)(u'(a)v(a) - u(a)v'(a)) \\
&= p(b)\left(\frac{-\alpha_2 u(b)}{\beta_2}v(b) - u(b)v'(b)\right) - p(a)\left(\frac{-\alpha_1 u(a)}{\beta_1}v(a) - u(a)v'(a)\right) \\
&= -p(b)u(b)\left(\frac{\alpha_2 v(b) + \beta_2 v'(b)}{\beta_2}\right) + p(a)u(a)\left(\frac{\alpha_1 v(a) + \beta_1 v'(a)}{\beta_1}\right) \\
&= 0.
\end{aligned}
$$

Therefore the Sturm-Liouville linear differential operator is self-adjoint.

12.1.7 Use integration by parts.

$$\begin{aligned}
\int_0^x (y+1)^2 \sin(4y)\, dy &= \left[\frac{-(y+1)^2}{4}\cos(4y)\right]_{y=0}^{y=x} + \int_0^x \frac{y+1}{2}\cos(4y)\, dy \\
&= \frac{1}{4} - \frac{(x+1)^2}{4}\cos(4x) + \int_0^x \frac{y+1}{2}\cos(4y)\, dy \\
&= \frac{1}{4} - \frac{(x+1)^2}{4}\cos(4x) + \left[\frac{y+1}{8}\sin(4y)\right]_{y=0}^{y=x} - \int_0^x \frac{1}{8}\sin(4y)\, dy \\
&= \frac{1}{4} - \frac{(x+1)^2}{4}\cos(4x) + \frac{x+1}{8}\sin(4x) - \int_0^x \frac{1}{8}\sin(4y)\, dy \\
&= \frac{1}{4} - \frac{(x+1)^2}{4}\cos(4x) + \frac{x+1}{8}\sin(4x) + \left[\frac{1}{32}\cos(4y)\right]_{y=0}^{y=x} \\
&= \left(\frac{1}{32} - \frac{(x+1)^2}{4}\right)\cos(4x) + \frac{x+1}{8}\sin(4x) + \frac{7}{32} \\
\int_x^1 (y+1)^2 \cos(4y-4)\, dy &= \left[\frac{(y+1)^2}{4}\sin(4y-4)\right]_{y=x}^{y=1} - \int_x^1 \frac{y+1}{2}\sin(4y-4)\, dy \\
&= -\frac{(x+1)^2}{4}\sin(4x-4) - \int_x^1 \frac{y+1}{2}\sin(4y-4)\, dy \\
&= -\frac{(x+1)^2}{4}\sin(4x-4) + \left[\frac{y+1}{8}\cos(4y-4)\right]_{y=x}^{y=1} - \int_x^1 \frac{1}{8}\cos(4y-4)\, dy \\
&= \frac{1}{4} - \frac{(x+1)^2}{4}\sin(4x-4) - \frac{x+1}{8}\cos(4x-4) - \int_x^1 \frac{1}{8}\cos(4y-4)\, dy \\
&= \frac{1}{4} - \frac{(x+1)^2}{4}\sin(4x-4) - \frac{x+1}{8}\cos(4x-4) - \left[\frac{1}{32}\sin(4y-4)\right]_{y=x}^{y=1} \\
&= \left(\frac{1}{32} - \frac{(x+1)^2}{4}\right)\sin(4x-4) - \frac{x+1}{8}\cos(4x-4) + \frac{1}{4}
\end{aligned}$$

Consequently the solution may be expressed as

$$\begin{aligned}
u(x) = &\frac{-\cos(4x-4)}{4\cos(4)}\left[\left(\frac{1}{32} - \frac{(x+1)^2}{4}\right)\cos(4x) + \frac{x+1}{8}\sin(4x) + \frac{7}{32}\right] \\
&- \frac{\sin(4x)}{4\cos(4)}\left[\left(\frac{1}{32} - \frac{(x+1)^2}{4}\right)\sin(4x-4) - \frac{x+1}{8}\cos(4x-4) + \frac{1}{4}\right] \\
= &\frac{-1}{4}\left(\frac{1}{32} - \frac{(x+1)^2}{4}\right) - \frac{7\cos(4x-4)}{128\cos(4)} - \frac{\sin(4x)}{16\cos(4)} \\
= &\frac{1}{16}\left((x+1)^2 - \frac{1}{8}\right) - \frac{7\cos(4x-4)}{128\cos(4)} - \frac{\sin(4x)}{16\cos(4)}.
\end{aligned}$$

12.1.9 Integrating the ordinary differential equation twice produces

$$u(x) = \frac{1}{12}x^4 + Ax + B$$

where A and B are constants to be determined from the boundary conditions. If $u(a) = 0$ then

$$\frac{1}{12}a^4 + Aa + B = 0 \implies B = \frac{-1}{12}a^4 - Aa.$$

If $u(b) = 0$ then

$$\frac{1}{12}b^4 + Ab - \frac{-1}{12}a^4 - Aa = 0 \implies A = \frac{-1}{12}(a^2 + b^2(a + b).$$

Hence the solution may be expressed as

$$\begin{aligned} u(x) &= \frac{1}{12}(x^4 - a^4) - \frac{1}{12}(a^2 + b^2)(a + b)(x - a) \\ &= \frac{1}{12}(x - a)(x - b)(x^2 + (a + b)x + a^2 + ab + b^2). \end{aligned}$$

12.1.11 After multiplying the ODE by e^{2x} it can be written as

$$e^{2x}u''(x) + 2e^{2x}u'(x) + e^{2x}u(x) = \left[e^{2x}u'(x)\right]' + e^{2x}u(x) = e^{2x}\cos x.$$

Applying Thm. 12.1 the Green's function $G(x; y)$ must solve the given ordinary differential equation for $x \neq y$, be continuous along the line where $x = y$, and solve the given boundary conditions. This leads to a general solution of

$$G(x; y) = \begin{cases} a_1(y)e^{-x} + a_2(y)xe^{-x} & \text{for } 0 \leq x < y \leq 1, \\ b_1(y)e^{-x} + b_2(y)xe^{-x} & \text{for } 0 \leq y < x \leq 1. \end{cases}$$

If $G(x; y)$ is continuous on the line $x = y$, then

$$(a_1(y) - b_1(y))e^{-y} + (a_2(y) - b_2(y))ye^{-y} = 0.$$

Satisfying the boundary conditions results in two more equations.

$$\begin{aligned} -a_1(y) + a_2(y) &= 0 \\ -b_1(y)e^{-1} &= 0 \end{aligned}$$

The partial derivative of $G(x; y)$ with respect to x must have a jump discontinuity across the line $x = y$. This condition produces a fourth linear equation of

$$(a_1(y) - b_1(y))e^{-y} + (b_2(y) - a_2(y))(1 - y)e^{-y} = \frac{1}{e^{2y}}.$$

Solving the system of four equations produces

$$\begin{aligned} a_1 &= ye^{-y} \\ a_2 &= ye^{-y} \\ b_1 &= 0 \\ b_2 &= (1 + y)e^{-y}. \end{aligned}$$

Therefore the Green's function for this boundary value problem is found to be

$$G(x; y) = e^{-x-y} \begin{cases} (1 + x)y & \text{for } 0 \leq x < y \leq 1 \\ x(1 + y) & \text{for } 0 \leq y < c \leq 1. \end{cases}$$

The solution to the boundary value problem may be found using the integral below.

$$u(x) = \int_0^1 G(x; y)e^{2y}\cos y\, dy = \frac{\cos 1}{2}(1 + x)e^{1-x} - \frac{x}{2}e^{-x} + \frac{1}{2}\sin x$$

12.2.1 Let $u(x)$ and $v(x)$ be n-times differentiable functions on $[a, b]$ and let c be a constant.

$$\begin{aligned}
L[u + c\,v] &= a_n(x)\frac{d^n}{dx^n}[u + c\,v] + a_{n-1}(x)\frac{d^{n-1}}{dx^{n-1}}[u + c\,v] + \cdots + a_1(x)\frac{d}{dx}[u + c\,v] + a_0(x)[u + c\,v] \\
&= a_n(x)\frac{d^n}{dx^n}[u] + a_{n-1}(x)\frac{d^{n-1}}{dx^{n-1}}[u] + \cdots + a_1(x)\frac{d}{dx}[u] + a_0(x)[u] \\
&\quad + ca_n(x)\frac{d^n}{dx^n}[v] + ca_{n-1}(x)\frac{d^{n-1}}{dx^{n-1}}[v] + \cdots + ca_1(x)\frac{d}{dx}[v] + ca_0(x)[v] \\
&= L[u] + c\,L[v]
\end{aligned}$$

12.2.3 Use a property of the definite integral.

$$\begin{aligned}
u(x) = &\int_a^x (y - x)\underbrace{H(y - x)}_{=0} f(y)\,dy + \int_x^b (y - x)\underbrace{H(y - x)}_{=1} f(y)\,dy \\
&+ \int_a^x \frac{(x - b)(y - a)}{b - a} f(y)\,dy + \int_x^b \frac{(x - b)(y - a)}{b - a} f(y)\,dy \\
= &\frac{x - b}{b - a}\int_a^x (y - a)f(y)\,dy + \int_x^b \left[y - x + \frac{(x - b)(y - a)}{b - a}\right] f(y)\,dy \\
= &\frac{x - b}{b - a}\int_a^x (y - a)f(y)\,dy + \frac{x - a}{b - a}\int_x^b (y - b)f(y)\,dy
\end{aligned}$$

12.2.5

(a) Using the piecewise-defined formula for $G(x; y)$ given in Eq. (12.22) yields the following.

$$\begin{aligned}
\frac{\partial G}{\partial x} &= \frac{1}{n\pi}\cos\left(\frac{n\pi x}{L}\right)\sin\left(\frac{n\pi y}{L}\right) - \frac{x}{L}\sin\left(\frac{n\pi x}{L}\right)\sin\left(\frac{n\pi y}{L}\right) - \cos\left(\frac{n\pi x}{L}\right)\cos\left(\frac{n\pi y}{L}\right) \\
\lim_{x\to y^-}\frac{\partial G}{\partial x} &= \frac{1}{n\pi}\cos\left(\frac{n\pi y}{L}\right)\sin\left(\frac{n\pi y}{L}\right) - \frac{y}{L}\sin\left(\frac{n\pi y}{L}\right)\sin\left(\frac{n\pi y}{L}\right) - \cos\left(\frac{n\pi y}{L}\right)\cos\left(\frac{n\pi y}{L}\right) \\
&= \frac{1}{2n\pi}\sin\left(\frac{2n\pi y}{L}\right) - \frac{y}{L}\sin^2\left(\frac{n\pi y}{L}\right) - \cos^2\left(\frac{n\pi y}{L}\right)
\end{aligned}$$

(b) Using the piecewise-defined formula for $G(x; y)$ given in Eq. (12.22) yields the following.

$$\begin{aligned}
\frac{\partial G}{\partial x} &= \frac{1}{n\pi}\cos\left(\frac{n\pi x}{L}\right)\sin\left(\frac{n\pi y}{L}\right) - \frac{x}{L}\sin\left(\frac{n\pi x}{L}\right)\sin\left(\frac{n\pi y}{L}\right) + \sin\left(\frac{n\pi x}{L}\right)\sin\left(\frac{n\pi y}{L}\right) \\
\lim_{x\to y^+}\frac{\partial G}{\partial x} &= \frac{1}{n\pi}\cos\left(\frac{n\pi y}{L}\right)\sin\left(\frac{n\pi y}{L}\right) - \frac{y}{L}\sin\left(\frac{n\pi y}{L}\right)\sin\left(\frac{n\pi y}{L}\right) + \sin\left(\frac{n\pi y}{L}\right)\sin\left(\frac{n\pi y}{L}\right) \\
&= \frac{1}{2n\pi}\sin\left(\frac{2n\pi x}{L}\right) - \frac{y - L}{L}\sin^2\left(\frac{n\pi x}{L}\right)
\end{aligned}$$

12.2.7

$$\begin{aligned}
G(0; y) &= (0)\cos(0)\sin\left(\frac{n\pi y}{L}\right) + B(y)\sin(0) - \frac{L}{n\pi}\sin(0)\cos\left(\frac{n\pi y}{L}\right) \\
&= 0 \\
G(L; y) &= \frac{L}{n\pi}\cos(n\pi)\sin\left(\frac{n\pi y}{L}\right) + B(y)\sin(n\pi) - \frac{L}{n\pi}\cos(n\pi)\sin\left(\frac{n\pi y}{L}\right) \\
&= 0
\end{aligned}$$

12.2.9 Assuming u and G satisfy the homogeneous boundary conditions,

$$\begin{aligned}\int_0^L (u''(x)+\sigma^2 u(x))G(x;y)\,dx &= -\int_0^L u'(x)G_x(x;y)\,dx + \int_0^L \sigma^2 u(x)G(x;y)\,dx\\ &= \int_0^L u(x)G_{xx}(x;y)\,dx + \int_0^L \sigma^2 u(x)G(x;y)\,dx\\ &= \int_0^L u(x)(G_{xx}(x;y)+\sigma^2 G(x;y))\,dx.\end{aligned}$$

12.2.11 Let $\phi(x)$ be a test function, then $\phi'(x)$ and $\phi''(x)$ are also test functions.

$$\begin{aligned}\int_{-\infty}^{\infty} H(x)\phi''(x)\,dx &= [H(x)\phi'(x)]_{x\to-\infty}^{x\to\infty} - \int_{-\infty}^{\infty} H'(x)\phi'(x)\,dx\\ &= -\int_{-\infty}^{\infty} \delta(x)\phi'(x)\,dx\\ -\phi'(0) &= -\left[\delta(x)\phi(x)\right]_{x\to-\infty}^{x\to\infty} + \int_{-\infty}^{\infty} \delta'(x)\phi(x)\,dx\\ &= \int_{-\infty}^{\infty} \delta'(x)\phi(x)\,dx\end{aligned}$$

Since the test function ϕ is arbitrary then the symbolic second derivative of the Heaviside function is $\delta'(x)$.
12.2.13 Consider the improper integral

$$\int_{-\infty}^{\infty} |c|\delta(cx)f(x)\,dx,$$

where $f(x)$ is any function continuous in an open interval containing 0. Assume that integration by substitution is valid for generalized functions and distributions. If $c > 0$ then make the substitution $x = u/c$ and $dx = (1/c)du$.

$$\int_{-\infty}^{\infty} |c|\delta(cx)f(x)\,dx = \int_{-\infty}^{\infty} \frac{|c|}{c}\delta(u)f\left(\frac{u}{c}\right)\,du = \int_{-\infty}^{\infty} \frac{c}{c}\delta(u)f\left(\frac{u}{c}\right)\,du = \int_{-\infty}^{\infty} \delta(u)f\left(\frac{u}{c}\right)\,du = f(0)$$

If $c < 0$ then make the substitution $x = u/c$ and $dx = (1/c)du$.

$$\int_{-\infty}^{\infty} |c|\delta(cx)f(x)\,dx = \int_{\infty}^{-\infty} \frac{|c|}{c}\delta(u)f\left(\frac{u}{c}\right)\,du = \int_{\infty}^{-\infty} \frac{-c}{c}\delta(u)f\left(\frac{u}{c}\right)\,du = \int_{-\infty}^{\infty} \delta(u)f\left(\frac{u}{c}\right)\,du = f(0)$$

Since this holds for any continuous function, $|c|\delta(cx) = \delta(x)$.
12.2.15 Note that the sign function can be written as

$$\operatorname{sgn}(x) = -1 + 2H(x)$$

therefore the symbolic derivative $\frac{d}{dx}[\operatorname{sgn}(x)] = 2\delta(x)$.
12.2.17 The general solution of the ordinary differential equation

$$u'''(x) = e^x$$

has the form $u(x) = e^x + c_0 + c_1 x + c_2 x^2$. If $u(0) = 0$ then $c_0 = -1$. If $u'(0) = 0$ then $c_1 = -1$. If $u''(1) = 0$ then $c_2 = -e/2$. Hence the solution to the boundary value problem is

$$u(x) = e^x - 1 - x - \frac{ex^2}{2}.$$

12.3.1 The Green's function has the form

$$G(t;\tau) = \begin{cases} 0 & \text{if } 0 \le t < \tau < \infty, \\ A(\tau)\cos(\sigma t) + B(\tau)\sin(\sigma \tau) & \text{if } 0 \le \tau < t < \infty. \end{cases}$$

The continuity of $G(t;\tau)$ and the jump discontinuity of $G_t(t;\tau)$ imply the following two equations.

$$\begin{aligned} A(\tau)\cos(\sigma\tau) + B(\tau)\sin(\sigma\tau) &= 0 \\ -\sigma A(\tau)\sin(\sigma\tau) + \sigma B(\tau)\cos(\sigma\tau) &= 1 \end{aligned}$$

These equations are satisfied simultaneously when $A(\tau) = \frac{-1}{\sigma}\sin(\sigma\tau)$ and $B(\tau) = \frac{1}{\sigma}\cos(\sigma\tau)$. Using a difference of angles formula for the sine function,

$$G(t;\tau) = \frac{1}{\sigma}\begin{cases} 0 & \text{if } 0 \le t < \tau < \infty, \\ \sin(\sigma(t-\tau)) & \text{if } 0 \le \tau < t < \infty. \end{cases}$$

12.3.3 By inspection the Green's function is of the form $G(t;\tau) = H(t-\tau)(At+B)$ where A and B are functions of τ which must be chosen to satisfy the continuity and jump conditions of $G(\tau;\tau)$ and $G_t(\tau;\tau)$ respectively. If $G(\tau;\tau) = A\tau + B = 0$ then $B = -A\tau$. If $G_t(\tau;\tau) = A = 1$ this implies $B = -\tau$. Hence the Green's function is $G(t;\tau) = H(t-\tau)(t-\tau)$. The solution to the initial value problem is

$$\begin{aligned} u(t) &= \int_0^\infty H(t-\tau)(t-\tau)e^{3\tau}\,d\tau \\ &= \int_0^t (t-\tau)e^{3\tau}\,d\tau \\ &= \frac{1}{9}e^{3t} - \frac{t}{3} - \frac{1}{9} \end{aligned}$$

12.3.5 Consider the related ordinary differential equation with homogeneous initial conditions. A fundamental set of solutions to the equation is $\{e^{3t}, e^{4t}\}$. A Green's function for this ordinary differential equation is

$$G(t;\tau) = H(t-\tau)\left(Ae^{3t} + Be^{4t}\right)$$

where A and B are functions of τ. To satisfy the continuity condition of $G(\tau;\tau) = 0$ requires

$$Ae^{3\tau} + Be^{4\tau} = 0 \implies A = -Be^{\tau}.$$

The jump condition $G_t(\tau;\tau) = 1$ requires

$$3Ae^{3\tau} + 4Be^{4\tau} = 1 \implies B = e^{-4\tau} \text{ and } A = -e^{-3\tau}.$$

Hence the Green's function for this ordinary differential equation is

$$G(t;\tau) = H(t-\tau)\left(e^{4(t-\tau)} - e^{3(t-\tau)}\right).$$

A particular solution to the ordinary differential equation is

$$\begin{aligned} u_p(t) &= \int_0^\infty H(t-\tau)\left(e^{4(t-\tau)} - e^{3(t-\tau)}\right)5e^{4\tau}\,d\tau \\ &= \int_0^t \left(e^{4(t-\tau)} - e^{3(t-\tau)}\right)5e^{4\tau}\,d\tau \\ &= 5(t-1)e^{4t} + 5e^{3t}. \end{aligned}$$

The general solution to the ordinary differential equation can be written as

$$u(t) = c_1e^{3t} + c_2e^{4t} + 5(t-1)e^{4t} + 5e^{3t}.$$

Using the initial conditions specified produces a system of two linear equations in the two unknowns c_1 and c_2.

$$\begin{aligned} c_1 + c_2 &= 0 \\ 3c_1 + 4c_2 &= 1 \end{aligned}$$

The solutions are $c_1 = -1$ and $c_2 = 1$. Hence the solution to the original initial value problem is

$$u(t) = (5t-4)e^{4t} + 4e^{3t}.$$

12.3.7 Consider the related ordinary differential equation with homogeneous initial conditions. A fundamental set of solutions to the equation is $\{e^{-t}, e^{-2t}\}$. A Green's function for this ordinary differential equation is

$$G(t;\tau) = H(t-\tau)\left(Ae^{-t} + Be^{-2t}\right)$$

where A and B are functions of τ. To satisfy the continuity condition of $G(\tau;\tau) = 0$ requires

$$Ae^{-\tau} + Be^{-2\tau} = 0 \implies A = -Be^{-\tau}.$$

The jump condition $G_t(\tau;\tau) = 1$ requires

$$-Ae^{-\tau} - 2Be^{-2\tau} = 1 \implies B = -e^{2\tau} \text{ and } A = e^{\tau}.$$

Hence the Green's function for this ordinary differential equation is

$$G(t;\tau) = H(t-\tau)\left(e^{-(t-\tau)} - e^{-2(t-\tau)}\right).$$

A particular solution to the ordinary differential equation is

$$\begin{aligned} u_p(t) &= \int_0^\infty H(t-\tau)\left(e^{-(t-\tau)} - e^{-2(t-\tau)}\right)\tau\cos\tau\,d\tau \\ &= \int_0^t \left(e^{-(t-\tau)} - e^{-2(t-\tau)}\right)\tau\cos\tau\,d\tau \\ &= -\frac{3}{25}e^{-2t} + \frac{5t+6}{50}\cos t + \frac{15t-17}{50}\sin t. \end{aligned}$$

The general solution to the ordinary differential equation can be written as

$$u(t) = c_1e^{-t} + c_2e^{-2t} + \frac{5t+6}{50}\cos t + \frac{15t-17}{50}\sin t.$$

Using the initial conditions specified produces a system of two linear equations in the two unknowns c_1 and c_2.

$$\begin{aligned} c_1 + c_2 + \frac{3}{25} &= 1 \\ -c_1 - 2c_2 - \frac{6}{25} &= 0 \end{aligned}$$

The solutions are $c_1 = 2$ and $c_2 = -28/25$. Hence the solution to the original initial value problem is

$$u(t) = 2e^{-t} - \frac{28}{25}e^{-2t} + \frac{5t+6}{50}\cos t + \frac{15t-17}{50}\sin t.$$

12.3.9 A fundamental set of solutions to the associated 4th order homogeneous ordinary differential equation is $\{\cosh t, \sinh t, \cos t, \sin t\}$. The Green's function has the form,

$$G(t;\tau) = H(t-\tau)(A\cosh t + B\sinh t + C\cos t + D\sin t)$$

where A, B, C, and D are functions of τ. These expressions must be chosen to satisfy the continuity and jump conditions of the Green's function when $t = \tau$. This produces the system of equations below.

$$\begin{aligned}
G(\tau;\tau) &= A\cosh t + B\sinh t + C\cos\tau + D\sin\tau = 0 \\
G_t(\tau;\tau) &= A\sinh t + B\cosh t - C\sin\tau + D\cos\tau = 0 \\
G_{tt}(\tau;\tau) &= A\cosh t + B\sinh t - C\cos\tau - D\sin\tau = 0 \\
G_{ttt}(\tau;\tau) &= A\sinh t + B\cosh t + C\sin\tau - D\cos\tau = 1
\end{aligned}$$

Adding the third equation to the first and adding the second equation to the fourth yield a system of two equations in the unknowns A and B.

$$\begin{aligned}
2A\cosh\tau + 2B\sinh\tau &= 0 \\
2A\sinh\tau + 2B\cosh\tau &= 1
\end{aligned}$$

The solutions are $A = -\frac{1}{2}\sinh\tau$ and $B = \frac{1}{2}\cosh\tau$. Subtracting the third equation from the first and subtracting the second equation from the fourth yield a system of two equations in the unknowns C and D.

$$\begin{aligned}
2C\cos\tau + 2D\sin\tau &= 0 \\
2C\sin\tau - 2D\cos\tau &= 1
\end{aligned}$$

The solutions are $C = \frac{1}{2}\sin\tau$ and $D = -\frac{1}{2}\cos\tau$.

The Green's function can be expressed as

$$\begin{aligned}
G(t;\tau) &= H(t-\tau)\left(-\frac{1}{2}\sinh\tau\cosh t + \frac{1}{2}\cosh\tau\sinh t + \frac{1}{2}\sin\tau\cos t - \frac{1}{2}\cos\tau\sin t\right) \\
&= \frac{1}{2}H(t-\tau)(\sinh(t-\tau) - \sin(t-\tau)).
\end{aligned}$$

The solution to the initial value problem is

$$\begin{aligned}
u(t) &= \int_0^\infty \frac{1}{2}H(t-\tau)(\sinh(t-\tau) - \sin(t-\tau))\tau^2\,d\tau \\
&= \frac{1}{2}\int_0^t (\sinh(t-\tau) - \sin(t-\tau))\tau^2\,d\tau \\
&= \cosh t - \cos t - t^2.
\end{aligned}$$

12.4.1 Define the vector field $\mathbf{F} = \langle vBu_y, (vB)_x u\rangle$ and apply Thm. 12.4.

$$\begin{aligned}
\iint_R [vBu_{xy} - (vB)_{xy}u]\,dA &= \oint_{\partial R} [vBu_y\mathbf{i} - (vB)_x u\mathbf{j}]\cdot\mathbf{N}\,ds \\
\iint_R vBu_{xy}\,dA &= \oint_{\partial R} [vBu_y\mathbf{i} - (vB)_x y\mathbf{j}]\cdot\mathbf{N}\,ds + \iint_R (vB)_{xy}u\,dA
\end{aligned}$$

12.4.3 Define the vector field $\mathbf{F} = \langle 0, vEu\rangle$ and apply Thm. 12.4.

$$\begin{aligned}
\iint_R [vEu_y + (vE)_y u]\,dA &= \oint_{\partial R} vEu\mathbf{j}\cdot\mathbf{N}\,ds \\
\iint_R vEu_y\,dA &= \oint_{\partial R} vEu\mathbf{j}\cdot\mathbf{N}\,ds - \iint_R (vE)_y u\,dA
\end{aligned}$$

12.4.5 Suppose f and g are scalar functions defined on $R \subset \mathbb{R}^2$ with continuous second partial derivatives on R.

$$\begin{aligned}
\nabla \cdot (g \nabla f) &= \left\langle \frac{\partial}{\partial x}, \frac{\partial}{\partial y} \right\rangle \cdot \left(g \left\langle \frac{\partial f}{\partial x}, \frac{\partial f}{\partial y} \right\rangle \right) \\
&= \frac{\partial g}{\partial x}\frac{\partial f}{\partial x} + g\frac{\partial^2 f}{\partial x^2} + \frac{\partial g}{\partial y}\frac{\partial f}{\partial y} + g\frac{\partial^2 f}{\partial y^2} \\
&= (\nabla f) \cdot (\nabla g) + g(\Delta f).
\end{aligned}$$

Taking the double integral of both sides of this equation, applying the flux form of Green's Theorem given in Thm. 12.4 to the left-hand side, and splitting the right-hand side into two double integrals at the addition sign produce

$$\oint_{\partial R} (g \nabla f) \cdot \mathbf{N}\, ds = \iint_R (\nabla f) \cdot (\nabla g)\, dA + \iint_R g(\Delta f)\, dA.$$

12.4.7 By construction each function $\gamma_k(x, y) > 0$ for all x and y. Without loss of generality it can be assumed that $\epsilon < 1$. Fix $R > 0$ and once again consider the double integral in polar coordinates.

$$\begin{aligned}
\int_0^{2\pi} \int_R^{\infty} \frac{k}{\pi} r e^{-kr^2}\, dr\, d\theta &= 2k \int_R^{\infty} r e^{-kr^2}\, dr \\
&= \int_{kR^2}^{\infty} e^{-u}\, dr \\
&= e^{-kR^2}
\end{aligned}$$

This expression can be made smaller than $0 < \epsilon < 1$ when

$$\begin{aligned}
e^{-kR^2} &< \epsilon \\
-kR^2 &< \ln \epsilon \\
k &> \frac{\ln \epsilon}{-R^2}.
\end{aligned}$$

The last expression is positive since $\ln \epsilon < 0$.

12.4.9

$$\begin{aligned}
\iint_{\mathbb{R}^2} \delta(x-\xi)\delta(y-\eta)h(x,y)\, dA &= \int_{-\infty}^{\infty} \delta(x-\xi) \left[\int_{-\infty}^{\infty} \delta(y-\eta)h(x,y)\, dy \right] dx \\
&= \int_{-\infty}^{\infty} \delta(x-\xi)h(x,\eta)\, dx \\
&= h(\xi, \eta)
\end{aligned}$$

12.4.11 Adopting variable t in place of variable y in Eq. (12.26) implies $A = -c^2$ and $C = 1$ in the general second-order linear partial differential equation. Using this in Eq. (12.29) produces,

$$\iint_R L[u]v\, dA = \oint_{\partial R} [(-c^2 u_x v + c^2 u v_x)\mathbf{i} + (u_t v - u v_t)\mathbf{j}] \cdot \mathbf{N}\, ds + \iint_R u L^*[v]\, dA$$

where $L^*[v] = -c^2 v_{xx} + v_{tt}$. By comparing $L[u]$ and $L^*[v]$ one may see that the linear differential operator for the wave equation is formally self-adjoint.

12.5.2

$$\begin{aligned}\iint_{\mathbb{R}^2} \Delta U_\epsilon \, dA &= \lim_{R\to\infty} \int_0^R \int_0^{2\pi} \frac{c_1\epsilon}{r(r+\epsilon)^2} r \, d\theta \, dr \\ &= 2\pi c_1 \epsilon \lim_{R\to\infty} \int_0^R \frac{1}{(r+\epsilon)^2} \, dr \\ &= 2\pi c_1 \epsilon \lim_{R\to\infty} \left[\frac{-1}{r+\epsilon}\right]_{r=0}^{r=R} \\ &= 2\pi c_1 \epsilon \lim_{R\to\infty} \left(\frac{1}{\epsilon} - \frac{1}{R+\epsilon}\right) \\ &= 2\pi c_1\end{aligned}$$

12.5.4 Use integration by parts.

$$\begin{aligned}\int_0^\epsilon rY_0(kr)\, dr &\approx \lim_{R\to 0^+} \int_R^\epsilon rY_0(kr)\, dr \\ &= \lim_{R\to 0^+} \int_R^\epsilon \frac{2}{\pi} r \ln r \, dr \text{ (for small } r > 0) \\ &= \frac{2}{\pi} \lim_{R\to 0^+} \left(\frac{1}{2}\epsilon^2 \ln \epsilon - \frac{1}{2} R^2 \ln R - \frac{1}{4}\epsilon^2 + \frac{1}{4} R^2\right) \\ &= \frac{2}{\pi}\left(\frac{1}{2}\epsilon^2 \ln\epsilon - \frac{1}{4}\epsilon^2\right)\end{aligned}$$

This limit holds according to the result of Exercise 12.5.3. Applying the same limit result again reveals

$$\lim_{\epsilon\to 0^+} \int_0^\epsilon rY_0(kr)\, dr = 0$$

12.5.6 Replacing the r and r' in Eq. (12.45) with the expressions found in Exercise 12.5.5 yields the desired result.

$$\begin{aligned}G &= \frac{1}{4\pi} \ln \frac{R^2(\hat{\rho}^2 + \rho^2 - 2\hat{\rho}\rho\cos(\hat{\theta}-\theta))}{\rho^2\left(\frac{R^4}{\rho^2} + \hat{\rho}^2 - 2\hat{\rho}\left(\frac{R^2}{\rho}\right)\rho\cos(\hat{\theta}-\theta)\right)} \\ &= \frac{1}{4\pi} \ln \frac{R^2(\hat{\rho}^2 + \rho^2 - 2\hat{\rho}\rho\cos(\hat{\theta}-\theta))}{R^4 + \hat{\rho}^2\rho^2 - 2R^2\hat{\rho}\rho\cos(\hat{\theta}-\theta)}\end{aligned}$$

12.5.8 If the source term $\phi = 0$ in Eq. (12.47) the formula for the solution reduces to the following.

$$\begin{aligned}u(\rho,\theta) &= \int_0^{2\pi} f(\hat{\theta}) \left[\frac{\partial G}{\partial \hat{\rho}}\right]_{\hat{\rho}=R} R \, d\hat{\theta} \\ &= \int_0^{2\pi} f(\hat{\theta}) \frac{R^2 - \rho^2}{2\pi R[R^2 + \rho^2 - 2R\rho\cos(\hat{\theta}-\theta)]} R\, d\hat{\theta} \\ &= \frac{1}{2\pi}\int_0^{2\pi} f(\hat{\theta}) \frac{R^2-\rho^2}{R^2+\rho^2 - 2R\rho\cos(\hat{\theta}-\theta)}\, d\hat{\theta}\end{aligned}$$

This is equivalent to the formula stated in Eq. (6.46).

12.5.10 Let $x = \rho\cos\theta$ and $y = \rho\sin\theta$ and note that

$$\begin{aligned}\sqrt{(\xi - x)^2 + (\eta - y)^2} &= \sqrt{(\hat\rho\cos\hat\theta - r\cos\theta)^2 + (\hat\rho\sin\hat\theta - r\sin\theta)^2} \\ &= \sqrt{\hat\rho^2\cos^2\hat\theta - 2\hat\rho r\cos\hat\theta\cos\theta + r^2\cos^2\theta + \hat\rho^2\sin^2\hat\theta - 2\hat\rho r\sin\hat\theta\sin\theta + r^2\sin^2\theta} \\ &= \sqrt{\hat\rho^2 - 2\hat\rho r(\cos\hat\theta\cos\theta + \sin\hat\theta\sin\theta) + r^2} \\ &= \sqrt{\hat\rho^2 + r^2 - 2\hat\rho r\cos(\hat\theta - \theta)}.\end{aligned}$$

When $\hat\rho = R$ the boundary condition for g is

$$g(R, \hat\theta) = -\frac{1}{4}Y_0(k\sqrt{\hat\rho^2 + r^2 - 2\hat\rho r\cos(\hat\theta - \theta)}).$$

12.6.1 By definition,

$$U = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}\hat U(\omega)e^{i\omega\xi}\,d\omega.$$

Substituting the given function $\hat U$ gives

$$\begin{aligned}U &= \frac{H(t-\tau)}{2\pi}\int_{-\infty}^{\infty}e^{-i\omega x - \omega^2\kappa(t-\tau)}e^{i\omega\xi}\,d\omega \\ &= \frac{H(t-\tau)}{2\pi}\int_{-\infty}^{\infty}e^{i\omega(\xi - x) - \omega^2\kappa(t-\tau)}\,d\omega\end{aligned}$$

Complete the square in the exponent with respect to the variable ω.

$$\begin{aligned}U &= \frac{H(t-\tau)}{2\pi}\int_{-\infty}^{\infty}e^{-\kappa(t-\tau)\left(\omega - \frac{i(\xi-x)}{2\kappa(t-\tau)}\right)^2 - \frac{(\xi-x)^2}{4\kappa(t-\tau)}}\,d\omega \\ &= \frac{H(t-\tau)}{2\pi}e^{-\frac{(\xi-x)^2}{4\kappa(t-\tau)}}\int_{-\infty}^{\infty}e^{-\kappa(t-\tau)\left(\omega - \frac{i(\xi-x)}{2\kappa(t-\tau)}\right)^2}\,d\omega\end{aligned}$$

Now make the substitutions

$$\gamma = \sqrt{\kappa(t-\tau)}\left(\omega - \frac{i(\xi - x)}{2\kappa(t-\tau)}\right),$$

$$\frac{1}{\sqrt{\kappa(t-\tau)}}\,d\gamma = d\omega.$$

The improper integral becomes

$$\begin{aligned}U &= \frac{H(t-\tau)}{2\pi\sqrt{\kappa(t-\tau)}}e^{-\frac{(\xi-x)^2}{4\kappa(t-\tau)}}\int_{-\infty - \frac{i(\xi-x)}{2\sqrt{\kappa(t-\tau)}}}^{\infty - \frac{i(\xi-x)}{2\sqrt{\kappa(t-\tau)}}}e^{-\gamma^2}\,d\gamma \\ &= \frac{H(t-\tau)}{\sqrt{4\pi\kappa(t-\tau)}}e^{-\frac{(\xi-x)^2}{4\kappa(t-\tau)}}.\end{aligned}$$

12.6.3

$$\begin{aligned}G(x, t; 0, \tau) &= U(x, t; 0, \tau) - U(-x, t; 0, \tau) \\ &= \frac{H(t-\tau)}{\sqrt{4\pi\kappa(t-\tau)}}e^{-\frac{(0-x)^2}{4\kappa(t-\tau)}} - \frac{H(t-\tau)}{\sqrt{4\pi\kappa(t-\tau)}}e^{-\frac{(0+x)^2}{4\kappa(t-\tau)}} \\ &= 0\end{aligned}$$

12.6.5 Let the reference solution $r(x,t) = 1-x$ and suppose $u(x,t) = v(x,t)+r(x,t)$. The unknown function $v(x,t)$ must satisfy the following initial boundary value problem with homogeneous Dirichlet boundary conditions.

$$\begin{aligned}
v_t - v_{xx} &= \sin(\pi x) \text{ for } 0 < x < 1 \text{ and } t > 0, \\
v(0,t) &= 0 \text{ for } t > 0, \\
v(1,t) &= 0 \text{ for } t > 0, \\
v(x,0) &= \cos\left(\frac{\pi x}{2}\right) + x - 1 \text{ for } 0 < x < 1.
\end{aligned}$$

The eigenfunctions of the boundary value problem are $\phi_n(x) = \sin(n\pi x)$ with corresponding eigenvalues $\lambda_n = n^2\pi^2$ for $n \in \mathbb{N}$. According to Eq. (12.58) the Green's function for this initial boundary value problem is

$$G(x,t;\xi,\tau) = 2\sum_{n=1}^{\infty} e^{-n^2\pi^2(t-\tau)} \sin(n\pi x)\sin(n\pi\xi).$$

Using this Green's function,

$$\begin{aligned}
u(x,t) &= 1 - x + \int_0^1 2\left(\cos\left(\frac{\pi\xi}{2}\right) + x - 1\right)\sum_{n=1}^{\infty} e^{-n^2\pi^2 t}\sin(n\pi x)\sin(n\pi\xi)\,d\xi \\
&\quad + \int_0^t\int_0^1 2\sin(\pi\xi)\sum_{n=1}^{\infty} e^{-n^2\pi^2(t-\tau)}\sin(n\pi x)\sin(n\pi\xi)\,d\xi\,d\tau \\
&= 1 - x + 2\sum_{n=1}^{\infty} e^{-n^2\pi^2 t}\sin(n\pi x)\int_0^1\left(\cos\left(\frac{\pi\xi}{2}\right) + x - 1\right)\sin(n\pi\xi)\,d\xi \\
&\quad + 2\sum_{n=1}^{\infty}\sin(n\pi x)\int_0^t e^{-n^2\pi^2(t-\tau)}\int_0^1 \sin(\pi\xi)\sin(n\pi\xi)\,d\xi\,d\tau \\
&= 1 - x + \sum_{n=1}^{\infty}\frac{2}{n\pi(4n^2-1)}e^{-n^2\pi^2 t}\sin(n\pi x) + \sin(\pi x)\int_0^t e^{-\pi^2(t-\tau)}\,d\tau \\
&= 1 - x + \sum_{n=1}^{\infty}\frac{2e^{-n^2\pi^2 t}}{n\pi(4n^2-1)}\sin(n\pi x) + \frac{(1-e^{-\pi^2 t})}{\pi^2}\sin(\pi x).
\end{aligned}$$

12.6.7 Using the Green's function in Eq. (**??**)s a model,

$$G(x,t;\xi,\tau) = \frac{2}{L}\sum_{n=1}^{\infty}\cos\left(\frac{(2n-1)\pi x}{2L}\right)\cos\left(\frac{(2n-1)\pi\xi}{2L}\right)e^{-\kappa(2n-1)^2\pi^2(t-\tau)/(4L^2)}.$$

12.7.1 If $t < \tau$ then the expression above is zero and the argument of the Heaviside function in Eq. (12.66) is less than zero making the Green's function zero. If $t > \tau$ then $H(t-\tau) = 1$ and

$$H(\xi - (x + c(t-\tau))) - H(\xi - (x - c(t-\tau))) = 1$$

if and only if

$$\begin{aligned}
x - c(t-\tau) &< \xi < x + c(t-\tau) \\
-x + c(t-\tau) &> -\xi > -x - c(t-\tau) \\
c(t-\tau) &> x - \xi > -c(t-\tau) \\
|x - \xi| &< c(t-\tau)
\end{aligned}$$

and zero otherwise. This implies for $t > \tau$ that

$$H(\xi - (x + c(t-\tau))) - H(\xi - (x - c(t-\tau))) = H(c(t-\tau) - |x-\xi|).$$

12.7.3 Let $f(x) = \sin x$, $g(x) = \cos x$, and $F(x,t) = e^{-(x-t)}$. Using Exercise 12.7.2 and the Green's function given in Eq. (12.66), the solution can be written as

$$\begin{aligned}
u(x,t) &= \frac{1}{2}(\sin(x+ct) + \sin(x-ct)) + \frac{1}{2c}\int_{x-ct}^{x+ct} \cos\xi \, d\xi \\
&\quad + \frac{1}{2c}\int_0^t \int_{x-c(t-\tau)}^{x+c(t-\tau)} e^{-(\xi-\tau)} \, d\xi \, d\tau \\
&= \cos(ct)\sin(x) + \frac{1}{c}\cos(x)\sin(ct) + \frac{1}{c}\int_0^t e^{-(x-\tau)} \sinh(c(t-\tau)) \, d\tau \\
&= \cos(ct)\sin(x) + \frac{1}{c}\cos(x)\sin(ct) + \frac{e^{-x}}{c(c^2-1)}(c\cosh(ct) - ce^t + \sinh(ct))
\end{aligned}$$

12.7.5 The nth orthonormal eigenfunction for the associated boundary value problem is $\phi_n(x) = \sqrt{\frac{2}{L}}\sin(\frac{n\pi x}{L})$ with corresponding eigenvalue $\frac{n^2\pi^2}{L^2}$ for $n \in \mathbb{N}$. According to Eq. (12.73) the appropriate Green's function is

$$\begin{aligned}
G(x,t;\xi,\tau) &= H(t-\tau)\sum_{n=1}^{\infty} \frac{\sin\left(\frac{cn\pi(t-\tau)}{L}\right)\sqrt{\frac{2}{L}}\sin\left(\frac{n\pi\xi}{L}\right)\sqrt{\frac{2}{L}}\sin\left(\frac{n\pi x}{L}\right)}{cn\pi/L} \\
&= 2H(t-\tau)\sum_{n=1}^{\infty} \frac{\sin\left(\frac{cn\pi(t-\tau)}{L}\right)\sin\left(\frac{n\pi\xi}{L}\right)\sin\left(\frac{n\pi x}{L}\right)}{cn\pi}.
\end{aligned}$$

13.1.1 Let $u(x,t) = u_1(x,t) + u_2(x,t)$, then

$$\begin{aligned}
u(x,t) &= A\left(\cos(kx - \omega t) + \cos(kx + \omega t + \phi)\right) \\
&= A\left(\cos(kx + \frac{\phi}{2} - (\omega t + \frac{\phi}{2})) + \cos(kx + \frac{\phi}{2} + \omega t + \frac{\phi}{2})\right) \\
&= A\left(\cos(kx + \frac{\phi}{2})\cos(\omega t + \frac{\phi}{2}) + \sin(kx + \frac{\phi}{2})\sin(\omega t + \frac{\phi}{2}) + \cos(kx + \frac{\phi}{2})\cos(\omega t + \frac{\phi}{2}) - \sin(kx + \frac{\phi}{2})\sin(\omega t + \frac{\phi}{2})\right) \\
&= 2A\cos(kx + \frac{\phi}{2})\cos(\omega t + \frac{\phi}{2})
\end{aligned}$$

13.1.3 Setting $u(x,t) = U(x - c\,t)$ where $\xi = x - c\,t$ and substituting into the Klein-Gordon equation results in the following

$$c^2 U'' - \alpha^2 U'' + U = 0.$$

If $c^2 = \alpha^2$ then $U \equiv 0$. If $c^2 < \alpha^2$ the solution of ordinary differential equation above is

$$U(\xi) = Ae^{\xi/\sqrt{\alpha^2 - c^2}} + Be^{-\xi/\sqrt{\alpha^2-c^2}}$$

which is unbounded unless $A = B = 0$. If $c^2 > \alpha^2$ the solution of the ordinary differential equation above is

$$U(\xi) = A\cos\frac{\xi}{\sqrt{c^2-\alpha^2}} + B\sin\frac{\xi}{\sqrt{c^2-\alpha^2}}.$$

Therefore the bounded traveling wave solutions of the Klein-Gordon equation have the form

$$u(x,t) = A\cos\frac{x - c\,t}{\sqrt{c^2-\alpha^2}} + B\sin\frac{x - c\,t}{\sqrt{c^2-\alpha^2}}$$

where A and B are arbitrary constants.

13.1.5

(a) The linear Klein-Gordon equation: assume that $u(x,t) = Ae^{i(kx-\omega t)}$ is a solution of the linear Klein-Gordon equation, then

$$\begin{aligned} -A\omega^2 e^{i(kx-\omega t)} + Ae^{i(kx-\omega t)} &= -A\alpha^2 k^2 e^{i(kx-\omega t)} \\ 1-\omega^2 &= -\alpha^2 k^2 \\ \omega^2 &= \alpha^2 k^2 + 1. \end{aligned}$$

(b) The beam equation: assume that $u(x,t) = Ae^{i(kx-\omega t)}$ is a solution of the beam equation, then

$$\begin{aligned} -A\omega^2 e^{i(kx-\omega t)} + A\alpha^2 k^4 e^{i(kx-\omega t)} &= 0 \\ -\omega^2 + \alpha^2 k^4 &= 0 \\ \omega^2 &= \alpha^2 k^4. \end{aligned}$$

(c) The linear Korteweg-de Vries equation: assume that $u(x,t) = Ae^{i(kx-\omega t)}$ is a solution of the linear KdV equation, then

$$\begin{aligned} -iA\,\omega e^{i(kx-\omega t)} + iA\alpha k e^{i(kx-\omega t)} - iA\beta k^3 e^{i(kx-\omega t)} &= 0 \\ -\omega + \alpha k - \beta k^3 &= 0 \\ \omega &= k(\alpha - \beta k^2). \end{aligned}$$

13.2.1

(a) Let $f(U) = U^2 - 2cU - 2A$. If $f(U) = 0$ has complex roots, then $c^2 + 2A < 0$. In this case the absolute minimum of $f(U)$ is $-c^2 - 2A > 0$. This implies in Eq. (13.14) that

$$\frac{dU}{d\xi} \geq -\frac{c^2+2A}{2\nu}$$

which implies

$$U(\xi) \geq -\frac{(c^2+2A)\xi}{2\nu}$$

which is unbounded as $\xi \to \pm\infty$.

(b) Let $f(U) = U^2 - 2cU - 2A$. If $f(U) = 0$ has a repeated real root, then $c^2 + 2A = 0$ and $f(c) = 0$. In this case the absolute minimum of $f(U)$ is 0. This implies in Eq. (13.14) that

$$\begin{aligned} \frac{dU}{d\xi} &= \frac{1}{2\nu}(U-c)^2 \\ \frac{1}{(U-c)^2}\,dU &= \frac{1}{2\nu}\,d\xi \text{ (assuming } U(\xi) \neq c) \\ \frac{-1}{U-c} &= \frac{\xi+B}{2\nu} \text{ (where } B \text{ is a constant)} \\ U(\xi) &= c - \frac{2\nu}{\xi+B}. \end{aligned}$$

Note that as $\xi \to B^-$ and as $\xi \to B^+$ the solution $U(\xi)$ grows unbounded if $\nu > 0$. On the other hand if $U(\xi) = c$ then U is bounded and solves the ordinary differential equation.

13.2.3 Suppose $u(x,t) = \dfrac{U_1 + U_2 e^{\alpha(x-c\,t)+B}}{1 + e^{\alpha(x-c\,t)+B}}$, then

$$\begin{aligned} u_t &= \frac{\alpha\, c(U_1 - U_2)e^{\alpha(x-c\,t)+B}}{(1 + e^{\alpha(x-c\,t)+B})^2} \\ u_x &= \frac{-\alpha(U_1 - U_2)e^{\alpha(x-c\,t)+B}}{(1 + e^{\alpha(x-c\,t)+B})^2} \\ u_{xx} &= \frac{-\alpha^2(U_1 - U_2)e^{\alpha(x-c\,t)+B}(1 - e^{\alpha(x-c\,t)+B})}{(1 + e^{\alpha(x-c\,t)+B})^3} \end{aligned}$$

If $u(x,t)$ is a solution to the viscous Burgers' equation, then substituting these partial derivatives in Eq. (13.8) produces the equation

$$\begin{aligned} 0 &= \frac{\alpha\, c(U_1 - U_2)e^{\alpha(x-c\,t)+B}}{(1 + e^{\alpha(x-c\,t)+B})^2} - \frac{\alpha(U_1 - U_2)e^{\alpha(x-c\,t)+B}(U_1 + U_2 e^{\alpha(x-c\,t)+B})}{(1 + e^{\alpha(x-c\,t)+B})^3} \\ &\quad + \frac{\alpha^2 \nu(U_1 - U_2)e^{\alpha(x-c\,t)+B}(1 - e^{\alpha(x-c\,t)+B})}{(1 + e^{\alpha(x-c\,t)+B})^3} \\ &= c(1 + e^{\alpha(x-c\,t)+B}) - (U_1 + U_2 e^{\alpha(x-c\,t)+B}) + \alpha\nu(1 - e^{\alpha(x-c\,t)+B}) \\ c &= \frac{U_1 - \alpha\,\nu + (U_2 + \alpha\,\nu)e^{\alpha(x-c\,t)+B}}{1 + e^{\alpha(x-c\,t)+B}} \end{aligned}$$

Recall that $\alpha = (U_1 - U_2)/(2\nu)$ and thus

$$c = \frac{\left(\frac{U_1+U_2}{2}\right) + \left(\frac{U_1+U_2}{2}\right) e^{\alpha(x-c\,t)+B}}{1 + e^{\alpha(x-c\,t)+B}} = \frac{U_1 + U_2}{2}.$$

13.2.5 Let $u(x,t) = Kv(z,\tau)$ where $z = x\sqrt{\gamma}$ and $\tau = \gamma t$, then

$$\begin{aligned} u_t &= K\gamma v_\tau \\ u_x &= K\sqrt{\gamma} v_z \\ u_{xx} &= K\gamma v_{zz}. \end{aligned}$$

Substituting these expressions into Fisher's equation produces,

$$\begin{aligned} K\gamma v_\tau - K\gamma v_{zz} &= \gamma\, K\, v \left(1 - \frac{K\, v}{K}\right) \\ v_\tau - v_{zz} &= v(1 - v). \end{aligned}$$

Renaming τ by t, z by x, and v by u results in the nondimensional form of Fisher's equation.

Now suppose $u(x,t) = U(\xi)$ where $\xi = x - c\,t$. Substituting this in the nondimensional form of Fisher's equation produces,

$$\begin{aligned} -c\,U' - U'' &= U(1 - U) \\ U'' + c\,U' + U - U^2 &= 0. \end{aligned}$$

13.2.7 According to Eq. (13.28)

$$u(x,t) = \begin{cases} -1 & \text{if } x < -t \\ 1 & \text{if } x > t. \end{cases}$$

For the region where $-t < x < t$ the solution must be constant along the characteristic lines emanating from $x = 0$. A continuous solution would "interpolate" between the values of -1 and 1. Therefore a suitable solution would be

$$u(x,t) = \begin{cases} -1 & \text{if } x < -t \\ \frac{x}{t} & \text{if } -t \le x \le t \\ 1 & \text{if } x > t. \end{cases}$$

13.3.1 Since $V_T + \dfrac{6B\,C}{A} V\,V_X + \dfrac{B}{A^3} V_{XXX} = 0$, equating coefficients

- $\frac{6B\,C}{A} = 1$ and $\frac{B}{A^3} = k$ implies $A = 1/\sqrt{6k}$, $B = 1/(6\sqrt{6k})$, and $C = 1$.
- $\frac{6B\,C}{A} = -6$ and $\frac{B}{A^3} = 1$ implies $A = 1$, $B = 1$, and $C = -1$.
- $\frac{6B\,C}{A} = 1$ and $\frac{B}{A^3} = 1$ implies $A = 1/\sqrt{6}$, $B = 1/(6\sqrt{6})$, and $C = 1$.
- $\frac{6B\,C}{A} = \alpha$ and $\frac{B}{A^3} = \beta$ implies $A = \sqrt{\alpha/(6\beta)}$, $B = \frac{\alpha}{6}\sqrt{\alpha/(6\beta)}$, and $C = 1$ if $\alpha\beta > 0$). When $\alpha\beta < 0$ then $C = -1$.

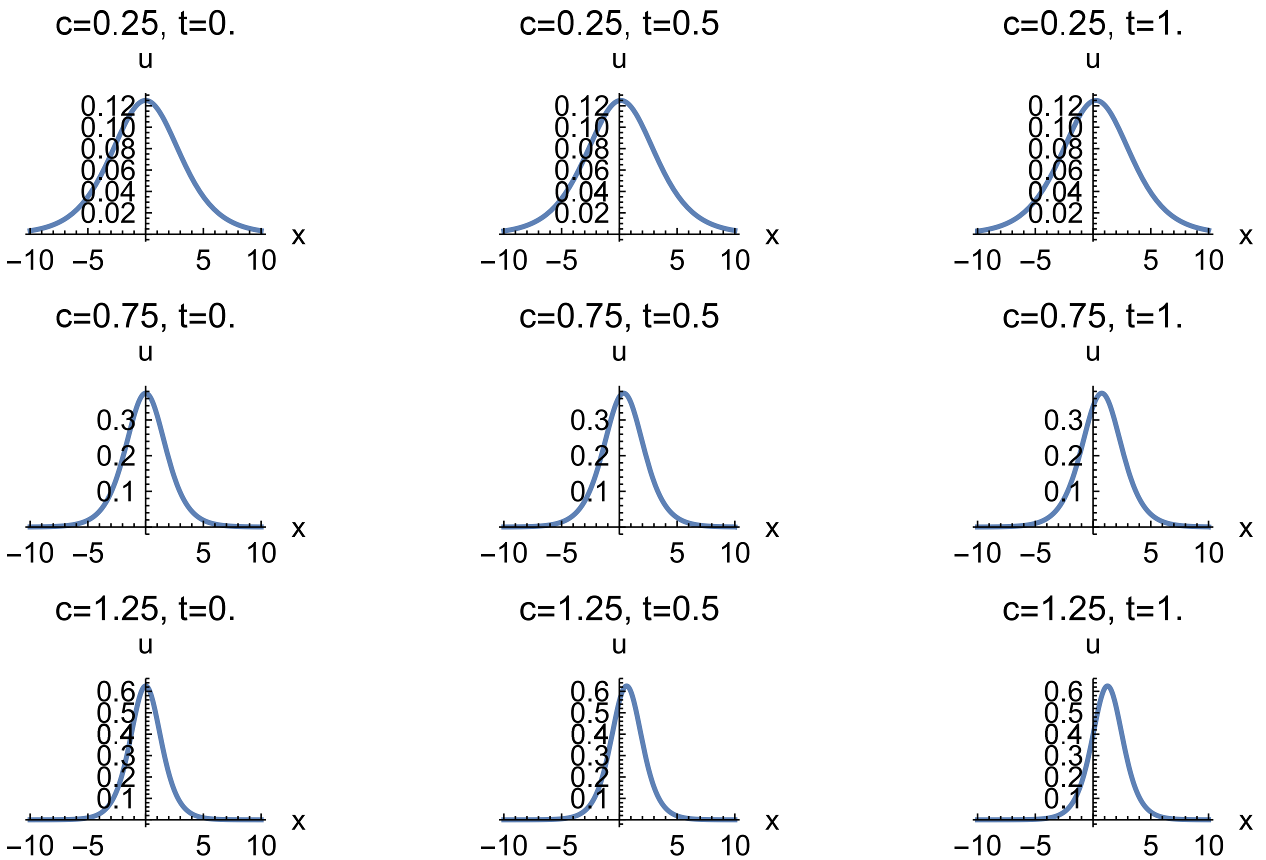
13.3.3 Differentiating the function above produces,

$$\begin{aligned}
u_t &= \frac{c^2\sqrt{c}}{2}\operatorname{sech}^2\left(\frac{\sqrt{c}}{2}(x - c\,t) + A\right)\tanh\left(\frac{\sqrt{c}}{2}(x - c\,t) + A\right)\\
u_x &= -\frac{c\sqrt{c}}{2}\operatorname{sech}^2\left(\frac{\sqrt{c}}{2}(x - c\,t) + A\right)\tanh\left(\frac{\sqrt{c}}{2}(x - c\,t) + A\right)\\
u_{xx} &= \frac{c^2}{4}\operatorname{sech}^4\left(\frac{\sqrt{c}}{2}(x - c\,t) + A\right)\left(\cosh\left(\sqrt{c}(x - c\,t) + 2A\right) - 2\right)\\
u_{xxx} &= -\frac{c^2\sqrt{c}}{4}\operatorname{sech}^4\left(\frac{\sqrt{c}}{2}(x - c\,t) + A\right)\tanh\left(\frac{\sqrt{c}}{2}(x - c\,t) + A\right)\left(\cosh\left(\sqrt{c}(x - c\,t) + 2A\right) - 5\right).
\end{aligned}$$

For the sake of compactness of notation, let $z = \dfrac{\sqrt{c}}{2}(x - c\,t) + A$ and substitute these partial derivatives in the KdV equation.

$$\begin{aligned}
u_t + 6u\,u_x + u_{xxx} &= \frac{c^2\sqrt{c}}{2}\operatorname{sech}^2(z)\tanh(z) - \frac{3c^2\sqrt{c}}{2}\operatorname{sech}^4(z)\tanh(z) - \frac{c^2\sqrt{c}}{4}\operatorname{sech}^4(z)\tanh(z)\,(\cosh(2z) - 5)\\
&= \frac{c^2\sqrt{c}}{2}\operatorname{sech}^2(z)\tanh(z)\left(1 - 3\operatorname{sech}^2(z) - \frac{1}{2}\operatorname{sech}^2(z)(\cosh(2z) - 5)\right)\\
&= \frac{c^2\sqrt{c}}{2}\operatorname{sech}^2(z)\tanh(z)\left(1 - 3\operatorname{sech}^2(z) - \frac{1}{2}\operatorname{sech}^2(z)(2\cosh^2(z) - 6)\right)\\
&= 0
\end{aligned}$$

Hence $u(x, t)$ is a traveling wave solution to the KdV equation. Several plots of the solution appear below.



13.3.5 Let $v(x,t) = V(x - c\,t) = V(\xi)$ be a solution to the modified KdV equation, then

$$V''' - 6V^2V' - c\,V' = 0.$$

Integrating both sides of the ordinary differential equation produces,

$$V'' - 2V^3 - c\,V = A$$

where A is a constant of integration. If it is assumed that as $\xi \to \pm\infty$ that $V \to 0$, $V' \to 0$, and $V'' \to 0$, then $A = 0$ and hence

$$\frac{d^2V}{d\xi^2} - 2V^3 - c\,V = 0.$$

Multiplying the ordinary differential equation by V' and integrating both sides of the resulting equation yields

$$\frac{1}{2}\left(\frac{dV}{d\xi}\right)^2 - \frac{1}{2}V^4 - \frac{c}{2}V^2 = B$$

where B is again a constant of integration. The assumptions made earlier that $V \to 0$ and $V' \to 0$ as $\xi \to \infty$ and $\xi \to -\infty$ imply that $B = 0$, and therefore,

$$\frac{1}{2}\left(\frac{dV}{d\xi}\right)^2 - \frac{1}{2}V^4 - \frac{c}{2}V^2 = 0$$

$$\left(\frac{dV}{d\xi}\right)^2 = V^4 + c\,V^2.$$

After solving for $dV/d\xi$,

$$\frac{dV}{d\xi} = \pm V\sqrt{c+V^2}.$$

The sign associated with the right-hand side is unimportant and will be dropped. Separating the variables and integrating both sides of the equation results in

$$\begin{aligned}\frac{1}{V\sqrt{c+V^2}}\,dV &= d\xi \\ \int \frac{1}{V\sqrt{c+V^2}}\,dV &= \xi + \xi_0,\end{aligned}$$

Where ξ_0 is a constant of integration. If $c = 1$, the integral above can be simplified.

$$\begin{aligned}\int \frac{1}{V\sqrt{1+V^2}}\,dV &= \xi + \xi_0 \\ \ln\left|\frac{V}{1+\sqrt{1+V^2}}\right| &= \xi + \xi_0 \\ \frac{V}{1+\sqrt{1+V^2}} &= e^{\xi+\xi_0}.\end{aligned}$$

Solving for V yields

$$V = \frac{2e^{\xi+\xi_0}}{1-e^{2(\xi+\xi_0)}}.$$

Simplifying further produces

$$\begin{aligned}V(\xi) &= \frac{2}{e^{-(\xi+\xi_0)} - e^{\xi+\xi_0}} \\ v(x,t) &= -\operatorname{csch}(x-t).\end{aligned}$$

13.3.7 Let $u(x,t) = v_x(x,t) - (v(x,t))^2$ where $v(x,t)$ is a solution to Eq. (13.44). Taking partial derivatives yields

$$\begin{aligned}u_t &= v_{xt} - 2v\,v_t \\ u_x &= v_{xx} - 2v\,v_x \\ u_{xx} &= v_{xxx} - 2(v_x)^2 - 2v\,v_{xx} \\ u_{xxx} &= v_{xxxx} - 6v_x v_{xx} - 2v\,v_{xxx}.\end{aligned}$$

Therefore,

$$\begin{aligned}u_t + 6u\,u_x + u_{xxx} &= v_{xt} - 2v\,v_t + 6(v_x - v^2)(v_{xx} - 2v\,v_x) + v_{xxxx} - 6v_x v_{xx} - 2v\,v_{xxx} \\ &= v_{xt} - 2v\,v_t + 12v^3 v_x - 12v(v_x)^2 - 6v^2 v_{xx} - 2v\,v_{xxx} + v_{xxxx} \\ &= -2v(v_t - 6v^2 v_x + v_{xxx}) + v_{tx} - 12v(v_x)^2 - 6v^2 v_{xx} + 12v_x v_{xx} + v_{xxxx} \\ &= (v_t - 6v^2 v_x + v_{xxx})_x \\ &= 0\end{aligned}$$

13.4.1 Using the function/inverse function relationship, for $0 < x < 1$

$$
\begin{aligned}
\operatorname{sech}\left(\operatorname{sech}^{-1} x\right) &= x \\
\frac{d}{dx}\left[\operatorname{sech}\left(\operatorname{sech}^{-1} x\right)\right] &= \frac{d}{dx}[x] \\
-\operatorname{sech}\left(\operatorname{sech}^{-1} x\right)\tanh\left(\operatorname{sech}^{-1} x\right)\frac{d}{dx}\left[\operatorname{sech}^{-1} x\right] &= 1 \\
\tanh\left(\operatorname{sech}^{-1} x\right)\frac{d}{dx}\left[\operatorname{sech}^{-1} x\right] &= \frac{-1}{x} \\
\sqrt{\tanh^2\left(\operatorname{sech}^{-1} x\right)}\frac{d}{dx}\left[\operatorname{sech}^{-1} x\right] &= \frac{-1}{x} \\
\sqrt{1-\operatorname{sech}^2\left(\operatorname{sech}^{-1} x\right)}\frac{d}{dx}\left[\operatorname{sech}^{-1} x\right] &= \frac{-1}{x} \\
\sqrt{1-x^2}\frac{d}{dx}\left[\operatorname{sech}^{-1} x\right] &= \frac{-1}{x} \\
\frac{d}{dx}\left[\operatorname{sech}^{-1} x\right] &= \frac{-1}{x\sqrt{1-x^2}}.
\end{aligned}
$$

13.4.3 Note that

$$
\begin{aligned}
\psi_t &= ie^{it}\operatorname{sech}(x) \\
\psi_{xx} &= e^{it}(\operatorname{sech}(x)\tanh^2(x) - \operatorname{sech}^3(x)) \\
|\psi|^2 &= \operatorname{sech}^2(x).
\end{aligned}
$$

Substituting these expressions into the partial differential equation reveals,

$$
\begin{aligned}
i\psi_t + \psi_{xx} + 2|\psi|^2\psi &= -e^{it}\operatorname{sech}(x) + e^{it}(\operatorname{sech}(x)\tanh^2(x) - \operatorname{sech}^3(x)) + 2\operatorname{sech}^2(x)e^{it}\operatorname{sech}(x) \\
&= e^{it}\left(-\operatorname{sech}(x) + \operatorname{sech}(x)\tanh^2(x) - \operatorname{sech}^3(x) + 2\operatorname{sech}^3(x)\right) \\
&= e^{it}\left(-\operatorname{sech}(x) + \operatorname{sech}(x)(1-\operatorname{sech}^2(x)) + \operatorname{sech}^3(x)\right) \\
&= 0.
\end{aligned}
$$

Hence $\psi(x,t)$ solves the nonlinear Schrödinger equation.

13.4.5 Making use of the product and chain rules for differentiation produces.

$$
\begin{aligned}
\psi_t &= \psi(x,t)\left[-i\omega + 2k(k^2-\omega)^{\frac{1}{2}}\tanh((k^2-\omega)^{\frac{1}{2}}(x-2kt+A))\right] \\
\psi_x &= \psi(x,t)\left[ik - (k^2-\omega)^{\frac{1}{2}}\tanh((k^2-\omega)^{\frac{1}{2}}(x-2kt+A))\right] \\
\psi_{xx} &= \psi(x,t)\left(-k^2 - 2ik(k^2-\omega)^{\frac{1}{2}}\tanh((k^2-\omega)^{\frac{1}{2}}(x-2kt+A)) + (k^2-\omega)(1-2\operatorname{sech}^2((k^2-\omega)^{\frac{1}{2}}(x-2kt+A)))\right)
\end{aligned}
$$

Substituting these partial derivatives into Eq. (13.45) produces

$$
\begin{aligned}
i\psi_t + \psi_{xx} + \gamma|\psi|^2\psi &= i\psi(x,t)\left[-i\omega + 2k(k^2-\omega)^{\frac{1}{2}}\tanh((k^2-\omega)^{\frac{1}{2}}(x-2kt+A))\right] \\
&\quad + \psi(x,t)\Big(-k^2 - 2ik(k^2-\omega)^{\frac{1}{2}}\tanh((k^2-\omega)^{\frac{1}{2}}(x-2kt+A)) \\
&\quad + (k^2-\omega)(1-2\,\mathrm{sech}^2((k^2-\omega)^{\frac{1}{2}}(x-2kt+A)))\Big) \\
&\quad + 2\gamma\frac{k^2-\omega}{\gamma}\psi(x,t)\,\mathrm{sech}^2(\sqrt{k^2-\omega}(x-2kt+A)) \\
&= \psi(x,t)\Big[\omega - k^2 + (k^2-\omega)(1-2\,\mathrm{sech}^2((k^2-\omega)^{\frac{1}{2}}(x-2kt+A))) \\
&\quad + 2(k^2-\omega)\,\mathrm{sech}^2(\sqrt{k^2-\omega}(x-2kt+A))\Big] \\
&= \psi(x,t)\left[-2(k^2-\omega)\,\mathrm{sech}^2((k^2-\omega)^{\frac{1}{2}}(x-2kt+A))) + 2(k^2-\omega)\,\mathrm{sech}^2(\sqrt{k^2-\omega}(x-2kt+A))\right] \\
&= 0.
\end{aligned}
$$

Hence the solution given in Eq. (13.51) solves the nonlinear Schrödinger equation given in Eq. (13.45).
13.4.7 Suppose the solution takes the form of $\psi(x,t) = f(x-ct)e^{i(kx-\omega t)}$. Differentiating and substituting into Eq. (13.45) produce,

$$
\begin{aligned}
0 &= i(-cf' - i\omega f)e^{i(kx-\omega t)} + (f'' + 2ikf' - k^2 f)e^{i(kx-\omega t)} - 2f^3 e^{i(kx-\omega t)} \\
0 &= -cif' + \omega f + f'' + 2ikf' - k^2 f - 2f^3 \\
0 &= i(2k-c)f' + (\omega - k^2)f + f'' - 2f^3
\end{aligned}
$$

Assume again the imaginary portion of the equation vanishes, meaning that $c = 2k$. This implies,

$$
\begin{aligned}
0 &= f'' + (\omega - k^2)f - 2f^3 \\
0 &= 2f'f'' + 2(\omega - k^2)ff' - 4f^3 f' \\
\int 0\,d\xi &= \int 2f'(\xi)f''(\xi)\,d\xi + (\omega - k^2)\int 2f(\xi)f'(\xi)\,d\xi - \int 4(f(\xi))^3 f'(\xi)\,d\xi \\
A &= (f'(\xi))^2 + (\omega - k^2)(f(\xi))^2 - (f(\xi))^4 \\
(f'(\xi))^2 &= A - (f(\xi))^2\left[(\omega - k^2) - (f(\xi))^2\right].
\end{aligned}
$$

If the solution $\psi(x,t)$ and its derivative $\psi_x(x,t)$ vanish as $x \to \pm\infty$ then $A = 0$. This implies

$$
\begin{aligned}
f'(\xi) &= \pm f(\xi)\sqrt{(k^2-\omega) + (f(\xi))^2} \\
\int \frac{f'(\xi)}{f(\xi)\sqrt{(k^2-\omega) + (f(\xi))^2}}\,d\xi &= \int 1\,d\xi \\
\frac{-1}{\sqrt{k^2-\omega}}\tanh^{-1}\left(\frac{\sqrt{(k^2-\omega) + (f(\xi))^2}}{\sqrt{k^2-\omega}}\right) &= \xi + A \\
-\sqrt{(k^2-\omega) + (f(\xi))^2} &= \sqrt{k^2-\omega}\tanh(\sqrt{k^2-\omega}(\xi + A)) \\
(f(\xi))^2 &= (k^2-\omega)\left[-1 + \tanh^2(\sqrt{k^2-\omega}(\xi + A))\right] \\
(f(\xi))^2 &= (k^2-\omega)\left[-\,\mathrm{sech}^2(\sqrt{k^2-\omega}(\xi + A))\right] \\
f(\xi) &= i\sqrt{k^2-\omega}\,\mathrm{sech}(\sqrt{k^2-\omega}(\xi + A)) \\
\psi(x,t) &= i\sqrt{k^2-\omega}e^{i(kx-\omega t)}\,\mathrm{sech}(\sqrt{k^2-\omega}(x - 2kt + A))
\end{aligned}
$$

where A is a constant of integration.
13.4.9 Let $\xi = x - ct$ and differentiate the proposed solution.

$$\begin{aligned}\psi_t &= (-cf'(\xi) + i\omega f(\xi))e^{i(-kx+\omega t+\theta_0)} \\ \psi_x &= (f'(\xi) - ikf(\xi))e^{i(-kx+\omega t+\theta_0)} \\ \psi_{xx} &= (f''(\xi) - 2ikf'(\xi) - k^2 f(\xi))e^{i(-kx+\omega t+\theta_0)}\end{aligned}$$

Substitute into the nonlinear PDE.

$$\begin{aligned}0 &= (-cf'(\xi) + i\omega f(\xi))e^{i(-kx+\omega t+\theta_0)} + \alpha(f''(\xi) - 2ikf'(\xi) - k^2 f(\xi))e^{i(-kx+\omega t+\theta_0)} \\ &\quad + \gamma \left| f(\xi)e^{i(-kx+\omega t+\theta_0} \right|^{2m} f(\xi)e^{i(-kx+\omega t+\theta_0} \\ &= (-cf'(\xi) + i\omega f(\xi)) + \alpha(f''(\xi) - 2ikf'(\xi) - k^2 f(\xi)) + \gamma \left| f(\xi) \right|^{2m} f(\xi) \\ &= \alpha f''(\xi) - (\omega + \alpha k^2)f(\xi) + \gamma(f(\xi))^{2m+1} - i(c + 2\alpha k)f'(\xi)\end{aligned}$$

The imaginary part of the equation implies $c = -2\alpha k$. Consider the real part of the equation.

$$\begin{aligned}\alpha f''(\xi) - (\omega + \alpha k^2)f(\xi) + \gamma(f(\xi))^{2m+1} &= 0 \\ \alpha f''(\xi)f'(\xi) - (\omega + \alpha k^2)f(\xi)f'(\xi) + \gamma(f(\xi))^{2m+1}f'(\xi) &= 0\end{aligned}$$

Integrate with respect to ξ over the real number line and assume that $\lim_{\xi\to\pm\infty} f(\xi) = 0$ and $\lim_{\xi\to\pm\infty} f'(\xi) = 0$.

$$\begin{aligned}\frac{\alpha}{2}(f'(\xi))^2 - \frac{1}{2}(\omega + \alpha k^2)(f(\xi))^2 + \frac{\gamma}{2m+2}(f(\xi))^{2m+2} &= A \\ \alpha(f'(\xi))^2 - (\omega + \alpha k^2)(f(\xi))^2 + \frac{\gamma}{m+1}(f(\xi))^{2m+2} &= 0\end{aligned}$$

Make the substitution $g(\xi) = (f(\xi))^2$ then solve for $g'(\xi)$.

$$\begin{aligned}\alpha\left(\frac{g'(\xi)}{2f(\xi)}\right)^2 - (\omega + \alpha k^2)g(\xi) + \frac{\gamma}{m+1}(g(\xi))^{m+1} &= 0 \\ \frac{(g'(\xi))^2}{4g(\xi)} - \left(\frac{\omega}{\alpha} + k^2\right)g(\xi) + \frac{\gamma}{\alpha(m+1)}(g(\xi))^{m+1} &= 0 \\ (g'(\xi))^2 - 4\left(\frac{\omega}{\alpha} + k^2\right)(g(\xi))^2 + \frac{4\gamma}{\alpha(m+1)}(g(\xi))^{m+2} &= 0 \\ \frac{dg}{d\xi} &= 2g(\xi)\sqrt{\left(\frac{\omega}{\alpha} + k^2\right) - \frac{\gamma}{\alpha(m+1)}(g(\xi))^m} \\ \frac{dg}{g\sqrt{\left(\frac{\omega}{\alpha} + k^2\right) - \frac{\gamma}{\alpha(m+1)}g^m}} &= 2\,d\xi\end{aligned}$$

Make the substitution $u = g^{m/2}$ and $\frac{2}{mu}\,du = \frac{1}{g}\,dg$.

$$\frac{2\,du}{mu\sqrt{\left(\frac{\omega}{\alpha} + k^2\right) - \frac{\gamma}{\alpha(m+1)}u^2}} = 2\,d\xi$$

Simplify and integrate both sides of the last equation.

$$\int \frac{1}{mu\sqrt{\left(\frac{\omega}{\alpha}+k^2\right)-\frac{\gamma}{\alpha(m+1)}u^2}}\,du = \int 1\,d\xi$$

$$\frac{-1}{m}\sqrt{\frac{\alpha}{\alpha k^2+\omega}}\tanh^{-1}\left(\sqrt{\frac{\alpha}{\alpha k^2+\omega}}\sqrt{\frac{\omega}{\alpha}+k^2-\frac{\gamma}{\alpha(m+1)}u^2}\right) = \xi + C$$

$$-\tanh^{-1}\left(\sqrt{\frac{\alpha}{\alpha k^2+\omega}}\sqrt{\frac{\omega}{\alpha}+k^2-\frac{\gamma}{\alpha(m+1)}u^2}\right) = m\sqrt{\frac{\omega}{\alpha}+k^2}\xi + C$$

$$\sqrt{\frac{\alpha}{\alpha k^2+\omega}}\sqrt{\frac{\omega}{\alpha}+k^2-\frac{\gamma}{\alpha(m+1)}u^2} = -\tanh\left(m\sqrt{\frac{\omega}{\alpha}+k^2}\xi + C\right)$$

$$\frac{\omega}{\alpha}+k^2-\frac{\gamma}{\alpha(m+1)}u^2 = \left(\frac{\omega}{\alpha}+k^2\right)\tanh^2\left(m\sqrt{\frac{\omega}{\alpha}+k^2}\xi + C\right)$$

$$\frac{\gamma}{\alpha(m+1)}u^2 = \left(\frac{\omega}{\alpha}+k^2\right)\operatorname{sech}^2\left(m\sqrt{\frac{\omega}{\alpha}+k^2}\xi + C\right)$$

$$u(\xi) = \sqrt{\frac{m+1}{\gamma}(\omega+\alpha k^2)}\operatorname{sech}\left(m\sqrt{\frac{\omega}{\alpha}+k^2}\xi + C\right)$$

$$(g(\xi))^{m/2} = \sqrt{\frac{m+1}{\gamma}(\omega+\alpha k^2)}\operatorname{sech}\left(m\sqrt{\frac{\omega}{\alpha}+k^2}\xi + C\right)$$

$$(f(\xi))^{m} = \sqrt{\frac{m+1}{\gamma}(\omega+\alpha k^2)}\operatorname{sech}\left(m\sqrt{\frac{\omega}{\alpha}+k^2}\xi + C\right)$$

$$f(\xi) = \left(\sqrt{\frac{m+1}{\gamma}(\omega+\alpha k^2)}\operatorname{sech}\left[m\sqrt{\frac{\omega}{\alpha}+k^2}\xi + C\right]\right)^{\frac{1}{m}}$$

The expression C is a constant of integration. Finally

$$\psi(x,t) = \left(\sqrt{\frac{m+1}{\gamma}(\omega+\alpha k^2)}\operatorname{sech}\left[m\sqrt{\frac{\omega}{\alpha}+k^2}(x+2\alpha k t) + C\right]\right)^{\frac{1}{m}} e^{i(-kx+\omega t+\theta_0)}.$$

13.5.1 Compute the partial derivatives of the proposed solution.

$$u_t = -8\alpha^3\left(-3\alpha(x+4\alpha^2 t)\right)^{-1/3}$$
$$u_{xxx} = -8\alpha^3\left(-3\alpha(x+4\alpha^2 t)\right)^{-7/3}$$

Hence we see that

$$-8\alpha^3\left(-3\alpha(x+4\alpha^2 t)\right)^{-1/3} = \left[\left(-3\alpha(x+4\alpha^2 t)\right)^{2/3}\right]^3 - 8\alpha^3\left(-3\alpha(x+4\alpha^2 t)\right)^{-7/3}$$
$$u_t = u^3 u_{xxx}.$$

13.5.3

(a) Setting $\xi = x - c\,t$ and substituting $U(\xi)$ into Eq. (13.53), produce

$$(c^2-1)\frac{d^2U}{d\xi^2} + \sin U = 0 \iff \frac{d^2U}{d\xi^2} = \frac{1}{1-c^2}\sin U.$$

Multiplying both sides of this equation by $2dU/d\xi$ and integrating both sides yield

$$\left(\frac{dU}{d\xi}\right)^2 = \frac{2}{1-c^2}(A - \cos U)$$

where A is an arbitrary constant of integration. This the ordinary differential equation for $U(\xi)$.

(b) Assume $U \to 0$ and $U' \to 0$ as $\xi \to \pm\infty$ and solve the ordinary differential equation in the previous part.

Under the assumptions that $U \to 0$ and $U' \to 0$ as $\xi \to \pm\infty$ it must be the case that $A = 1$ and thus

$$\left(\frac{dU}{d\xi}\right)^2 = \frac{2}{1-c^2}(1-\cos U) = \frac{\sin(U/2)}{1-c^2}$$

where we have used a half-angle formula. Solving for $dU/d\xi$,

$$\frac{1}{\sin(U/2)}\frac{dU}{d\xi} = \pm\frac{2}{\sqrt{1-c^2}}$$

and therefore

$$\pm\frac{2(\xi-\xi_0)}{\sqrt{1-c^2}} = \int \frac{1}{\sin(U/2)}\,dU = \int \frac{\sin(U/2)}{1-\cos^2(U/2)}\,dU.$$

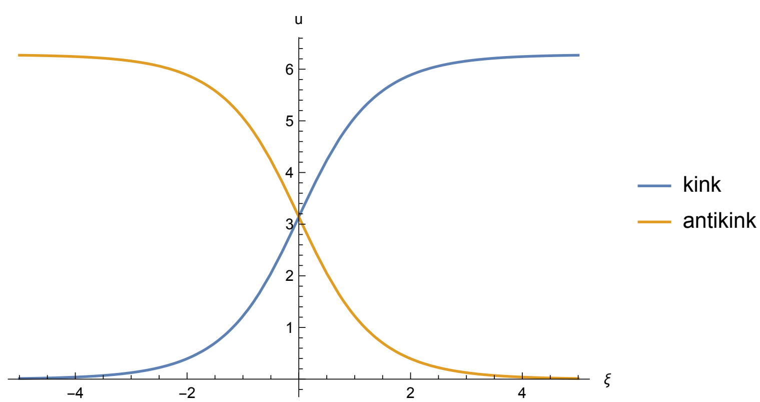
The integral can be evaluated by making the change of variable $z = \cos(U/2)$.

$$\begin{aligned}\pm\frac{2(\xi-\xi_0)}{\sqrt{1-c^2}} &= -2\int \frac{1}{1-z^2}\,dz \\ &= -2\int\left(\frac{1}{1+z} - \frac{1}{1-z}\right)dz = \ln\left|\frac{1-z}{1+z}\right| \\ &= \ln\frac{1-\cos(U/2)}{1+\cos(U/2)} = \ln\frac{\sin^2(U/4)}{\cos^2(U/4)} = 2\ln\tan\frac{U}{4}\end{aligned}$$

Finally solve for $U(\xi)$ to obtain

$$U(\xi) = 4\tan^{-1}\left(e^{\pm(\xi-\xi_0)/\sqrt{1-c^2}}\right).$$

The soliton solution above is called a "kink" if the "+" sign is used, and is called an "antikink" if the " − " is used. The graph below illustrates the kink and antikink when $c = 1/2$ and $\xi_0 = 0$.



13.5.5

(a) Suppose $\xi = x - ct$, then

$$\begin{aligned}-cU'(\xi) &= U''(\xi) + U(\xi)(U(\xi)-a)(1-U(\xi)) \\ -cU'(\xi) &= U''(\xi) - (U(\xi))^3 + (1+a)(U(\xi))^2 - aU(\xi) \\ U''(\xi) &= -cU'(\xi) + (U(\xi))^3 - (1+a)(U(\xi))^2 + aU(\xi)\end{aligned}$$

(b) Define a new function $\Psi(U) = U'(\xi)$, then $U''(\xi) = \Psi'(U)U'(\xi) = \Psi'(U)\Psi(U)$. Thus the previous second-order ordinary differential equation can be written as a first-order ordinary differential equation in Ψ.

$$\begin{aligned}
\Psi'(U)\Psi(U) &= -c\Psi(U) + (U(\xi))^3 - (1+a)(U(\xi))^2 + aU(\xi) \\
2\Psi'(U)\Psi(U) &= -2c\Psi(U) + 2(U(\xi))^3 - 2(1+a)(U(\xi))^2 + 2aU(\xi) \\
\left[(\Psi(U))^2\right]' &= -2c\Psi(U) + 2(U(\xi))^3 - 2(1+a)(U(\xi))^2 + 2aU(\xi)
\end{aligned}$$

(c) Assume that $\Psi(U) = \alpha U^2 + \beta U + \gamma$, then

$$\begin{aligned}
\left[(\alpha U^2 + \beta U + \gamma)^2\right]' &= -2c(\alpha U^2 + \beta U + \gamma) + 2U^3 - 2(1+a)U^2 + 2aU \\
4\alpha^2 U^3 + 6\alpha\beta U^2 + (2\beta^2 + 4\alpha\gamma)U + 2\beta\gamma &= 2U^3 + (-2 - 2a - 2\alpha c)U^2 + (2a - 2\beta c)U - 2\gamma c
\end{aligned}$$

Equating coefficients of U on both sides of the last equation produces the following system of equations.

$$\begin{aligned}
4\alpha^2 &= 2 \\
6\alpha\beta &= -2 - 2a - 2\alpha c \\
2\beta^2 + 4\alpha\gamma &= 2a - 2\beta c \\
2\beta\gamma &= -2\gamma c
\end{aligned}$$

(d) Assume without loss of generality that $\alpha = 1/\sqrt{2}$. Substituting this into the remaining three equations yields

$$\begin{aligned}
\frac{3}{\sqrt{2}}\beta &= -1 - a - \frac{c}{\sqrt{2}} \\
\beta^2 + \sqrt{2}\gamma &= a - \beta c \\
\beta\gamma &= -\gamma c.
\end{aligned}$$

Solve the first equation for c and substitute into the last equation. This produces the equation,

$$\gamma(2\beta + \sqrt{2}(1+a)) = 0.$$

There are two cases to consider.

Case $\gamma = 0$: the second equation can then be written as

$$\begin{aligned}
\beta^2 &= a - \beta c \\
\beta^2 + \beta c - a &= 0 \\
\beta^2 - 3\beta^2 - \sqrt{2}(1+a)\beta - a &= 0 \\
2\beta^2 + \sqrt{2}(1+a)\beta + a &= 0 \\
\beta &= \frac{-1}{\sqrt{2}} \text{ or } \beta = \frac{-a}{\sqrt{2}}.
\end{aligned}$$

If $\beta = -1/\sqrt{2}$ then $\Psi(U) = \frac{1}{\sqrt{2}}U^2 - \frac{1}{\sqrt{2}}U = U'(\xi) \neq 0$ for $0 < U < 1$. Solving the separable ordinary differential equation,

$$\begin{aligned}
\frac{dU}{d\xi} &= \frac{1}{\sqrt{2}}U(U-1) \\
\frac{1}{U(U-1)}\,dU &= \frac{1}{\sqrt{2}}\,d\xi \\
\int \frac{1}{U}\,dU + \int \frac{1}{1-U}\,dU &= \frac{\xi}{2\sqrt{2}} + C \\
\ln\frac{U}{1-U} &= \frac{\xi}{2\sqrt{2}} + C \\
U(\xi) &= \frac{\hat{C}e^{\frac{\xi}{\sqrt{2}}}}{1+\hat{C}e^{\frac{\xi}{\sqrt{2}}}} \\
u(x,t) &= \frac{\hat{C}e^{\frac{x-ct}{\sqrt{2}}}}{1+\hat{C}e^{\frac{x-ct}{\sqrt{2}}}}
\end{aligned}$$

where C and $\hat{C}$ are constants. On the other hand if $\beta = -a/\sqrt{2}$ then $\Psi(U) = \frac{1}{\sqrt{2}}U(U-a) = U'(\xi) = 0$ for $0 < U = a < 1$. This implies the change of variable $\Psi(U) = U'(\xi)$ may not be invertible. Hence this solution will be ignored.

Case $\gamma \neq 0$: then $\beta = -c = \frac{-1}{\sqrt{2}}(1+a)$ which in turn implies $\gamma = a/\sqrt{2}$. In this case

$$\Psi(U) = \frac{1}{\sqrt{2}}U^2 - \frac{1}{\sqrt{2}}(1+a)U + \frac{a}{\sqrt{2}} = \frac{1}{\sqrt{2}}(a-U)(1-U).$$

Again $\Psi(U) = 0$ when $0 < U = a < 1$ and this case can be ignored.

14.1.1

(a)

$$\begin{aligned}
f'(1) &= \frac{f(1.1)-f(1)}{0.1} - \frac{0.1}{2}f''(z) \\
f'(1) &\approx \frac{f(1.1)-f(1)}{0.1} = 1.38857
\end{aligned}$$

(b)

$$E \leq \frac{0.1}{2}\max_{1\leq z\leq 1.1}|f''(z)| = (0.05)(0.239134) = 0.0119567$$

14.1.3

(a)

$$\begin{aligned}
f''(1) &= \frac{f(1.1)-2f(1)+f(0.9)}{(0.1)^2} - \frac{(0.1)^2}{12}f^{(4)}(z) \\
f''(1) &\approx \frac{f(1.1)-2f(1)+f(0.9)}{0.01} = 0.238035
\end{aligned}$$

(b)

$$E \leq \frac{(0.1)^2}{12}\max_{0.9\leq z\leq 1.1}\left|f^{(4)}(z)\right| = \frac{0.01}{12}(1.78145) = 0.00148454$$

14.1.5 First express $f(x)$ as a Taylor polynomial of degree 2 centered at x_0.

$$f(x) = f(x_0) + f'(x_0)(x - x_0) + \frac{1}{2}f''(x_0)(x - x_0)^2 + O((x - x_0)^3)$$

This implies

$$\begin{aligned} f(x_0 + \alpha h) &= f(x_0) + \alpha h f'(x_0) + \frac{\alpha^2 h^2}{2} f''(x_0) + \alpha^3 O(h^3) \\ f(x_0 + 2h) &= f(x_0) + 2hf'(x_0) + +4h^2 f''(x_0) + 8O(h^3). \end{aligned}$$

Multiply the first equation by 4 and the second equation by α^2 and then subtract.

$$\begin{aligned} 4f(x_0 + \alpha h) - \alpha^2 f(x_0 + 2h) &= (4 - \alpha^2)f(x_0) + (4\alpha h - 2\alpha^2 h)f'(x_0) + (4\alpha^3 - 8\alpha^2)O(h^3) \\ 2\alpha h(2 - \alpha)f'(x_0) &= 4f(x_0 + \alpha h) - \alpha^2 f(x_0 + 2h) - (4 - \alpha^2)f(x_0) - (4\alpha^3 - 8\alpha^2)O(h^3) \\ f'(x_0) &= \frac{4}{2\alpha h(2 - \alpha)} f(x_0 + \alpha h) - \frac{\alpha^2}{2\alpha h(2 - \alpha)} f(x_0 + 2h) - \frac{4 - \alpha^2}{2\alpha h(2 - \alpha)} f(x_0) - \frac{4\alpha^3 - 8\alpha^2}{2\alpha h(2 - \alpha)} O(h^3) \\ &= \frac{1}{2h}\left[-\frac{2 + \alpha}{\alpha} f(x_0) + \frac{4}{\alpha(2 - \alpha)} f(x_0 + \alpha h) - \frac{\alpha}{2 - \alpha} f(x_0 + 2j) \right] + O(h^2) \end{aligned}$$

14.1.7

$$\begin{aligned} f(x - 2h) &= f(x) + f'(x)(-2h) + \frac{f''(x)}{2!}(-2h)^2 + \frac{f'''(x)}{3!}(-2h)^3 + \frac{f^{(4)}(x)}{4!}(-2h)^4 + \frac{f^{(5)}(x)}{5!}(-2h)^5 + \frac{f^{(6)}(z_{-2h})}{6!}(-2h)^6 \\ f(x - h) &= f(x) + f'(x)(-h) + \frac{f''(x)}{2!}(-h)^2 + \frac{f'''(x)}{3!}(-h)^3 + \frac{f^{(4)}(x)}{4!}(-h)^4 + \frac{f^{(5)}(x)}{5!}(-h)^5 + \frac{f^{(6)}(z_{-h})}{6!}(-h)^6 \\ f(x + h) &= f(x) + f'(x)h + \frac{f''(x)}{2!}h^2 + \frac{f'''(x)}{3!}h^3 + \frac{f^{(4)}(x)}{4!}h^4 + \frac{f^{(5)}(x)}{5!}h^5 + \frac{f^{(6)}(z_h)}{6!}h^6 \\ f(x + 2h) &= f(x) + f'(x)(2h) + \frac{f''(x)}{2!}(2h)^2 + \frac{f'''(x)}{3!}(2h)^3 + \frac{f^{(4)}(x)}{4!}(2h)^4 + \frac{f^{(5)}(x)}{5!}(2h)^5 + \frac{f^{(6)}(z_{2h})}{6!}(2h)^6 \end{aligned}$$

Form the following linear combination of the Taylor polynomials.

$$\begin{aligned} & f(x - 2h) - 4f(x - h) + 6f(x) - 4f(x + h) + f(x + 2h) \\ &= 8f(x) + 4h^2 f''(x) + \frac{4}{3}h^4 f^{(4)}(x) - 4f(x) + 4hf'(x) - 2h^2 f''(x) + \frac{2}{3}h^3 f'''(x) \\ &\quad - \frac{1}{6}h^4 f^{(4)}(x) + \frac{1}{30}h^5 f^{(5)}(x) - 4f(x) - 4hf'(x) - 2h^2 f''(x) - \frac{2}{3}h^3 f'''(x) \\ &\quad - \frac{1}{6}h^4 f^{(4)}(x) - \frac{1}{30}h^5 f^{(5)}(x) + \frac{4h^6}{45} f^{(6)}(z_{-2h}) - \frac{h^6}{180} f^{(6)}(z_{-h}) - \frac{h^6}{180} f^{(6)}(z_h) + \frac{4h^6}{45} f^{(6)}(z_{2h}) \\ &= h^4 f^{(4)}(x) + \frac{4h^6}{45} f^{(6)}(z_{-2h}) - \frac{h^6}{180} f^{(6)}(z_{-h}) - \frac{h^6}{180} f^{(6)}(z_h) + \frac{4h^6}{45} f^{(6)}(z_{2h}) \end{aligned}$$

If $f^{(6)}(u)$ is continuous on $(x - 2h, x + 2h)$ there exists $z \in (x - 2h, x + 2h)$ such that

$$\frac{4}{45} f^{(6)}(z_{-2h}) - \frac{1}{180} f^{(6)}(z_{-h}) - \frac{1}{180} f^{(6)}(z_h) + \frac{4}{45} f^{(6)}(z_{2h}) = \frac{1}{6} f^{(6)}(z).$$

Rearranging terms yields

$$f^{(4)}(x) = \frac{1}{h^4}\left(f(x - 2h) - 4f(x - h) + 6f(x) - 4f(x + h) + f(x + 2h) \right) - \frac{h^2}{6} f^{(6)}(z)$$

where $x - 2h < z < x + 2h$.

14.2.2

$$\begin{aligned} u_i^0 &= 1 + \sin\left(\frac{\pi(a + \frac{i(b-a)}{N} - a)}{b-a}\right) = 1 + \sin\left(\frac{\pi\, i}{N}\right) \\ u_0^j &= e^{-j\Delta t} \\ u_N^j &= \frac{1}{1 + j\Delta t} \end{aligned}$$

for $i = 0, 1, \ldots, N$ and $j \in \mathbb{N}$.
14.2.4

$$\begin{aligned} u_i^0 &= 1 + \frac{i}{N}\left(1 - \frac{i}{N}\right) \\ u_0^j &= \frac{N}{N+1} u_1^j + (N+1)\sin(j\Delta t) \\ u_N^j &= \frac{N}{N+1} u_{N-1}^j + (N+1)(1 - e^{-j\Delta t}) \end{aligned}$$

for $i = 0, 1, \ldots, N$ and $j \in \mathbb{N}$.
14.2.6

$$\begin{aligned} u_i^0 &= \cos\left(\frac{2\pi i}{N}\right) \\ u_{-1}^j &= \frac{N}{2} u_0^j + u_1^j \\ u_{N+1}^j &= u_{N-1}^j - \frac{N}{2} u_N^j \end{aligned}$$

for $i = 0, 1, \ldots, N$ and $j \in \mathbb{N}$.
14.3.2 Define $\xi = (x - a)/(b - a)$ and $\tau = \kappa t/(b - a)^2$, then

$$\begin{aligned} u_\xi &= u_x x_\xi = u_x(b - a) \\ u_{\xi\xi} &= u_{xx}(b - a)^2 \\ u_\tau &= u_t t_\tau = u_t \frac{(b-a)^2}{\kappa}. \end{aligned}$$

Therefore the partial differential equation in the new variables becomes

$$\frac{\kappa}{(b-a)^2} u_\tau = \frac{\kappa}{(b-a)^2} u_{\xi\xi} \iff u_\tau = u_{\xi\xi},$$

where $x \in [a, b]$ implies $\xi \in [0, 1]$.
14.3.4

$$\begin{aligned} \frac{u_i^{j+1} - u_i^j}{k} &= \frac{u_{i-1}^j - 2u_i^j + u_{i+1}^j}{h^2} + 2 \cdot \frac{u_{i+1}^j - u_{i-1}^j}{2h} \\ u_i^{j+1} &= \frac{k(1-h)u_{i-1}^j + (h^2 - 2k)u_i^j + k(1+h)u_{i+1}^j}{h^2} \end{aligned}$$

for $i = 1, 2, \ldots, N-1$ and $j = 0, 1, \ldots$. At $x = 0$ or equivalently $i = 0$,

$$\frac{u_1^j - u_{-1}^j}{2h} = 0 \iff u_{-1}^j = u_1^j,$$

which implies

$$u_0^{j+1} = \frac{k(1-h)u_{-1}^j + (h^2-2k)u_0^j + k(1+h)u_1^j}{h^2} = \frac{(h^2-2k)u_0^j + 2ku_1^j}{h^2}.$$

At $x = 1$ or equivalently $i = N$,

$$\frac{u_{N+1}^j - u_{N-1}^j}{2h} = 0 \iff u_{N+1}^j = u_{N-1}^j.$$

which implies

$$u_N^{j+1} = \frac{k(1-h)u_{N-1}^j + (h^2-2k)u_N^j + k(1+h)u_{N+1}^j}{h^2} = \frac{2ku_{N-1}^j + (h^2-2k)u_N^j}{h^2}.$$

The table below lists the approximations to the solution for ten time steps.

t	$u(0,t)$	$u(0.1,t)$	$u(0.2,t)$	$u(0.3,t)$	$u(0.4,t)$	$u(0.5,t)$	$u(0.6,t)$	$u(0.7,t)$	$u(0.8,t)$	$u(0.9,t)$	$u(1.0,t)$
0.000	1.0000	1.0000	1.0000	1.0000	1.0000	2.0000	2.0000	2.0000	2.0000	2.0000	2.0000
0.001	1.0000	1.0000	1.0000	1.0000	1.1100	1.9100	2.0000	2.0000	2.0000	2.0000	2.0000
0.002	1.0000	1.0000	1.0000	1.0121	1.1881	1.8479	1.9919	2.0000	2.0000	2.0000	2.0000
0.003	1.0000	1.0000	1.0013	1.0304	1.2448	1.8044	1.9798	1.9993	2.0000	2.0000	2.0000
0.004	1.0000	1.0001	1.0044	1.0513	1.2871	1.7733	1.9662	1.9976	1.9999	2.0000	2.0000
0.005	1.0000	1.0006	1.0092	1.0731	1.3194	1.7508	1.9523	1.9950	1.9997	2.0000	2.0000
0.006	1.0001	1.0015	1.0154	1.0944	1.3446	1.7341	1.9388	1.9917	1.9993	2.0000	2.0000
0.007	1.0004	1.0029	1.0229	1.1148	1.3650	1.7216	1.9262	1.9878	1.9987	1.9999	2.0000
0.008	1.0009	1.0049	1.0312	1.1341	1.3817	1.7120	1.9146	1.9834	1.9979	1.9998	2.0000
0.009	1.0017	1.0074	1.0401	1.1520	1.3957	1.7045	1.9039	1.9788	1.9968	1.9997	1.9999
0.010	1.0028	1.0105	1.0495	1.1688	1.4078	1.6987	1.8942	1.9741	1.9955	1.9994	1.9999

14.3.6 As in Exercise 14.3.5,

$$-ru_{i-1}^{j+1} + 2(1+r-k)u_i^{j+1} - ru_{i+1}^{j+1} = ru_{i-1}^j + 2(1-r+k)u_i^j + ru_{i+1}^j$$

for $i = 1, 2, \ldots, N-1$. The homogeneous Neumann boundary conditions are handled by introducing the fictitious points at $i = -1$ and $i = N+1$. At $x = 0$ or equivalently $i = 0$,

$$\frac{u_1^j - u_{-1}^j}{2h} = 0 \iff u_{-1}^j = u_1^j,$$

which implies

$$-ru_{-1}^{j+1} + 2(1+r-k)u_0^{j+1} - ru_1^{j+1} = ru_{-1}^j + 2(1-r+k)u_0^j + ru_1^j$$
$$(1+r-k)u_0^{j+1} - ru_1^{j+1} = (1-r+k)u_0^j + ru_1^j$$

At $x = 1$ or equivalently $i = N$,

$$\frac{u_{N+1}^j - u_{N-1}^j}{2h} = 0 \iff u_{N+1}^j = u_{N-1}^j.$$

which implies

$$-ru_{N-1}^{j+1} + 2(1+r-k)u_N^{j+1} - ru_{N+1}^{j+1} = ru_{N-1}^j + 2(1-r+k)u_N^j + ru_{N+1}^j$$
$$-ru_{N-1}^{j+1} + (1+r-k)u_N^{j+1} = ru_{N-1}^j + (1-r+k)u_N^j$$

The table below lists the approximations to the solution for ten time steps.

t	$u(0,t)$	$u(0.1,t)$	$u(0.2,t)$	$u(0.3,t)$	$u(0.4,t)$	$u(0.5,t)$	$u(0.6,t)$	$u(0.7,t)$	$u(0.8,t)$	$u(0.9,t)$	$u(1.0,t)$
0.00	−1.0000	−1.0000	−1.0000	−1.0000	−1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000
0.01	−1.0112	−1.0022	−0.9576	−0.7889	−0.1624	0.1625	0.7890	0.9579	1.0033	1.0153	1.0178
0.02	−0.9903	−0.9583	−0.8351	−0.5549	−0.2378	0.2381	0.5557	0.8374	0.9657	1.0129	1.0244
0.03	−0.9289	−0.8804	−0.7305	−0.4971	−0.1641	0.1655	0.5003	0.7391	0.9031	0.9857	1.0095
0.04	−0.8493	−0.8004	−0.6593	−0.4314	−0.1530	0.1574	0.4401	0.6793	0.8454	0.9427	0.9741
0.05	−0.7712	−0.7257	−0.5927	−0.3882	−0.1315	0.1419	0.4059	0.6280	0.7944	0.8947	0.9282
0.06	−0.6983	−0.6562	−0.5350	−0.3475	−0.1159	0.1353	0.3769	0.5870	0.7466	0.8458	0.8793
0.07	−0.6312	−0.5929	−0.4821	−0.3114	−0.0994	0.1304	0.3545	0.5509	0.7027	0.7980	0.8304
0.08	−0.5699	−0.5349	−0.4338	−0.2776	−0.0836	0.1281	0.3355	0.5190	0.6619	0.7523	0.7832
0.09	−0.5137	−0.4818	−0.3892	−0.2463	−0.0679	0.1271	0.3194	0.4903	0.6241	0.7092	0.7383
0.10	−0.4622	−0.4329	−0.3481	−0.2168	−0.0528	0.1273	0.3054	0.4643	0.5892	0.6689	0.6962

14.3.8 As in Exercise 14.3.5,

$$-ru_{i-1}^{j+1} + 2(1+r-k)u_i^{j+1} - ru_{i+1}^{j+1} = ru_{i-1}^j + 2(1-r+k)u_i^j + ru_{i+1}^j$$

for $i = 1, 2, \ldots, N$. The homogeneous Neumann boundary condition is handled by introducing the fictitious point at $i = -1$. At $x = 0$ or equivalently $i = 0$,

$$\frac{u_1^j - u_{-1}^j}{2h} = 0 \iff u_{-1}^j = u_1^j.$$

which implies

$$\begin{aligned} -ru_{-1}^{j+1} + 2(1+r-k)u_0^{j+1} - ru_1^{j+1} &= ru_{-1}^j + 2(1-r+k)u_0^j + ru_1^j \\ (1+r-k)u_0^{j+1} - ru_1^{j+1} &= (1-r+k)u_0^j + ru_1^j \end{aligned}$$

The table below lists the approximations to the solution for ten time steps.

t	$u(0,t)$	$u(0.1,t)$	$u(0.2,t)$	$u(0.3,t)$	$u(0.4,t)$	$u(0.5,t)$	$u(0.6,t)$	$u(0.7,t)$	$u(0.8,t)$	$u(0.9,t)$	$u(1.0,t)$
0.00	2.0000	1.9511	1.8090	1.5878	1.3090	1.0000	0.6910	0.4122	0.1910	0.0489	0.0000
0.01	1.9452	1.9000	1.7686	1.5639	1.3060	1.0201	0.7338	0.4746	0.2649	0.1148	0.0000
0.02	1.8965	1.8546	1.7330	1.5437	1.3049	1.0398	0.7731	0.5277	0.3190	0.1473	0.0000
0.03	1.8533	1.8146	1.7020	1.5266	1.3050	1.0579	0.8066	0.5695	0.3572	0.1706	0.0000
0.04	1.8153	1.7794	1.6751	1.5121	1.3056	1.0734	0.8338	0.6018	0.3861	0.1876	0.0000
0.05	1.7820	1.7486	1.6516	1.4996	1.3058	1.0860	0.8555	0.6271	0.4082	0.2005	0.0000
0.06	1.7528	1.7217	1.6310	1.4884	1.3055	1.0957	0.8725	0.6468	0.4254	0.2104	0.0000
0.07	1.7273	1.6981	1.6128	1.4783	1.3044	1.1030	0.8857	0.6621	0.4387	0.2182	0.0000
0.08	1.7049	1.6773	1.5967	1.4689	1.3026	1.1082	0.8957	0.6740	0.4490	0.2241	0.0000
0.09	1.6850	1.6589	1.5822	1.4601	1.3002	1.1115	0.9032	0.6830	0.4570	0.2288	0.0000
0.10	1.6674	1.6424	1.5690	1.4517	1.2971	1.1134	0.9085	0.6898	0.4631	0.2323	0.0000

14.3.10

$$\begin{aligned} \frac{u_i^{j+1} - u_i^j}{k} &= \frac{e^{-ih}}{2h^2}(u_{i-1}^j - 2u_i^j + u_{i+1}^j) \\ &\quad + \frac{e^{-ih}}{2h^2}(u_{i-1}^{j+1} - 2u_i^{j+1} + u_{i+1}^{j+1}) \end{aligned}$$

$$-re^{-ih}u_{i-1}^{j+1} + 2(1+re^{-ih})u_i^{j+1} - re^{-ih}u_{i+1}^{j+1} = re^{-ih}u_{i-1}^j + 2(1-re^{-ih})u_i^j + re^{-ih}u_{i+1}^j$$

where $r = k/h^2$ for $i = 1, 2, \ldots, N-1$. At $x = 0$ or equivalently $i = 0$,

$$\frac{u_1^j - u_{-1}^j}{2h} = 0 \iff u_{-1}^j = u_1^j,$$

which implies

$$-ru_{-1}^{j+1} + 2(1+r)u_0^{j+1} - ru_1^{j+1} = ru_{-1}^{j} + 2(1-r)u_0^{j} + ru_1^{j}$$
$$(1+r)u_0^{j+1} - ru_1^{j+1} = (1-r)u_0^{j} + ru_1^{j}.$$

At $x = 1$ or equivalently $i = N$,

$$\frac{u_{N+1}^j - u_{N-1}^j}{2h} = 0 \iff u_{N+1}^j = u_{N-1}^j,$$

which implies

$$-re^{-Nh}u_{N-1}^{j+1} + 2(1+re^{-Nh})u_N^{j+1} - re^{-Nh}u_{N+1}^{j+1} = re^{-Nh}u_{N-1}^{j} + 2(1-re^{-Nh})u_N^{j} + re^{-Nh}u_{N+1}^{j}$$
$$-re^{-Nh}u_{N-1}^{j+1} + (1+re^{-Nh})u_N^{j+1} = re^{-Nh}u_{N-1}^{j} + (1-re^{-Nh})u_N^{j}.$$

The table below lists the approximations to the solution for ten time steps.

t	$u(0.0,t)$	$u(0.1,t)$	$u(0.2,t)$	$u(0.3,t)$	$u(0.4,t)$	$u(0.5,t)$	$u(0.6,t)$	$u(0.7,t)$	$u(0.8,t)$	$u(0.9,t)$	$u(1.0,t)$
0.00	0.0000	0.8100	2.5600	4.4100	5.7600	6.2500	5.7600	4.4100	2.5600	0.8100	0.0000
0.01	1.1930	1.5760	2.7122	4.1203	5.2463	5.6994	5.3168	4.1789	2.6103	1.1658	0.5314
0.02	1.8595	2.1430	2.9266	3.9619	4.8516	5.2368	4.9322	3.9813	2.6651	1.4690	0.9541
0.03	2.3407	2.5383	3.1095	3.8760	4.5561	4.8591	4.6067	3.8145	2.7159	1.7232	1.2994
0.04	2.6847	2.8311	3.2512	3.8220	4.3332	4.5565	4.3372	3.6754	2.7600	1.9352	1.5844
0.05	2.9391	3.0471	3.3589	3.7830	4.1615	4.3155	4.1174	3.5611	2.7978	2.1118	1.8206
0.06	3.1273	3.2075	3.4385	3.7518	4.0266	4.1236	3.9401	3.4683	2.8302	2.2589	2.0168
0.07	3.2667	3.3260	3.4962	3.7248	3.9191	3.9705	3.7977	3.3942	2.8585	2.3817	2.1801
0.08	3.3693	3.4126	3.5365	3.7003	3.8323	3.8480	3.6839	3.3356	2.8836	2.4845	2.3162
0.09	3.4438	3.4749	3.5633	3.6773	3.7613	3.7496	3.5931	3.2899	2.9063	2.5710	2.4300
0.10	3.4968	3.5186	3.5798	3.6556	3.7026	3.6702	3.5208	3.2547	2.9271	2.6440	2.5254

14.4.2 The table below lists the approximations to the solution for ten time steps.

t	$u(0,t)$	$u(0.1,t)$	$u(0.2,t)$	$u(0.3,t)$	$u(0.4,t)$	$u(0.5,t)$	$u(0.6,t)$	$u(0.7,t)$	$u(0.8,t)$	$u(0.9,t)$	$u(1.0,t)$
0.00	0.0000	0.3600	0.6400	0.8400	0.9600	1.0000	0.9600	0.8400	0.6400	0.3600	0.0000
0.05	0.0000	0.3905	0.6776	0.8455	0.9206	0.9400	0.9206	0.8455	0.6776	0.3905	0.0000
0.10	0.0000	0.3951	0.6853	0.8277	0.8673	0.8703	0.8673	0.8277	0.6853	0.3951	0.0000
0.15	0.0000	0.3735	0.6561	0.7843	0.8048	0.7991	0.8048	0.7843	0.6561	0.3735	0.0000
0.20	0.0000	0.3292	0.5883	0.7139	0.7358	0.7307	0.7358	0.7139	0.5883	0.3292	0.0000
0.25	0.0000	0.2674	0.4871	0.6176	0.6600	0.6649	0.6600	0.6176	0.4871	0.2674	0.0000
0.30	0.0000	0.1936	0.3636	0.4993	0.5749	0.5966	0.5749	0.4993	0.3636	0.1936	0.0000
0.35	0.0000	0.1140	0.2315	0.3659	0.4763	0.5175	0.4763	0.3659	0.2315	0.1140	0.0000
0.40	0.0000	0.0352	0.1037	0.2266	0.3604	0.4177	0.3604	0.2266	0.1037	0.0352	0.0000
0.45	0.0000	−0.0352	−0.0105	0.0900	0.2254	0.2893	0.2254	0.0900	−0.0105	−0.0352	0.0000
0.50	0.0000	−0.0907	−0.1058	−0.0379	0.0725	0.1289	0.0725	−0.0379	−0.1058	−0.0907	0.0000

14.4.4 Using $h = 0.1$ and $k = 0.05$ the first ten time steps of the numerical solution are given below.

t	$u(0,t)$	$u(0.1,t)$	$u(0.2,t)$	$u(0.3,t)$	$u(0.4,t)$	$u(0.5,t)$	$u(0.6,t)$	$u(0.7,t)$	$u(0.8,t)$	$u(0.9,t)$	$u(1.0,t)$
0.00	0.0000	0.3090	0.5878	0.8090	0.9511	1.0000	0.9511	0.8090	0.5878	0.3090	0.0000
0.05	0.0000	0.3052	0.5806	0.7991	0.9394	0.9878	0.9394	0.7991	0.5806	0.3052	0.0000
0.10	0.0000	0.2941	0.5594	0.7699	0.9051	0.9516	0.9051	0.7699	0.5594	0.2941	0.0000
0.15	0.0000	0.2758	0.5246	0.7221	0.8488	0.8925	0.8488	0.7221	0.5246	0.2758	0.0000
0.20	0.0000	0.2509	0.4772	0.6568	0.7721	0.8118	0.7721	0.6568	0.4772	0.2509	0.0000
0.25	0.0000	0.2199	0.4182	0.5756	0.6767	0.7115	0.6767	0.5756	0.4182	0.2199	0.0000
0.30	0.0000	0.1835	0.3491	0.4805	0.5649	0.5940	0.5649	0.4805	0.3491	0.1835	0.0000
0.35	0.0000	0.1428	0.2716	0.3738	0.4395	0.4621	0.4395	0.3738	0.2716	0.1428	0.0000
0.40	0.0000	0.0986	0.1875	0.2581	0.3034	0.3190	0.3034	0.2581	0.1875	0.0986	0.0000
0.45	0.0000	0.0520	0.0989	0.1361	0.1600	0.1682	0.1600	0.1361	0.0989	0.0520	0.0000
0.50	0.0000	0.0041	0.0079	0.0108	0.0128	0.0134	0.0128	0.0108	0.0079	0.0041	0.0000

14.4.6 Using $h = 0.1$ and $k = 0.05$ the first ten time steps of the numerical solution are given below.

t	$u(0,t)$	$u(0.1,t)$	$u(0.2,t)$	$u(0.3,t)$	$u(0.4,t)$	$u(0.5,t)$	$u(0.6,t)$	$u(0.7,t)$	$u(0.8,t)$	$u(0.9,t)$	$u(1.0,t)$
0.00	0.0000	0.0814	0.1310	0.1556	0.1609	0.1516	0.1317	0.1043	0.0719	0.0366	0.0000
0.05	0.0000	0.0929	0.1433	0.1826	0.1995	0.1979	0.1808	0.1512	0.1120	0.0519	0.0000
0.10	0.0000	0.0955	0.1520	0.2042	0.2335	0.2402	0.2267	0.1955	0.1477	0.0684	0.0000
0.15	0.0000	0.0901	0.1585	0.2206	0.2619	0.2775	0.2681	0.2353	0.1769	0.0864	0.0000
0.20	0.0000	0.0801	0.1628	0.2319	0.2841	0.3086	0.3036	0.2687	0.1990	0.1046	0.0000
0.25	0.0000	0.0703	0.1639	0.2389	0.2996	0.3324	0.3315	0.2938	0.2151	0.1201	0.0000
0.30	0.0000	0.0648	0.1614	0.2419	0.3082	0.3478	0.3503	0.3093	0.2264	0.1301	0.0000
0.35	0.0000	0.0654	0.1559	0.2412	0.3102	0.3541	0.3585	0.3149	0.2333	0.1329	0.0000
0.40	0.0000	0.0708	0.1494	0.2364	0.3058	0.3507	0.3552	0.3112	0.2348	0.1287	0.0000
0.45	0.0000	0.0780	0.1444	0.2277	0.2952	0.3375	0.3403	0.2992	0.2291	0.1192	0.0000
0.50	0.0000	0.0831	0.1422	0.2154	0.2781	0.3146	0.3150	0.2794	0.2143	0.1070	0.0000

14.4.8 Using $h = 0.1$ and $k = 0.05$ the first ten time steps of the numerical solution are given below.

t	$u(0,t)$	$u(0.1,t)$	$u(0.2,t)$	$u(0.3,t)$	$u(0.4,t)$	$u(0.5,t)$	$u(0.6,t)$	$u(0.7,t)$	$u(0.8,t)$	$u(0.9,t)$	$u(1.0,t)$
0.00	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000
0.05	0.0000	0.0294	0.0476	0.0476	0.0294	0.0000	−0.0294	−0.0476	−0.0476	−0.0294	0.0000
0.10	0.0000	0.0560	0.0907	0.0907	0.0560	0.0000	−0.0560	−0.0907	−0.0907	−0.0560	0.0000
0.15	0.0000	0.0775	0.1253	0.1253	0.0775	0.0000	−0.0775	−0.1253	−0.1253	−0.0775	0.0000
0.20	0.0000	0.0917	0.1483	0.1483	0.0917	0.0000	−0.0917	−0.1483	−0.1483	−0.0917	0.0000
0.25	0.0000	0.0973	0.1574	0.1574	0.0973	0.0000	−0.0973	−0.1574	−0.1574	−0.0973	0.0000
0.30	0.0000	0.0939	0.1519	0.1519	0.0939	0.0000	−0.0939	−0.1519	−0.1519	−0.0939	0.0000
0.35	0.0000	0.0817	0.1322	0.1322	0.0817	0.0000	−0.0817	−0.1322	−0.1322	−0.0817	0.0000
0.40	0.0000	0.0619	0.1002	0.1002	0.0619	0.0000	−0.0619	−0.1002	−0.1002	−0.0619	0.0000
0.45	0.0000	0.0363	0.0588	0.0588	0.0363	0.0000	−0.0363	−0.0588	−0.0588	−0.0363	0.0000
0.50	0.0000	0.0074	0.0119	0.0119	0.0074	0.0000	−0.0074	−0.0119	−0.0119	−0.0074	0.0000

14.4.10 Using $h = 0.1$ and $k = 0.05$ the first ten time steps of the numerical solution are given below.

t	$u(0,t)$	$u(0.1,t)$	$u(0.2,t)$	$u(0.3,t)$	$u(0.4,t)$	$u(0.5,t)$	$u(0.6,t)$	$u(0.7,t)$	$u(0.8,t)$	$u(0.9,t)$	$u(1.0,t)$
0.00	0.0000	0.0000	0.0000	0.0000	1.0000	1.0000	1.0000	0.0000	0.0000	0.0000	0.0000
0.05	0.0000	0.0003	0.0059	0.1056	0.8441	0.9382	0.8441	0.1056	0.0059	0.0003	0.0000
0.10	0.0000	0.0034	0.0404	0.3470	0.5494	0.8192	0.5494	0.3470	0.0404	0.0034	0.0000
0.15	0.0000	0.0172	0.1343	0.5689	0.2619	0.5811	0.2619	0.5689	0.1343	0.0172	0.0000
0.20	0.0000	0.0568	0.2916	0.6365	0.0979	0.2148	0.0979	0.6365	0.2916	0.0568	0.0000
0.25	0.0000	0.1368	0.4652	0.5160	0.0670	−0.1886	0.0670	0.5160	0.4652	0.1368	0.0000
0.30	0.0000	0.2558	0.5730	0.2832	0.0795	−0.4736	0.0795	0.2832	0.5730	0.2558	0.0000
0.35	0.0000	0.3809	0.5467	0.0586	0.0280	−0.5199	0.0280	0.0586	0.5467	0.3809	0.0000
0.40	0.0000	0.4503	0.3769	−0.0777	−0.1206	−0.3334	−0.1206	−0.0777	0.3769	0.4503	0.0000
0.45	0.0000	0.3985	0.1204	−0.1290	−0.2972	−0.0555	−0.2972	−0.1290	0.1204	0.3985	0.0000
0.50	0.0000	0.1963	−0.1369	−0.1573	−0.3868	0.1247	−0.3868	−0.1573	−0.1369	0.1963	0.0000

14.5.2 The system of equations can be written in matrix/vector form as follows.

$$\begin{bmatrix} 4 & -1 & 0 & -1 & 0 & 0 \\ -1 & 4 & -1 & 0 & -1 & 0 \\ 0 & -1 & 4 & 0 & 0 & -1 \\ -1 & 0 & 0 & 4 & -1 & 0 \\ 0 & -1 & 0 & -1 & 4 & -1 \\ 0 & 0 & -1 & 0 & -1 & 4 \end{bmatrix} \begin{bmatrix} u_1^1 \\ u_2^1 \\ u_3^1 \\ u_1^2 \\ u_2^2 \\ u_3^2 \end{bmatrix} = \begin{bmatrix} -1 \\ -e \\ 4-e^2 \\ 3-e^{-1} \\ 5 \\ 17-e \end{bmatrix}$$

Express the system of equations as a 6×7 matrix where the additional 7th column is the vector of constants from the right-hand side of the equations.

$$\left[\begin{array}{rrrrrr|r} 4 & -1 & 0 & -1 & 0 & 0 & -1 \\ -1 & 4 & -1 & 0 & -1 & 0 & -e \\ 0 & -1 & 4 & 0 & 0 & -1 & 4-e^2 \\ -1 & 0 & 0 & 4 & -1 & 0 & 3-e^{-1} \\ 0 & -1 & 0 & -1 & 4 & -1 & 5 \\ 0 & 0 & -1 & 0 & -1 & 4 & 17-e \end{array}\right]$$

Use elementary row operations to convert this matrix to reduced, row-echelon form. The numerical approximation to the solution is

$$\begin{bmatrix} u_1^1 \\ u_2^1 \\ u_3^1 \\ u_1^2 \\ u_2^2 \\ u_3^2 \end{bmatrix} = \begin{bmatrix} 0.1096 \\ 0.0815 \\ 0.2491 \\ 1.3568 \\ 2.6856 \\ 4.3041 \end{bmatrix}$$

14.5.4 Proceeding as general development of the linear system $A\,\mathbf{u} = \mathbf{b}$, if $h = 1/4$ then $N = 4$ and

$$\mathbf{u}^T = \begin{bmatrix} u_1^1 & u_2^1 & u_3^1 & u_1^2 & u_2^2 & u_3^2 & u_1^3 & u_2^3 & u_3^3 \end{bmatrix}.$$

Matrix $\hat{A}$ has the form

$$\hat{A} = \begin{bmatrix} 4 & -1 & 0 \\ -1 & 4 & -1 \\ 0 & -1 & 4 \end{bmatrix}$$

so that

$$A = \begin{bmatrix} 4 & -1 & 0 & -1 & 0 & 0 & 0 & 0 & 0 \\ -1 & 4 & -1 & 0 & -1 & 0 & 0 & 0 & 0 \\ 0 & -1 & 4 & -1 & 0 & -1 & 0 & 0 & 0 \\ -1 & 0 & 0 & 4 & -1 & 0 & -1 & 0 & 0 \\ 0 & -1 & 0 & -1 & 4 & -1 & 0 & -1 & 0 \\ 0 & 0 & -1 & 0 & -1 & 4 & 0 & 0 & -1 \\ 0 & 0 & 0 & -1 & 0 & 0 & 4 & -1 & 0 \\ 0 & 0 & 0 & 0 & -1 & 0 & -1 & 4 & -1 \\ 0 & 0 & 0 & 0 & 0 & -1 & 0 & -1 & 4 \end{bmatrix}.$$

The vector $\mathbf{b}$ is found as

$$\mathbf{b} = \frac{-1}{16} \begin{bmatrix} -2\pi^2 \sin\left(\frac{\pi}{4}\right) \sin\left(\frac{\pi}{4}\right) \\ -2\pi^2 \sin\left(\frac{\pi}{2}\right) \sin\left(\frac{\pi}{4}\right) \\ -2\pi^2 \sin\left(\frac{3\pi}{4}\right) \sin\left(\frac{\pi}{4}\right) \\ -2\pi^2 \sin\left(\frac{\pi}{4}\right) \sin\left(\frac{\pi}{2}\right) \\ -2\pi^2 \sin\left(\frac{\pi}{2}\right) \sin\left(\frac{\pi}{2}\right) \\ -2\pi^2 \sin\left(\frac{3\pi}{4}\right) \sin\left(\frac{\pi}{2}\right) \\ -2\pi^2 \sin\left(\frac{\pi}{4}\right) \sin\left(\frac{3\pi}{4}\right) \\ -2\pi^2 \sin\left(\frac{\pi}{2}\right) \sin\left(\frac{3\pi}{4}\right) \\ -2\pi^2 \sin\left(\frac{3\pi}{4}\right) \sin\left(\frac{3\pi}{4}\right) \end{bmatrix} = \frac{\pi^2}{8} \begin{bmatrix} 1/2 \\ 1/\sqrt{2} \\ 1/2 \\ 1/\sqrt{2} \\ 1 \\ 1/\sqrt{2} \\ 1/2 \\ 1/\sqrt{2} \\ 1/2 \end{bmatrix} \approx \begin{bmatrix} 0.61685 \\ 0.872358 \\ 0.61685 \\ 0.872358 \\ 1.2337 \\ 0.872358 \\ 0.61685 \\ 0.872358 \\ 0.61685 \end{bmatrix}$$

14.5.6 The linear system can be expressed as $A\,\mathbf{u} = \mathbf{b}$ where matrix A is as shown in Eq. (14.29). The vector $\mathbf{b} = \mathbf{b}_1 + \mathbf{b}_2 + \mathbf{b}_3$ where

$$\mathbf{b}_1 = \left(f_1^1, \ldots, f_{N-1}^1, f_1^2, \ldots, f_{N-1}^2, \ldots, f_1^{M-1}, \ldots, f_{N-1}^{M-1}\right)^T$$

$$\mathbf{b}_2 = \frac{h^2}{12} \left((\Delta f)_1^1, \ldots, (\Delta f)_{N-1}^1, (\Delta f)_1^2, \ldots, (\Delta f)_{N-1}^2, \ldots, (\Delta f)_1^{M-1}, \ldots, (\Delta f)_{N-1}^{M-1}\right)^T$$

and $\mathbf{b}_3 = \mathbf{g}_4 + \mathbf{g}$ where $\mathbf{g}_4$ and $\mathbf{g}$ may be found in Eqs. (14.30) and (14.31) respectively.
14.6.2 The iterates of the Gauss-Seidel method are shown in the table below.

k	$u_1^{(k)}$	$u_2^{(k)}$	$u_3^{(k)}$	$u_4^{(k)}$	$u_5^{(k)}$
0	0.0000	0.0000	0.0000	0.0000	0.0000
1	1.2500	−2.0833	0.9167	1.2708	2.2031
2	0.9909	−1.2678	1.5520	1.5688	1.8797
3	0.7090	−0.8628	1.5786	1.6062	1.6745
4	0.6524	−0.8225	1.5597	1.5974	1.6518
5	0.6528	−0.8319	1.5551	1.5940	1.6562
6	0.6551	−0.8353	1.5551	1.5937	1.6580
7	0.6556	−0.8356	1.5552	1.5938	1.6582
8	0.6556	−0.8355	1.5553	1.5938	1.6581

14.6.4 The first five iterates of the Jacobi method are shown in the table below.

k	$u_1^{(k)}$	$u_2^{(k)}$	$u_3^{(k)}$
0	0.0000	0.0000	0.0000
1	1.0000	−5.0000	0.5000
2	16.5000	−2.0000	3.0000
3	10.0000	44.5000	1.5000
4	−131.0000	25.0000	−21.7500
5	−95.7500	−398.0000	−12.0000

14.6.6 The first five iterates of the Gauss-Seidel method are shown in the table below.

k	$u_1^{(k)}$	$u_2^{(k)}$	$u_3^{(k)}$
0	0.0000	0.0000	0.0000
1	1.0000	−2.0000	1.5000
2	8.5000	20.5000	−9.7500
3	−70.2500	−215.7500	108.3750
4	756.6250	2264.8750	−1131.9375
5	−7925.5625	−23781.6875	11891.3438

14.6.8 The iterates of the SOR method are shown in the table below.

k	$u_1^{(k)}$	$u_2^{(k)}$	$u_3^{(k)}$	$u_4^{(k)}$	$u_5^{(k)}$	$u_5^{(k)}$
0	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000
1	1.5000	0.4500	1.9350	0.0000	1.8000	−0.0600
2	1.0650	0.8100	1.6560	0.5400	1.5840	−0.1128
3	1.0440	0.6480	1.6632	0.3672	1.5595	−0.1096
4	1.0968	0.6984	1.6769	0.3944	1.5735	−0.1060
5	1.0711	0.6847	1.6700	0.3932	1.5714	−0.1074
6	1.0804	0.6882	1.6724	0.3928	1.5713	−0.1071
7	1.0775	0.6873	1.6717	0.3928	1.5714	−0.1071
8	1.0783	0.6875	1.6719	0.3929	1.5714	−0.1071

14.6.10 In this exercise choose $N = M = 10$ which implies $h = 1/10$. The homogeneous Neumann boundary condition along the edge where $x = 1$ implies that $u_{N+1}^j = u_{N-1}^j$ for $j = 1, 2, \ldots, M-1$. Likewise homogeneous Neumann boundary condition along the edge where $y = 1$ implies that $u_i^{M+1} = u_i^{M-1}$ for $i = 1, 2, \ldots, N-1$. With $\epsilon = 10^{-6}$ the Gauss-Seidel method requires 400 iterations from an initial approximation of $\mathbf{u}^{(0)} = \mathbf{0}$ to converge to the solution in the table below.

	0	1	2	3	4	5	6	7	8	9	10
0	2.7183	2.4596	2.2255	2.0138	1.8221	1.6487	1.4918	1.3499	1.2214	1.1052	1.0000
1	2.4596	2.2851	2.1105	1.9445	1.7899	1.6482	1.5198	1.4056	1.3077	1.2312	1.1907
2	2.2255	2.1105	1.9871	1.8636	1.7450	1.6342	1.5335	1.4452	1.3726	1.3213	1.3003
3	2.0138	1.9445	1.8636	1.7780	1.6923	1.6102	1.5348	1.4690	1.4163	1.3810	1.3680
4	1.8221	1.7899	1.7450	1.6923	1.6359	1.5796	1.5265	1.4798	1.4426	1.4183	1.4097
5	1.6487	1.6482	1.6342	1.6102	1.5796	1.5457	1.5117	1.4809	1.4561	1.4399	1.4342
6	1.4918	1.5198	1.5335	1.5348	1.5265	1.5117	1.4939	1.4761	1.4611	1.4510	1.4474
7	1.3499	1.4056	1.4452	1.4690	1.4798	1.4809	1.4761	1.4685	1.4610	1.4555	1.4536
8	1.2214	1.3077	1.3726	1.4163	1.4426	1.4561	1.4611	1.4610	1.4588	1.4566	1.4557
9	1.1052	1.2312	1.3213	1.3810	1.4183	1.4399	1.4510	1.4555	1.4566	1.4564	1.4562
10	1.0000	1.1907	1.3003	1.3680	1.4097	1.4342	1.4474	1.4536	1.4557	1.4562	1.4562

14.7.2

$$\begin{aligned} A\,\mathbf{v} &= \lambda\,\mathbf{v} \\ A^{-1}A\,\mathbf{v} &= \lambda\,A^{-1}\mathbf{v} \\ \mathbf{v} &= \lambda\,A^{-1}\mathbf{v} \\ \frac{1}{\lambda}\,\mathbf{v} &= A^{-1}\mathbf{v} \end{aligned}$$

Since A is invertible $\lambda \neq 0$.

14.7.4 Equation (14.19) can be written as

$$\mathbf{u}^{j+1} = (A(r))^{-1}B(r)\mathbf{u}^j - \mathbf{u}^{j-1}.$$

Suppose $\mathbf{u}^j$ is the exact solution to the linear system for $j \in \mathbb{N}$ and $\hat{\mathbf{u}}^j$ is the numerical solution. Thus $\hat{\mathbf{u}}^j = \mathbf{u}^j + \mathbf{e}^j$ where the vector $\mathbf{e}^j$ is interpreted as the error in the jth numerical solution. Substituting this into the linear system yields

$$\begin{aligned} \mathbf{u}^{j+1} + \mathbf{e}^{j+1} &= (A(r))^{-1}B(r)(\mathbf{u}^j + \mathbf{e}^j) - (\mathbf{u}^{j-1} + \mathbf{e}^{j-1}) \\ \mathbf{e}^{j+1} &= (A(r))^{-1}B(r)\mathbf{e}^j - \mathbf{e}^{j-1}. \end{aligned}$$

The finite difference scheme will be stable provided the eigenvalues of $(A(r))^{-1}B(r)$ lie inside the unit disk in the complex plane. According to Lemma 14.1 the eigenvectors of $A(r)$ and $B(r)$ are for $j = 1, 2, \ldots, N-1$,

$$\mathbf{v}_j = \left(\sin\frac{j\pi}{N}, \sin\frac{2j\pi}{N}, \ldots, \sin\frac{(N-1)j\pi}{N}\right)^T.$$

The corresponding eigenvalues of $A(r)$ are

$$\lambda_j = 2(2 + r^2) + 2r^2\cos\frac{j\pi}{N}$$

and the corresponding eigenvalues of $B(r)$ are

$$\mu_j = 4(2 - r^2) + 4r^2\cos\frac{j\pi}{N}.$$

Thus the eigenvalues of $(A(r))^{-1}B(r)$ are

$$\begin{aligned}\frac{\mu_j}{\lambda_j} &= \frac{4-2r^2+2r^2\cos\frac{j\pi}{N}}{2+r^2+r^2\cos\frac{j\pi}{N}}\\ &= \frac{2-2r^2\sin^2\frac{j\pi}{2N}}{1+r^2\cos^2\frac{j\pi}{2N}}\\ \left|\frac{\mu_j}{\lambda_j}\right| &= \frac{2\left|1-r^2\sin^2\frac{j\pi}{2N}\right|}{1+r^2\cos^2\frac{j\pi}{2N}}\end{aligned}$$

The expression $|\mu_j/\lambda_j| \le 1$ when

$$\begin{aligned}2\left|1-r^2\sin^2\frac{j\pi}{2N}\right| &\le 1+r^2\cos^2\frac{j\pi}{2N}\\ -1-r^2\cos^2\frac{j\pi}{2N} &\le 2-2r^2\sin^2\frac{j\pi}{2N} \le 1+r^2\cos^2\frac{j\pi}{2N}\\ -3-r^2\cos^2\frac{j\pi}{2N} &\le -2r^2\sin^2\frac{j\pi}{2N} \le -1+r^2\cos^2\frac{j\pi}{2N}\\ \frac{1}{2}-\frac{r^2}{2}\cos^2\frac{j\pi}{2N} &\le r^2\sin^2\frac{j\pi}{2N} \le \frac{3}{2}+\frac{r^2}{2}\cos^2\frac{j\pi}{2N}\end{aligned}$$

The first inequality above implies

$$1 \le r^2\left(1+\sin^2\frac{j\pi}{2N}\right) = r^2\left(\frac{3}{2}-\frac{1}{2}\cos\frac{j\pi}{N}\right) \le 2r^2,$$

and thus $r^2 \ge 1/2$. The second inequality from above implies that

$$r^2\sin^2\frac{j\pi}{2N} \le 3+r^2\cos\frac{j\pi}{N} \iff r^2\left(\sin^2\frac{j\pi}{2N}-\cos\frac{j\pi}{N}\right) \le 3$$

and thus $r^2 \le 3/2$.

14.7.6

(a) By definition A is symmetric so the lower triangular portion of A, is the transpose of the upper triangular portion of matrix A. Let L be the negative of the strictly lower triangular portion of matrix A and let D be the diagonal matrix whose diagonal entries are the diagonal entries of matrix A, then $A = D - L - L^T$. By Thm. B.24 $(D)_{ii} = (A)_{ii} > 0$ for $i = 1, 2, \ldots, n$.

(b) $D - L$ is a lower triangular matrix with diagonal entries $(D - L)_{ii} = (D)_{ii} > 0$ for $i = 1, 2, \ldots, n$ and thus $D - L$ is nonsingular.

(c) Matrix P is symmetric since,

$$P^T = (A - S^TAS)^T = A^T - S^TA^T(S^T)^T = A - S^TA^TS = P.$$

(d) By definition,

$$S = (D-L)^{-1}L^T = (D-L)^{-1}(D-L-A) = (D-L)^{-1}(D-L) - (D-L)^{-1}A = I - (D-L)^{-1}A.$$

(e) If $Q = (D - L)^{-1}A$ then

$$S = I - (D-L)^{-1}A = I - Q.$$

Since $P = A - S^T AS$ then

$$\begin{aligned}
P &= A - (I-Q)^T A(I-Q) \\
&= A - (I-Q^T)(A-AQ) \\
&= A - (A - Q^T A) - (-AQ - Q^T AQ) \\
&= Q^T A - Q^T AQ + AQ \\
&= Q^T AQ^{-1}Q - Q^T AQ + Q^T (Q^T)^{-1} AQ \\
&= Q^T \left(AQ^{-1}Q - AQ + (Q^T)^{-1}AQ\right) \\
&= Q^T \left(AQ^{-1} - A + (Q^T)^{-1}A\right) Q.
\end{aligned}$$

(f) Recall that $Q = (D-L)^{-1}A$ and thus

$$\begin{aligned}
P &= Q^T \left(AQ^{-1} - A + (Q^T)^{-1}A\right) Q \\
&= Q^T \left(A\left[(D-L)^{-1}A\right]^{-1} - A + \left[\left((D-L)^{-1}A\right)^T\right]^{-1} A\right) Q \\
&= Q^T \left(AA^{-1}(D-L) - A + \left[\left((D-L)^{-1}A\right)^{-1}\right]^T A\right) Q \\
&= Q^T \left(D - L - A + \left[A^{-1}(D-L)\right]^T A\right) Q \\
&= Q^T \left(L^T + (D-L)^T (A^{-1})^T A\right) Q \\
&= Q^T \left(L^T + (D-L^T)(A^T)^{-1} A\right) Q \\
&= Q^T \left(L^T + (D-L^T)A^{-1} A\right) Q \\
&= Q^T \left(L^T + D - L^T\right) Q \\
&= Q^T D\, Q
\end{aligned}$$

Let the $n \times n$ matrix $B = \sqrt{D}Q$ where $\sqrt{D}$ is the $n \times n$ diagonal matrix for which $(\sqrt{D})_{ii} = \sqrt{(D)_{ii}}$ for $i = 1, 2, \ldots, n$. Then $P = B^T B$ and by Thm. B.24 matrix P is positive definite.

(g) Suppose $\mathbf{x} \neq \mathbf{0}$ is an eigenvector of S corresponding to eigenvalue λ. Earlier it was shown that matrix P is positive definite.

$$\begin{aligned}
\mathbf{x}^T P\, \mathbf{x} &> 0 \\
\mathbf{x}^T (A - S^T A\, S)\mathbf{x} &> 0 \\
\mathbf{x}^T A\, \mathbf{x} - \mathbf{x}^T S^T A(S\, \mathbf{x}) &> 0 \\
\mathbf{x}^T A\, \mathbf{x} - (\mathbf{x}^T S^T) A\, \lambda\, \mathbf{x} &> 0 \\
\mathbf{x}^T A\, \mathbf{x} - \lambda (S\, \mathbf{x})^T A\, \mathbf{x} &> 0 \\
\mathbf{x}^T A\, \mathbf{x} - \lambda^2 \mathbf{x}^T A\, \mathbf{x} &> 0 \\
\mathbf{x}^T A\, \mathbf{x}(1 - \lambda^2) &> 0 \\
1 - \lambda^2 &> 0
\end{aligned}$$

Since A is positive definite. In order to satisfy the inequality $|\lambda| < 1$.

(h) Since the strictly upper triangular portion of a symmetric matrix is just the transpose of the strictly lower triangular portion of the same matrix, the matrix S defined earlier is the Gauss-Seidel matrix for the symmetric matrix A.