Solving the Black-Scholes Equation

An Undergraduate Introduction to Financial Mathematics

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Initial Value Problem for the European Call

The main objective of this lesson is solving the Black-Scholes initial boundary value problem.

For (S, t) in $[0, \infty) \times [0, T]$,

$$\begin{aligned} rF &= F_t + rSF_S + \frac{1}{2}\sigma^2 S^2 F_{SS} \\ F(S, T) &= (S(T) - K)^+ \quad \text{for } S > 0, \\ F(0, t) &= 0 \quad \text{for } 0 \leq t < T, \\ F(S, t) &= S - Ke^{-r(T-t)} \quad \text{as } S \to \infty. \end{aligned}$$

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\end{aligned}
$$

We will solve this system of equations using Fourier Transforms.

Fourier Transform of a Function

Definition

If $f : \mathbb{R} \to \mathbb{R}$ then the **Fourier Transform** of f is

$$\mathcal{F}\{f(x)\} = \hat{f}(\omega) = \int_{-\infty}^{\infty} f(x) e^{-i\omega x} \, dx,$$

where $i = \sqrt{-1}$ and ω is a parameter. The Fourier Transform exists only if the improper integral converges.

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The Fourier Transform of f will exist when

- f and f' are piecewise continuous on every interval of the form $[-M, M]$ for arbitrary $M > 0$, and
- $\displaystyle\int_{-\infty}^{\infty} |f(x)|\, dx$ converges.

Example

When working with complex-valued exponentials, the **Euler Identity** may be useful:

$$e^{i\theta} = \cos\theta + i\sin\theta.$$

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Example

Find the Fourier Transform of the piecewise-defined function

$$f(x) = \begin{cases} 1/2 & \text{if } |x| \le 1, \\ 0 & \text{otherwise.} \end{cases}$$

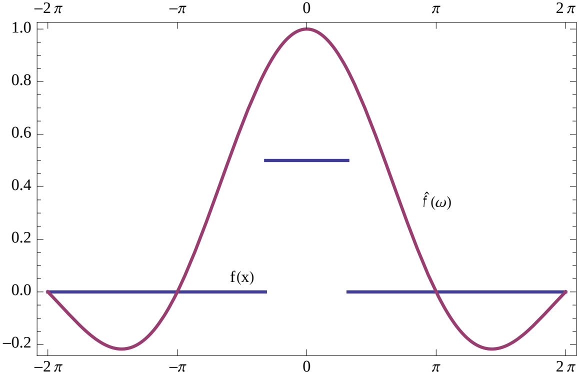
Solution

$$
\begin{aligned}
\hat{f}(\omega) &= \int_{-\infty}^{\infty} f(x) e^{-i\omega x}\,dx \\
&= \int_{-1}^{1} \frac{1}{2} e^{-i\omega x}\,dx \\
&= \left.\frac{-1}{2i\omega} e^{-i\omega x}\right|_{-1}^{1} \\
&= \frac{-1}{2i\omega}\left(e^{-i\omega} - e^{i\omega}\right) \\
&= \frac{1}{\omega}\left(\frac{e^{i\omega} - e^{-i\omega}}{2i}\right) \\
&= \frac{1}{\omega}\left(\frac{\cos\omega + i\sin\omega - \cos\omega + i\sin\omega}{2i}\right) \\
\hat{f}(\omega) &= \frac{\sin\omega}{\omega}
\end{aligned}
$$

Illustration

$$f(x) = \begin{cases} 1/2 & \text{if } |x| \leq 1, \\ 0 & \text{otherwise.} \end{cases}$$

$$\hat{f}(\omega) = \frac{\sin \omega}{\omega}$$



Fourier Transform of a Derivative

Suppose the Fourier Transform of f exists and that f' exists, find $\mathcal{F}\{f'(x)\}$.

Hint: use integration by parts.

Solution

Applying integration by parts with

$$\begin{aligned} u &= e^{-i\omega x} & v &= f(x) \\ du &= -i\omega e^{-i\omega x}\,dx & dv &= f'(x)\,dx \end{aligned}$$

yields

$$\begin{aligned} \mathcal{F}\left\{f'(x)\right\} &= \int_{-\infty}^{\infty} f'(x) e^{-i\omega x}\,dx \\ &= f(x)e^{-i\omega x}\Big|_{-\infty}^{\infty} - \int_{-\infty}^{\infty} f(x)(-i\omega)e^{-i\omega x}\,dx \\ &= i\omega \int_{-\infty}^{\infty} f(x) e^{-i\omega x}\,dx \\ &= i\omega \hat{f}(\omega). \end{aligned}$$

Fourier Transforms and Derivatives

Theorem

If $f(x), f'(x), \ldots, f^{(n-1)}(x)$ are all Fourier transformable and if $f^{(n)}(x)$ exists (where $n \in \mathbb{N}$) then $\mathcal{F}\{f^{(n)}(x)\} = (i\omega)^n \hat{f}(\omega)$.

Proof

The previous example demonstrates the result is true for $n = 1$. Suppose the result is true for $n = k \geq 1$. By definition

$$
\begin{aligned}
\mathcal{F}\left\{f^{(k+1)}(x)\right\} &= \int_{-\infty}^{\infty} f^{(k+1)}(x) e^{-i\omega x}\,dx \\
&= f^{(k)}(x) e^{-i\omega x}\Big|_{-\infty}^{\infty} - \int_{-\infty}^{\infty} f^{(k)}(x)(-i\omega) e^{-i\omega x}\,dx \\
&= (i\omega) \int_{-\infty}^{\infty} f^{(k)}(x) e^{-i\omega x}\,dx \\
&= (i\omega)(i\omega)^k \hat{f}(\omega) \\
&= (i\omega)^{k+1} \hat{f}(\omega).
\end{aligned}
$$

The result follows by induction on k.

Fourier Convolution

Definition

The **Fourier Convolution** of two functions f and g is

$$(f * g)(x) = \int_{-\infty}^{\infty} f(x - z)g(z)\, dz,$$

provided the improper integral converges.

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Theorem

$\mathcal{F}\{(f*g)(x)\} = \hat{f}(\omega)\hat{g}(\omega)$, in other words the Fourier Transform of the Fourier Convolution of f and g is the product of the Fourier Transforms of f and g.

Proof (1 of 2)

$$
\begin{aligned}
\mathcal{F}\{(f * g)(x)\} &= \int_{-\infty}^{\infty} \left[\int_{-\infty}^{\infty} f(x-z) g(z)\, dz \right] e^{-i\omega x}\, dx \\
&= \int_{-\infty}^{\infty} \left[\int_{-\infty}^{\infty} f(x-z) g(z) e^{-i\omega x}\, dz \right] dx \\
&= \int_{-\infty}^{\infty} \left[\int_{-\infty}^{\infty} f(x-z) g(z) e^{-i\omega x}\, dx \right] dz \\
&= \int_{-\infty}^{\infty} g(z) \left[\int_{-\infty}^{\infty} f(x-z) e^{-i\omega x}\, dx \right] dz
\end{aligned}
$$

So far,

$$\begin{aligned}
\mathcal{F}\{(f * g)(x)\} &= \int_{-\infty}^{\infty} g(z) \left[\int_{-\infty}^{\infty} f(x-z) e^{-i\omega x} \, dx \right] dz \\
&= \int_{-\infty}^{\infty} g(z) \left[\int_{-\infty}^{\infty} f(u) e^{-i\omega(u+z)} \, du \right] dz \\
&= \int_{-\infty}^{\infty} g(z) e^{-i\omega z} \left[\int_{-\infty}^{\infty} f(u) e^{-i\omega u} \, du \right] dz \\
&= \hat{f}(\omega) \int_{-\infty}^{\infty} g(z) e^{-i\omega z} \, dz \\
&= \hat{f}(\omega) \hat{g}(\omega).
\end{aligned}$$

Example

Let

$$\begin{aligned} f(x) &= \cos x \\ g(x) &= \begin{cases} 1/2 & \text{if } |x| \leq 1, \\ 0 & \text{otherwise.} \end{cases} \end{aligned}$$

and find $(f * g)(x)$.

Solution

$$
\begin{aligned}
(f * g)(x) &= \int_{-\infty}^{\infty} f(x-z)\, g(z)\, dz \\
&= \int_{-1}^{1} \frac{1}{2} \cos(x-z)\, dz \\
&= \frac{1}{2} \int_{-1}^{1} (\cos x \cos z + \sin x \sin z)\, dz \\
&= \frac{1}{2} \cos x \int_{-1}^{1} \cos z\, dz \\
&= \sin(1) \cos x
\end{aligned}
$$

Example

Let

$$f(x) = \begin{cases} 1 - x & \text{if } 0 \leq x < 1, \\ 0 & \text{otherwise.} \end{cases}$$

$$g(x) = \begin{cases} e^{-x} & \text{if } x \geq 0, \\ 0 & \text{otherwise.} \end{cases}$$

and show $\mathcal{F}\{(f * g)(x)\} = \hat{f}(\omega)\hat{g}(\omega)$.

$$\begin{aligned}
\hat{f}(\omega) &= \int_{-\infty}^{\infty} f(x) e^{-i\omega x}\,dx \\
&= \int_0^1 (1-x) e^{-i\omega x}\,dx \\
&= \frac{-1}{i\omega}(1-x)e^{-i\omega x}\Big|_0^1 - \int_0^1 \frac{1}{i\omega} e^{-i\omega x}\,dx \\
&= \frac{1}{\omega^2}\left(1 - e^{-i\omega}\right) + \frac{1}{i\omega}
\end{aligned}$$

Solution (2 of 4)

$$\begin{aligned}
\hat{g}(\omega) &= \int_{-\infty}^{\infty} g(x) e^{-i\omega x} \, dx \\
&= \int_{0}^{\infty} e^{-x} e^{-i\omega x} \, dx \\
&= \left. \frac{-1}{1+i\omega} e^{-(1+i\omega)x} \right|_{0}^{\infty} \\
&= \frac{1}{1+i\omega}
\end{aligned}$$

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&= \frac{1}{1+i\omega}
\end{aligned}
$$

Thus

$$
\hat{f}(\omega)\hat{g}(\omega) = \frac{1 - e^{-i\omega} - i\omega}{\omega^2(1+i\omega)}.
$$

Now find the convolution.

$$
\begin{aligned}
(f * g)(x) &= \int_{-\infty}^{\infty} f(z)g(x-z)\,dz \\
&= \int_{0}^{\min(1,x)} (1-z)e^{-(x-z)}\,dz \\
&= (2-z)e^{-(x-z)}\Big|_{0}^{\min(1,x)} \\
h(x) &= \begin{cases} 0 & \text{if } x < 0, \\ 2 - x - 2e^{-x} & \text{if } 0 \le x < 1, \\ (e-2)e^{-x} & \text{if } x \ge 1. \end{cases}
\end{aligned}
$$

Now find the Fourier transform of the convolution.

$$\begin{aligned}
\hat{h}(\omega) &= \int_{-\infty}^{\infty} h(x) e^{-i\omega x}\,dx \\
&= \int_0^1 (2 - x - 2e^{-x}) e^{-i\omega x}\,dx + \int_1^{\infty} (e-2) e^{-x} e^{-i\omega x}\,dx \\
&= \frac{e^{-1-i\omega}(i(e + (e-2)\omega^2) - (i+\omega)e^{1+i\omega})}{\omega^2(\omega - i)} + \frac{(e-2)e^{-1-i\omega}}{1 - i\omega} \\
&= \frac{1 - e^{-i\omega} - i\omega}{\omega^2(1 + i\omega)}
\end{aligned}$$

Inverse Fourier Transform

Definition

The **inverse Fourier Transform** of $\hat{f}(\omega)$ is denoted $\mathcal{F}^{-1}\{\hat{f}(\omega)\}$ and is given by

$$\mathcal{F}^{-1}\{\hat{f}(\omega)\} = \frac{1}{2\pi}\int_{-\infty}^{\infty} \hat{f}(\omega)e^{i\omega x}\,d\omega.$$

Example

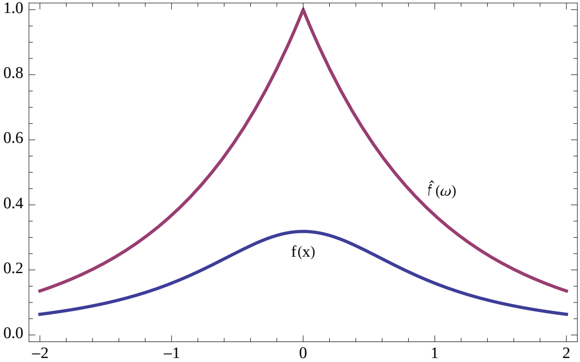
Find the inverse Fourier Transform of $e^{-|\omega|}$.

Solution

$$
\begin{aligned}
\mathcal{F}^{-1}\left\{e^{-|\omega|}\right\} &= \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{-|\omega|}e^{i\omega x}\,d\omega \\
&= \frac{1}{2\pi}\int_{-\infty}^{0} e^{(1+ix)\omega}\,d\omega + \frac{1}{2\pi}\int_{0}^{\infty} e^{(-1+ix)\omega}\,d\omega \\
&= \frac{1}{2\pi}\frac{1}{1+ix}e^{(1+ix)\omega}\Big|_{-\infty}^{0} + \frac{1}{2\pi}\frac{1}{-1+ix}e^{(-1+ix)\omega}\Big|_{0}^{\infty} \\
&= \frac{1}{2\pi(1+ix)} + \frac{1}{2\pi(1-ix)} \\
f(x) &= \frac{1}{\pi(1+x^2)}
\end{aligned}
$$

Illustration

$$\hat{f}(\omega) = e^{-|\omega|}$$
$$f(x) = \frac{1}{\pi(1+x^2)}$$



We will use the Fourier Transform and its inverse to solve the Black-Scholes PDE once we have performed a suitable change of variables on the PDE.

Define the new variables x, τ, and v as

$$\begin{aligned} x &= \ln \frac{S}{K} \\ \tau &= \frac{\sigma^2}{2}(T - t) \\ v(x, \tau) &= \frac{1}{K} F(S, t) \end{aligned}$$

and calculate F_t, F_S, and F_{SS}.

Change of Variables

By the Chain Rule:

$$
\begin{aligned}
F_t &= Kv_x x_t + Kv_\tau \tau_t = -\frac{K\sigma^2}{2} v_\tau \\
F_S &= Kv_x x_S + Kv_\tau \tau_S = e^{-x} v_x \\
F_{SS} &= -e^{-x} v_x x_S + e^{-x} v_{xx} x_S = \frac{e^{-2x}}{K}(v_{xx} - v_x).
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Substituting into the Black-Scholes PDE:

$$r\,F = F_t + r\,S\,F_S + \frac{1}{2}\sigma^2 S^2 F_{SS}$$

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\end{aligned}
$$

Substituting into the Black-Scholes PDE:

$$
\begin{aligned}
r\,F &= F_t + r\,S\,F_S + \frac{1}{2}\sigma^2 S^2 F_{SS} \\
v_\tau &= v_{xx} + (k-1)v_x - k\,v
\end{aligned}
$$

where $k = 2r/\sigma^2$.

Side Conditions (1 of 2)

Under the change of variables the final condition

$$
\begin{aligned}
F(S,T) &= (S(T)-K)^+ \\
Kv(x,0) &= (Ke^x-K)^+ \\
v(x,0) &= (e^x-1)^+
\end{aligned}
$$

becomes an initial condition.

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becomes an initial condition.

The boundary condition

$$\begin{aligned} F(0,t) &= \lim_{S\to 0^+} F(S,t) \\ 0 &= \lim_{x\to -\infty} K\, v(x,\tau) \\ 0 &= \lim_{x\to -\infty} v(x,\tau). \end{aligned}$$

The boundary at $S = 0$ has moved to a boundary as $x \to -\infty$.

The boundary condition

$$\begin{aligned}
\lim_{S\to\infty} F(S,t) &= S - Ke^{-r(T-t)} \\
\lim_{x\to\infty} K\, v(x,\tau) &= Ke^{x} - Ke^{-r(T-[T-2\tau/\sigma^2])} \\
\lim_{x\to\infty} v(x,\tau) &= e^{x} - e^{-k\tau}.
\end{aligned}$$

IBVP in New Variables

The initial value problem in the new variables is

$$\begin{aligned}
v_\tau &= v_{xx} + (k-1)v_x - kv \quad \text{for } x \in (-\infty,\infty),\ \tau \in (0, \tfrac{T\sigma^2}{2}) \\
v(x,0) &= (e^x - 1)^+ \quad \text{for } x \in (-\infty,\infty) \\
v(x,\tau) &\to 0 \quad \text{as } x \to -\infty \text{ and} \\
v(x,\tau) &\to e^x - e^{-k\tau} \quad \text{as } x \to \infty,\ \tau \in (0, \tfrac{T\sigma^2}{2}).
\end{aligned}$$

Another Change of Variables

Suppose α and β are constants and

$$v(x, \tau) = e^{\alpha x + \beta \tau} u(x, \tau).$$

Find v_x, v_{xx}, and v_τ in terms of u_x, u_{xx}, and u_τ.

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Suppose α and β are constants and

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Find v_x, v_{xx}, and v_τ in terms of u_x, u_{xx}, and u_τ.

$$\begin{aligned}
v_x &= e^{\alpha x+\beta\tau}\left(\alpha u(x,\tau) + u_x\right) \\
v_{xx} &= e^{\alpha x+\beta\tau}\left(\alpha^2 u(x,\tau) + 2\alpha u_x + u_{xx}\right) \\
v_\tau &= e^{\alpha x+\beta\tau}\left(\beta u(x,\tau) + u_\tau\right)
\end{aligned}$$

Substituting into the PDE

If we substitute function u in place of function v in the IBVP, we obtain:

$$
\begin{aligned}
v_\tau &= v_{xx} + (k-1)v_x - kv \\
u_\tau &= (\alpha^2 + (k-1)\alpha - k - \beta)u + (2\alpha + k - 1)u_x + u_{xx}
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If we choose α and β so that

$$\begin{aligned} 0 &= \alpha^2 + (k-1)\alpha - k - \beta \\ 0 &= 2\alpha + k - 1 \end{aligned}$$

then the first two terms on the right-hand side of the equation for u vanish.

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Find α and β.

If we let

$$\begin{aligned}\alpha &= \frac{1-k}{2}\\ \beta &= -\frac{(1+k)^2}{4}\end{aligned}$$

then the PDE:

$$u_\tau = (\alpha^2 + (k-1)\alpha - k - \beta)u + (2\alpha + k - 1)u_x + u_{xx}$$

can be written as

$$u_\tau = u_{xx}$$

which is known as the **Heat Equation.**

Side Conditions

If $\alpha = \dfrac{1-k}{2}$ and $\beta = -\dfrac{(k+1)^2}{4}$, then the initial condition becomes:

$$\begin{aligned} v(x,0) &= (e^x - 1)^+ \\ u(x,0) &= (e^{(k+1)x/2} - e^{(k-1)x/2})^+. \end{aligned}$$

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$$\begin{aligned} v(x,0) &= (e^x - 1)^+ \\ u(x,0) &= (e^{(k+1)x/2} - e^{(k-1)x/2})^+. \end{aligned}$$

The boundary conditions become

$$\begin{aligned} \lim_{x\to-\infty} v(x,\tau) &= 0 \\ \lim_{x\to-\infty} u(x,\tau) &= 0 \end{aligned}$$

and

$$\begin{aligned} \lim_{x\to\infty} v(x,\tau) &= e^x - e^{-k\tau} \\ \lim_{x\to\infty} u(x,\tau) &= e^{\frac{(k+1)}{2}[x+(k+1)\tau/2]} - e^{\frac{(k-1)}{2}[x+(k-1)\tau/2]}. \end{aligned}$$

Initial Boundary Value Problem for the Heat Equation

The final form of the Black-Scholes IBVP can be summarized as follows.

$$
\begin{aligned}
u_\tau &= u_{xx} \quad \text{for } x \in (-\infty, \infty) \text{ and } \tau \in (0, T\sigma^2/2) \\
u(x,0) &= (e^{(k+1)x/2} - e^{(k-1)x/2})^+ \quad \text{for } x \in (-\infty, \infty) \\
u(x,\tau) &\to 0 \quad \text{as } x \to -\infty \text{ for } \tau \in (0, T\sigma^2/2) \\
u(x,\tau) &\to e^{\frac{(k+1)}{2}[x+(k+1)\tau/2]} - e^{\frac{(k-1)}{2}[x+(k-1)\tau/2]} \\
&\qquad \text{as } x \to \infty \text{ for } \tau \in (0, T\sigma^2/2)
\end{aligned}
$$

We now turn to the Fourier Transform to solve the IBVP.

$$\begin{aligned}
u_\tau &= u_{xx} \\
\mathcal{F}\{u_\tau\} &= \mathcal{F}\{u_{xx}\} \\
\frac{d\hat{u}}{d\tau} &= -\omega^2 \hat{u} \\
\hat{u}(\omega, \tau) &= De^{-\omega^2 \tau}
\end{aligned}$$

where $D = \hat{f}(\omega)$ is the Fourier Transform of the initial condition.

Inverse Fourier Transforming the Solution

Recall the Fourier Convolution and the Fourier Transform of the Fourier Convolution.

$$\begin{aligned}
\mathcal{F}^{-1}\{\hat{u}(\omega,\tau)\} &= \mathcal{F}^{-1}\{\hat{f}(\omega)e^{-\omega^2\tau}\} \\
u(x,\tau) &= (e^{(k+1)x/2} - e^{(k-1)x/2})^{+} * \frac{1}{2\sqrt{\pi\tau}} e^{-x^2/(4\tau)} \\
&= \frac{1}{2\sqrt{\pi\tau}} \int_{-\infty}^{\infty} (e^{(k+1)\frac{z}{2}} - e^{(k-1)\frac{z}{2}})^{+} e^{-\frac{(x-z)^2}{4\tau}}\, dz
\end{aligned}$$

Make the substitutions:

$$
\begin{aligned}
z &= x + \sqrt{2\tau}\, y \\
dz &= \sqrt{2\tau}\, dy
\end{aligned}
$$

then

$$
\begin{aligned}
u(x,\tau) &= \frac{1}{2\sqrt{\pi\tau}} \int_{-\infty}^{\infty} \left(e^{(k+1)\frac{z}{2}} - e^{(k-1)\frac{z}{2}}\right)^{+} e^{-\frac{(x-z)^2}{4\tau}}\, dz \\
&= \frac{e^{(k+1)x/2} e^{(k+1)^2\tau/4}}{\sqrt{2\pi}} \int_{-x/\sqrt{2\tau}}^{\infty} e^{-(y-\frac{1}{2}(k+1)\sqrt{2\tau})^2/2}\, dy \\
&\quad - \frac{e^{(k-1)x/2} e^{(k-1)^2\tau/4}}{\sqrt{2\pi}} \int_{-x/\sqrt{2\tau}}^{\infty} e^{-(y-\frac{1}{2}(k-1)\sqrt{2\tau})^2/2}\, dy
\end{aligned}
$$

Undoing the Change of Variables (2 of 5)

Now make the substitutions $w = y - \frac{1}{2}(k+1)\sqrt{2\tau}$ in the first integral and $w' = y - \frac{1}{2}(k-1)\sqrt{2\tau}$ in the second.

$$\begin{aligned} u(x,\tau) &= \frac{e^{(k+1)x/2}e^{(k+1)^2\tau/4}}{\sqrt{2\pi}} \int_{-x/\sqrt{2\tau}}^{\infty} e^{-(y-\frac{1}{2}(k+1)\sqrt{2\tau})^2/2}\,dy \\ &\quad - \frac{e^{(k-1)x/2}e^{(k-1)^2\tau/4}}{\sqrt{2\pi}} \int_{-x/\sqrt{2\tau}}^{\infty} e^{-(y-\frac{1}{2}(k-1)\sqrt{2\tau})^2/2}\,dy \\ &= e^{(k+1)x/2+(k+1)^2\tau/4}\Phi\left(\frac{x}{\sqrt{2\tau}} + \frac{1}{2}(k+1)\sqrt{2\tau}\right) \\ &\quad - e^{(k-1)x/2+(k-1)^2\tau/4}\Phi\left(\frac{x}{\sqrt{2\tau}} + \frac{1}{2}(k-1)\sqrt{2\tau}\right) \end{aligned}$$

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$$
\begin{aligned}
u(x,\tau) &= \frac{e^{(k+1)x/2}e^{(k+1)^2\tau/4}}{\sqrt{2\pi}} \int_{-x/\sqrt{2\tau}}^{\infty} e^{-(y-\frac{1}{2}(k+1)\sqrt{2\tau})^2/2}\,dy \\
&\quad - \frac{e^{(k-1)x/2}e^{(k-1)^2\tau/4}}{\sqrt{2\pi}} \int_{-x/\sqrt{2\tau}}^{\infty} e^{-(y-\frac{1}{2}(k-1)\sqrt{2\tau})^2/2}\,dy \\
&= e^{(k+1)x/2+(k+1)^2\tau/4}\Phi\left(\frac{x}{\sqrt{2\tau}} + \frac{1}{2}(k+1)\sqrt{2\tau}\right) \\
&\quad - e^{(k-1)x/2+(k-1)^2\tau/4}\Phi\left(\frac{x}{\sqrt{2\tau}} + \frac{1}{2}(k-1)\sqrt{2\tau}\right)
\end{aligned}
$$

Recall: Φ is the cumulative normal distribution function.

Note that

$$\begin{aligned} e^{(k+1)\frac{x}{2}+(k+1)^2\frac{\tau}{4}} e^{-(k-1)\frac{x}{2}-(k+1)^2\frac{\tau}{4}} &= e^x \\ e^{(k-1)\frac{x}{2}+(k-1)^2\frac{\tau}{4}} e^{-(k-1)\frac{x}{2}-(k+1)^2\frac{\tau}{4}} &= e^{-k\tau} \end{aligned}$$

and therefore

$$\begin{aligned} v(x,\tau) &= e^{-(k-1)x/2-(k+1)^2\tau/4} u(x,\tau) \\ &= e^x \Phi\left(\frac{x}{\sqrt{2\tau}} + \frac{1}{2}(k+1)\sqrt{2\tau}\right) \\ &\quad - e^{-k\tau} \Phi\left(\frac{x}{\sqrt{2\tau}} + \frac{1}{2}(k-1)\sqrt{2\tau}\right). \end{aligned}$$

Undoing the Change of Variables (4 of 5)

Recall that

$$
\begin{aligned}
x &= \ln \frac{S}{K} \\
\tau &= \frac{\sigma^2}{2}(T-t) \\
k &= \frac{2r}{\sigma^2}
\end{aligned}
$$

and thus

$$
\begin{aligned}
\frac{x}{\sqrt{2\tau}} + \frac{1}{2}(k+1)\sqrt{2\tau} &= \frac{\ln(S/K) + (r+\sigma^2/2)(T-t)}{\sigma\sqrt{T-t}} = w \\
\frac{x}{\sqrt{2\tau}} + \frac{1}{2}(k-1)\sqrt{2\tau} &= w - \sigma\sqrt{T-t}.
\end{aligned}
$$

$$
\begin{aligned}
v(x,\tau) &= \frac{S}{K}\Phi\left(w\right) - e^{-r(T-t)}\Phi\left(w - \sigma\sqrt{T-t}\right) \\
F(S,t) &= S\Phi\left(w\right) - Ke^{-r(T-t)}\Phi\left(w - \sigma\sqrt{T-t}\right) \\
w &= \frac{\ln(S/K) + (r + \sigma^2/2)(T-t)}{\sigma\sqrt{T-t}}
\end{aligned}
$$

Undoing the Change of Variables (5 of 5)

$$
\begin{aligned}
v(x,\tau) &= \frac{S}{K}\Phi\left(w\right) - e^{-r(T-t)}\Phi\left(w - \sigma\sqrt{T-t}\right) \\
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w &= \frac{\ln(S/K) + (r+\sigma^2/2)(T-t)}{\sigma\sqrt{T-t}}
\end{aligned}
$$

Finally we have the formula for the European call.

$$
C(S,t) = S\Phi\left(w\right) - K\,e^{-r(T-t)}\Phi\left(w - \sigma\sqrt{T-t}\right)
$$

Undoing the Change of Variables (5 of 5)

$$\begin{aligned} v(x,\tau) &= \frac{S}{K}\Phi\left(w\right) - e^{-r(T-t)}\Phi\left(w - \sigma\sqrt{T-t}\right) \\ F(S,t) &= S\Phi\left(w\right) - Ke^{-r(T-t)}\Phi\left(w - \sigma\sqrt{T-t}\right) \\ w &= \frac{\ln(S/K) + (r + \sigma^2/2)(T-t)}{\sigma\sqrt{T-t}} \end{aligned}$$

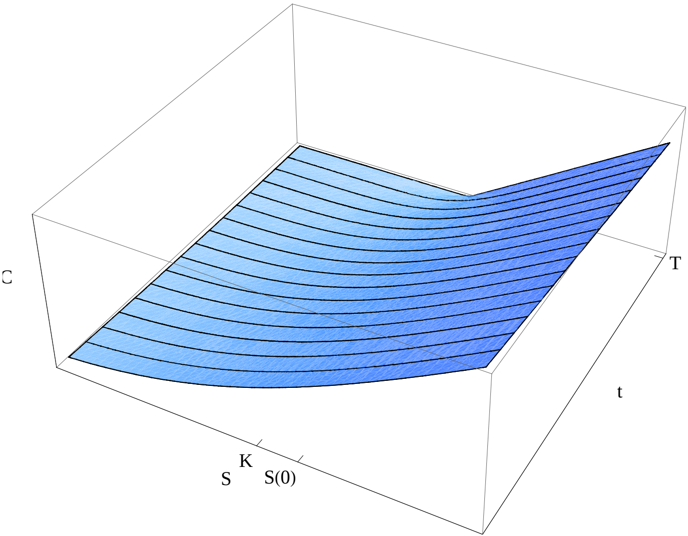
Finally we have the formula for the European call.

$$C(S,t) = S\Phi\left(w\right) - K\,e^{-r(T-t)}\Phi\left(w - \sigma\sqrt{T-t}\right)$$

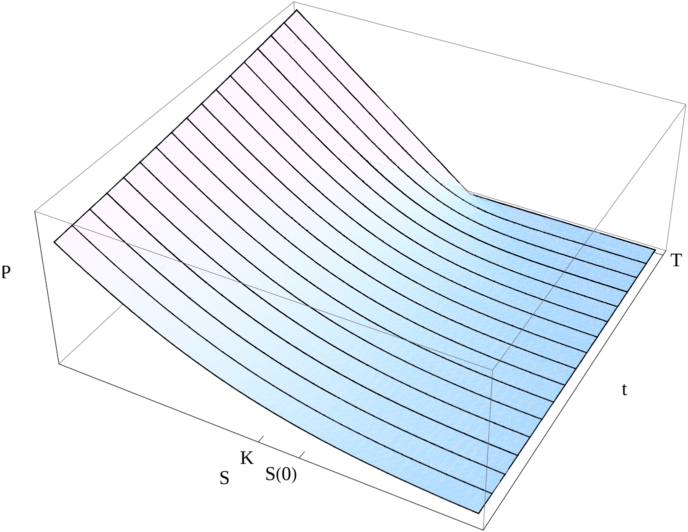
Using the Put-Call Parity Formula we can find the formula for the European put.

$$P(S,t) = K\,e^{-r(T-t)}\Phi\left(\sigma\sqrt{T-t} - w\right) - S\Phi\left(-w\right)$$

Plotting the Call Price



Plotting the Put Price



Example (1 of 2)

Suppose the current price of a security is \$62 per share. The continuously compounded interest rate is 10% per year. The volatility of the price of the security is $\sigma = 20\%$ per year. Find the cost of a five-month European call option with a strike price of \$60 per share.

Summary:

$$\begin{aligned} T &= 5/12, \quad t = 0, \quad r = 0.10, \\ \sigma &= 0.20, \quad S = 62, \quad \text{and} \quad K = 60. \end{aligned}$$

Example (2 of 2)

Summary:

$$\begin{aligned} T &= 5/12, \quad t = 0, \quad r = 0.10, \\ \sigma &= 0.20, \quad S = 62, \quad \text{and} \quad K = 60. \end{aligned}$$

$$\begin{aligned} w &= \frac{\ln(S/K) + (r + \sigma^2/2)(T - t)}{\sigma\sqrt{T - t}} \\ C &= S\Phi\left(w\right) - Ke^{-r(T-t)}\Phi\left(w - \sigma\sqrt{T - t}\right) \end{aligned}$$

Summary:

$$\begin{aligned} T &= 5/12, \quad t = 0, \quad r = 0.10, \\ \sigma &= 0.20, \quad S = 62, \quad \text{and} \quad K = 60. \end{aligned}$$

$$\begin{aligned} w &= \frac{\ln(S/K) + (r + \sigma^2/2)(T - t)}{\sigma\sqrt{T - t}} \approx 0.641287 \\ C &= S\Phi\left(w\right) - Ke^{-r(T-t)}\Phi\left(w - \sigma\sqrt{T - t}\right) \approx \$5.80 \end{aligned}$$

Suppose the current price of a security is \$97 per share. The continuously compounded interest rate is 8% per year. The volatility of the price of the security is $\sigma = 45\%$ per year. Find the cost of a three-month European put option with a strike price of \$95 per share.

Example

Summary:

$$
\begin{aligned}
T &= 1/4, \quad t = 0, \quad r = 0.08, \\
\sigma &= 0.45, \quad S = 97, \quad \text{and} \quad K = 95.
\end{aligned}
$$

Example

Summary:

$$\begin{aligned} T &= 1/4, \quad t = 0, \quad r = 0.08, \\ \sigma &= 0.45, \quad S = 97, \quad \text{and} \quad K = 95. \end{aligned}$$

$$\begin{aligned} w &= \frac{\ln(S/K) + (r + \sigma^2/2)(T - t)}{\sigma\sqrt{T - t}} \\ P &= Ke^{-r(T-t)}\Phi\left(\sigma\sqrt{T - t} - w\right) - S\Phi(-w) \end{aligned}$$

Example

Summary:

$$\begin{aligned} T &= 1/4, \quad t = 0, \quad r = 0.08, \\ \sigma &= 0.45, \quad S = 97, \quad \text{and} \quad K = 95. \end{aligned}$$

$$\begin{aligned} w &= \frac{\ln(S/K) + (r + \sigma^2/2)(T-t)}{\sigma\sqrt{T-t}} \approx 0.293985 \\ P &= Ke^{-r(T-t)}\Phi\left(\sigma\sqrt{T-t} - w\right) - S\Phi\left(-w\right) \approx \$6.71 \end{aligned}$$

Implied Volatility (1 of 3)

Each financial firm writing option contracts may have its own estimate of the volatility σ of a stock. If we know the price of a call option, its strike price, expiry, the current stock price, and the risk-free interest rate, we can determine the **implied volatility** of the stock.

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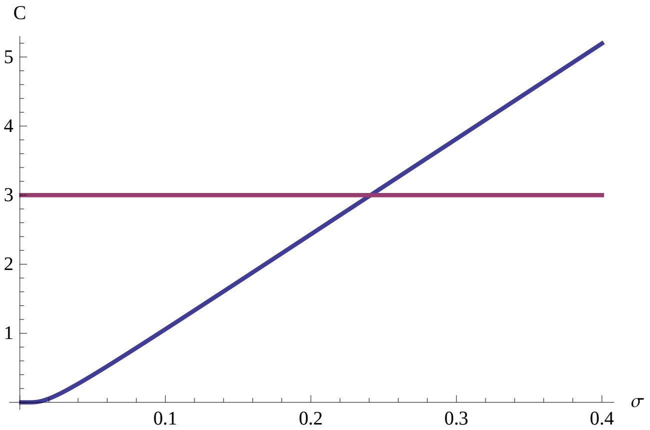
Example

Suppose the current price of a security is $60 per share. The continuously compounded interest rate is 6.25% per year. The cost of a four-month European call option with a strike price of $62 per share is $3. What is the implied volatility of the stock?

We must solve the equation

$$
\begin{aligned}
C &= S\Phi\left(w\right) - Ke^{-rT}\Phi\left(w - \sigma\sqrt{T}\right) \\
3 &= 60\Phi\left(\frac{\left(0.0625 + \frac{\sigma^2}{2}\right)\frac{4}{12} + \ln\frac{60}{62}}{\sigma\sqrt{\frac{4}{12}}}\right) \\
&\quad - 62e^{-(0.0625)\frac{4}{12}}\Phi\left(\frac{\left(0.0625 + \frac{\sigma^2}{2}\right)\frac{4}{12} + \ln\frac{60}{62}}{\sigma\sqrt{\frac{4}{12}}} - \sigma\sqrt{\frac{4}{12}}\right).
\end{aligned}
$$



Using Newton's Method, $\sigma \approx 0.241045$.

Binomial Model

The binomial model is a discrete approximation to the Black-Scholes initial value problem originally developed by Cox, Ross, and Rubinstein.

Assumptions:

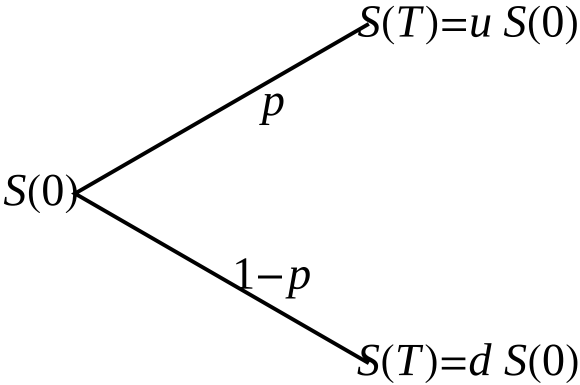
- Strike price of the call option is K.
- Exercise time of the call option is T.
- Present price of the security is $S(0)$.
- Continuously compounded interest rate is r.
- Price of the security follows a geometric Brownian motion with variance σ^2.
- Present time is t.

Binomial Lattice

If the value of the stock is $S(0)$ then at $t = T$

$$S(T) = \begin{cases} uS(0) & \text{with probability } p, \\ dS(0) & \text{with probability } 1 - p \end{cases}$$

where $0 < d < 1 < u$ and $0 < p < 1$.



Making the Continuous and Discrete Models Agree (1 of 2)

Continuous model:

$$
\begin{aligned}
dS &= \mu S\,dt + \sigma S\,dW(t) \\
d(\ln S) &= (\mu - \frac{1}{2}\sigma^2)\,dt + \sigma\,dW(t) \\
\mathbb{E}\left[\ln S(t)\right] &= \ln S(0) + (\mu - \frac{1}{2}\sigma^2)t \\
\mathbb{V}\left(\ln S(t)\right) &= \sigma^2 t
\end{aligned}
$$

Making the Continuous and Discrete Models Agree (1 of 2)

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\end{aligned}
$$

In the absence of arbitrage $\mu = r$, *i.e.* the return on the security should be the same as the return on an equivalent amount in savings.

Making the Continuous and Discrete Models Agree (2 of 2)

$$
\begin{aligned}
\ln S(0) + (r - \frac{1}{2}\sigma^2)\Delta t &= p \ln(uS(0)) + (1-p)\ln(dS(0)) \\
(r - \frac{1}{2}\sigma^2)\Delta t &= p \ln u + (1-p)\ln d
\end{aligned}
$$

Making the Continuous and Discrete Models Agree (2 of 2)

$$\begin{aligned}
\ln S(0) + (r - \frac{1}{2}\sigma^2)\Delta t &= p \ln(uS(0)) + (1-p)\ln(dS(0)) \\
(r - \frac{1}{2}\sigma^2)\Delta t &= p \ln u + (1-p)\ln d
\end{aligned}$$

The variance in the returns in the continuous and discrete models should also agree.

$$\begin{aligned}
\sigma^2 \Delta t &= p[\ln(uS(0))]^2 + (1-p)[\ln(dS(0))]^2 \\
&\quad - (p\ln(uS(0)) + (1-p)\ln(dS(0)))^2 \\
&= p(1-p)(\ln u - \ln d)^2
\end{aligned}$$

We would like to write p, u, and d as functions of r, σ, and Δt.

$$\begin{aligned} p \ln u + (1-p) \ln d &= (r - \frac{1}{2}\sigma^2)\Delta t \\ p(1-p)(\ln u - \ln d)^2 &= \sigma^2 \Delta t \end{aligned}$$

Summary

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- We need a third equation in order to solve this system.
- We are free to pick any equation consistent with the first two.
- We pick $d = 1/u$ (why?).

Solving the System

$$
\begin{aligned}
(2p - 1)\ln u &= (r - \frac{1}{2}\sigma^2)\Delta t \\
4p(1-p)(\ln u)^2 &= \sigma^2 \Delta t
\end{aligned}
$$

1. Square the first equation and add to the second.
2. Ignore terms involving $(\Delta t)^2$.

Solving the System

$$
\begin{aligned}
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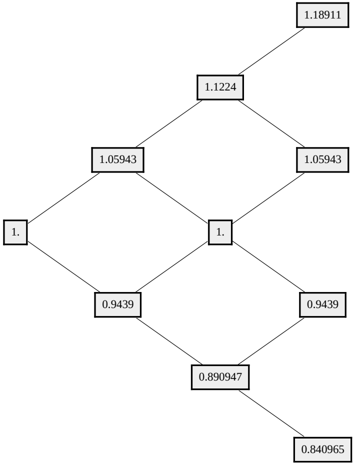
\end{aligned}
$$

1. Square the first equation and add to the second.
2. Ignore terms involving $(\Delta t)^2$.

$$
\begin{aligned}
u &= e^{\sigma\sqrt{\Delta t}} \\
d &= e^{-\sigma\sqrt{\Delta t}} \\
p &= \frac{1}{2}\left(1 + \left(\frac{r}{\sigma} - \frac{\sigma}{2}\right)\sqrt{\Delta t}\right)
\end{aligned}
$$

Example

Suppose $S(0) = 1$, $r = 0.10$, $\sigma = 0.20$, $T = 1/4$, $\Delta t = 1/12$, then the lattice of security prices resembles:



Determining a European Call Price

Payoff: $(S(T) - K)^+$

Let Y be a binomial random variable with probability of an UP step p and n total steps.

$$\begin{aligned}
C &= e^{-rT}\mathbb{E}\left[(u^Y d^{n-Y} S(0) - K)^+\right] \\
&= e^{-rT}\mathbb{E}\left[(e^{Y\sigma\sqrt{\Delta t}} e^{(Y-n)\sigma\sqrt{\Delta t}} S(0) - K)^+\right] \\
&= e^{-rT}\mathbb{E}\left[(e^{(2Y-n)\sigma\sqrt{\Delta t}} S(0) - K)^+\right] \\
&= e^{-rT}\mathbb{E}\left[(e^{(2Y-T/\Delta t)\sigma\sqrt{\Delta t}} S(0) - K)^+\right].
\end{aligned}$$

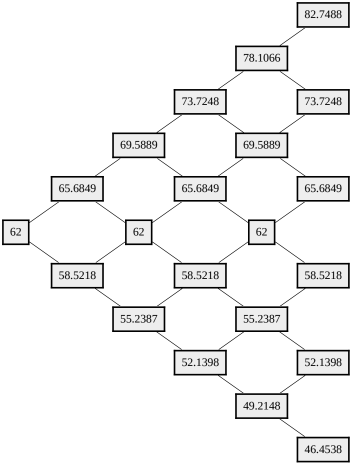
Example

The price of a security is \$62, the continuously compounded interest rate is 10% per year, the volatility of the price of the security is $\sigma = 20\%$ per year. If the strike price of a call option is \$60 per share with an expiry of 5 months, then $C = \$5.789$ according to the solution to the Black-Scholes equation.

The parameters of the discrete model are:

$$u = 1.05943, \quad d = 0.9439, \quad \text{and} \quad p = 0.557735.$$

Lattice of Security Prices



Payoffs of the Call Option

S	$(S-K)^+$	Prob.
82.7488	22.7488	$\binom{5}{5} u^5 d^0 \approx 0.0539684$
73.7248	13.7248	$\binom{5}{4} u^4 d^1 \approx 0.213976$
65.6849	5.6849	$\binom{5}{3} u^3 d^2 \approx 0.339351$
58.5218	0	$\binom{5}{2} u^2 d^3 \approx 0.269094$
52.1398	0	$\binom{5}{1} u^1 d^3 \approx 0.106691$
46.4538	0	$\binom{5}{0} u^0 d^5 \approx 0.0169205$

$$
\begin{aligned}
C &\approx \frac{(5.6849)(0.3394) + (13.7248)(0.2140) + (22.7488)(0.0540)}{e^{(0.10)(5/12)}} \\
&= 5.83509.
\end{aligned}
$$

Credits

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