Brownian Motion

An Undergraduate Introduction to Financial Mathematics

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Background

We have already seen that the limiting behavior of a **discrete random walk** yields a derivation of the **normal probability density function.**

Today we explore some further properties of the discrete random walk and introduce the concept of **stochastic processes.**

Discrete Steps

Assumptions:

- Initial value of random variable is $S(0)$.
- At each “tick” of a clock S may change by ± 1.
- Probabilities of a unit increase or decrease are the same, *i.e.*,

$$\begin{aligned}\mathbb{P}\left(S(t+1) = S(t) + 1\right) &= \frac{1}{2}\\ \mathbb{P}\left(S(t+1) = S(t) - 1\right) &= \frac{1}{2}.\end{aligned}$$

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We can relate $S(t)$ to a sequence of random variables taking on only the values ± 1. For all $i \in \mathbb{N}$

$$X_i = \begin{cases} +1 & \text{with probability } 1/2, \\ -1 & \text{with probability } 1/2 \end{cases}$$

Then

$$S(N) = S(0) + X_1 + X_2 + \cdots + X_N.$$

Further Assumptions

- X_i and X_j are independent when $i \neq j$.
- Out of the sequence $\{X_i\}_{i=1}^n$ random variable $X_i = +1$ exactly k times.

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Then

$$S(n) = S(0) + k - (n - k) = S(0) + 2k - n$$

and

$$\mathbb{P}\left(S(n) = S(0) + 2k - n\right) = \binom{n}{k}\left(\frac{1}{2}\right)^n.$$

Spatial Homogeneity

Define $T(i) = S(i) - S(0)$ for $i = 0, 1, \ldots, n$ then

- $T(0) = 0$,
- $S(n) = S(0) + 2k - n$ if and only if

$$T(n) = \sum_{i=1}^{n} X_i = 2k - n.$$

Remarks:

- The difference $S(n) - S(0)$ is independent of $S(0)$, a property of the random walk known as **spatial homogeneity.**
- We may assume $S(0) = 0$.

Visited States

Question: what states can $S(t)$ visit in n steps?

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Lemma

For the random walk $S(i)$ for $i = 0, 1, \ldots, n$ with initial state $S(0) = 0$,

1. $\mathbb{P}\left(S(n) = m\right) = 0$ *if* $|m| > n$,
2. $\mathbb{P}\left(S(n) = m\right) = 0$ *if* $n + m$ *is odd,*
3. $\mathbb{P}\left(S(n) = m\right) = \binom{n}{(n+m)/2}\left(\frac{1}{2}\right)^n$, *otherwise.*

Mean and Variance of Random Walk (1 of 2)

Theorem

For the random walk $\{S(i)\}_{i=0}^{n}$ *with initial state* $S(0) = 0$*,*

$$\mathbb{E}\left[S(n)\right] = 0 \qquad \textit{and} \qquad \mathbb{V}\left(S(n)\right) = n.$$

Mean and Variance of Random Walk (2 of 2)

Proof.

$$\begin{aligned} S(n) &= S(0) + \sum_{i=1}^{n} X_i \\ \mathbb{E}\left[S(n)\right] &= \mathbb{E}\left[S(0)\right] + \sum_{i=1}^{n} \mathbb{E}\left[X_i\right] = 0 \end{aligned}$$

since $\mathbb{E}\left[X_i\right] = 0$ for $i = 1, 2, \ldots, n$.

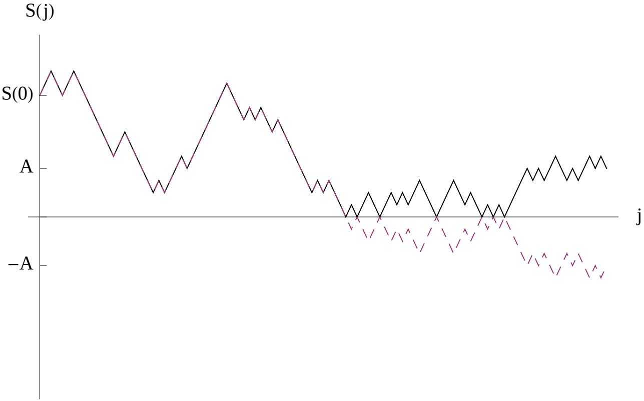
By assumption X_i and X_j are independent when $i \neq j$ and thus

$$\mathbb{V}\left(S(n)\right) = \mathbb{V}\left(S(0)\right) + \sum_{i=1}^{n} \mathbb{V}\left(X_i\right) = n.$$

☐

Reflections of Random Walks (1 of 2)

Consider a random walk $\{S(j)\}_{j=0}^{n}$ for which $S(k) = 0$ for some k.



Whenever the random walk crosses the horizontal axis, it has an equal probability of following either the solid or dashed path.

Reflections of Random Walks (2 of 2)

If $\{S(j)\}_{j=0}^{n}$ is a random walk with $S(k) = 0$ for some $k \in \{0, 1, \dots, n\}$ then define another random walk $\{\hat{S}(j)\}_{j=0}^{n}$ by

$$\hat{S}(j) = \begin{cases} S(j) & \text{for } j = 0, 1, \dots, k \\ -S(j) & \text{for } j = k+1, k+2, \dots, n. \end{cases}$$

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Since $\mathbb{P}\left(X_i = +1\right) = \mathbb{P}\left(X_i = -1\right) = 1/2$ then

$$\mathbb{P}\left(S(n) = A\right) = \mathbb{P}\left(\hat{S}(n) = -A\right).$$

Justification

Remarks:

- Random walks have no "memory" of how they arrive at a particular state. Only the current state influences the next state. This is known as the **Markov property.**
- If $\{S(j)\}_{j=0}^{n}$ is a random walk for which $S(k) = 0$ for some $k \in \{0, 1, \ldots, n\}$, then

$$\mathbb{P}\left(S(n) = A \,|\, S(k) = 0\right) = \mathbb{P}\left(T(n-k) = A\right).$$

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$$\mathbb{P}\left(S(n) = A \,|\, S(k) = 0\right) = \mathbb{P}\left(T(n-k) = A\right).$$

Therefore

$$\begin{aligned} \mathbb{P}\left(S(n) = A\right) &= \mathbb{P}\left(S(k) = 0\right)\mathbb{P}\left(T(n-k) = A\right) \\ &= \mathbb{P}\left(\hat{S}(k) = 0\right)\mathbb{P}\left(\hat{T}(n-k) = -A\right) \\ &= \mathbb{P}\left(\hat{S}(n) = -A\right). \end{aligned}$$

Unbiased Random Walks

Remark: as defined, the random walk $\{S(j)\}_{j=0}^{n}$ is **unbiased**, *i.e.*, equally likely to move up as well as down.

Theorem

If $\{S(j)\}_{j=0}^{n}$ is an unbiased random walk with initial state $S(0) = i$ and if $|A - i| \leq n$ and $|A + i| \leq n$ then

$$\mathbb{P}\left(S(n) = A \,|\, S(0) = i\right) = \mathbb{P}\left(S(n) = -A \,|\, S(0) = i\right).$$

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These probabilities are 0 if $n + A - i$ is odd (and consequently $n - A - i$ is odd).

Absorbing Boundary Conditions

Remark: so far we have considered only random walks which were free to wander unrestricted.

Question: what if there is a state A such that if $S(k) = A$ then $S(n) = A$ for all $n \geq k$? Such a state is called an **absorbing boundary condition.**

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Example

A gambler going broke and unable to borrow money has encountered an absorbing boundary condition.

Questions

Suppose random walk $\{S(j)\}_{j=0}^{n}$ has an absorbing boundary condition at 0. If $0 < S(0) < A$,

1. what is the probability that the state of the random walk crosses the threshold value of A before it hits the boundary at 0?

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Suppose random walk $\{S(j)\}_{j=0}^{n}$ has an absorbing boundary condition at 0. If $0 < S(0) < A$,

1. what is the probability that the state of the random walk crosses the threshold value of A before it hits the boundary at 0?
2. what is the expected value of the number of steps which will elapse before the state of the random variable first crosses the A threshold?

Answer to First Question (1 of 2)

Define $S_{\text{min}}(n) = \min_{0 \leq k \leq n}\{S(k)\}$ which can be thought of as the smallest value the random walk takes on.

The probability the state of the random walk crosses the threshold value of A before it hits the boundary at 0 is then

$$\mathbb{P}\left((S(n) = A) \wedge (S_{\text{min}}(n) > 0) \,|\, S(0) = i\right).$$

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$$\mathbb{P}\left((S(n) = A) \wedge (S_{\min}(n) > 0) \,|\, S(0) = i\right).$$

Remarks:

- $S_{\min}(n)$ is itself a random variable.
- For the sake of brevity, denote the probability above by $f_{A,i}(n)$.

Answer to First Question (2 of 2)

Lemma

Suppose a random walk $S(k) = S(0) + \sum_{i=1}^{k} X_i$ *in which the* X_i *for* $i = 1, 2, \ldots$ *are independent, identically distributed random variables taking on the values* ± 1*, each with probability* $p = 1/2$*. Suppose further that the boundary at* 0 *is absorbing, then if* $A, i > 0$*,*

$$f_{A,i}(n) = \left[\binom{n}{(n+A-i)/2} - \binom{n}{(n-A-i)/2} \right] \left(\frac{1}{2} \right)^n,$$

provided $|A - i| \leq n$*,* $|A + i| \leq n$*, and* $n + A - i$ *is even.*

Proof (1 of 3)

Consider a random walk with no boundary, that is, the random variable $S(n)$ has an initial state of $S(0) = i > 0$ and $S(k)$ is allowed to wander into negative territory (and back) arbitrarily. In this situation

$$\begin{aligned} & \mathbb{P}\left(S(n) = A \,|\, S(0) = i\right) \\ & \quad = \mathbb{P}\left((S(n) = A) \wedge (S_{\text{min}}(n) > 0) \,|\, S(0) = i\right) \\ & \qquad + \mathbb{P}\left((S(n) = A) \wedge (S_{\text{min}}(n) \leq 0) \,|\, S(0) = i\right) \end{aligned}$$

by the Addition Rule.

Proof (2 of 3)

Now consider the probability on the left-hand side of the equation:

$$\mathbb{P}\left(S(n) = A \,|\, S(0) = i\right).$$

It possesses no boundary condition and by the spatial homogeneity of the random walk

$$\mathbb{P}\left(S(n) = A \,|\, S(0) = i\right) = \mathbb{P}\left(T(n) = A - i\right)$$

where $\{T(j)\}_{j=0}^{n}$ is an unbiased random walk with initial state $T(0) = 0$. Hence $\mathbb{P}\left(T(n) = A - i\right) = 0$ unless $n + A - i$ is even and $|A - i| \leq n$, in which case

$$\mathbb{P}\left(S(n) = A \,|\, S(0) = i\right) = \binom{n}{(n + A - i)/2} \left(\frac{1}{2}\right)^{n}.$$

Proof (3 of 3)

On the other hand if the random walk starts at a positive state i and finishes at $-A < 0$ then it is certain that $S_{\min}(n) \leq 0$. Consequently

$$\begin{aligned}
&\mathbb{P}\left((S(n) = A) \wedge (S_{\min}(n) \leq 0) \,|\, S(0) = i\right) \\
&\quad = \quad \mathbb{P}\left(S(n) = -A \,|\, S(0) = i\right) \\
&\quad = \quad \binom{n}{(n-A-i)/2} \left(\frac{1}{2}\right)^n
\end{aligned}$$

provided $|A+i| \leq n$ and $n-A-i$ is even. Finally

$$f_{A,i}(n) = \binom{n}{(n+A-i)/2} \left(\frac{1}{2}\right)^n - \binom{n}{(n-A-i)/2} \left(\frac{1}{2}\right)^n.$$

Example

For an unbiased random walk with initial state $S(0) = 10$, what is the probability that $S(50) = 16$ and $S_{\text{min}}(50) > 0$?

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$$\begin{aligned} f_{16,10}(50) &= \mathbb{P}\left((S(50) = 16) \wedge (S_{\text{min}}(50) > 0) \,|\, S(0) = 10\right) \\ &= \left[\binom{50}{28} - \binom{50}{12}\right] 2^{-50} \\ &\approx 0.0787178 \end{aligned}$$

Another Example

For an unbiased random walk with initial state $S(0) = 10$, what is the probability that $S(50) \geq 16$ and $S_{\text{min}}(50) > 0$?

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$$p = \sum_{j=16}^{60} f_{j,10}(50) \approx 0.239791$$

Stopping Times

Define $\Omega_A = \min\{n \,|\, S(n) = A\}$ which is the first time that the random walk $S(n) = A$. This is called the **stopping time.**

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Suppose $A = 0$, then $\Omega_0 = n$ if and only if $S(0) = i > 0$, $S(n-1) = 1$, $S_{\min}(n-1) > 0$ and $X_n = -1$.

$$\begin{aligned}
&\mathbb{P}\left(\Omega_0 = n \,|\, S(0) = i\right) \\
&\quad = \mathbb{P}\left((X_n = -1) \wedge (S(n-1) = 1) \wedge (S_{\min}(n-1) > 0) \,|\, S(0) = i\right) \\
&\quad = \frac{1}{2}\mathbb{P}\left((S(n-1) = 1) \wedge (S_{\min}(n-1) > 0) \,|\, S(0) = i\right) \\
&\quad = \frac{1}{2} f_{1,i}(n-1).
\end{aligned}$$

Thus by spatial homogeneity

$$\mathbb{P}\left(\Omega_A = n \,|\, S(0) = i\right) = \frac{1}{2} f_{1,(i-A)}(n-1)$$

Paths

We can analyze the stopping time by thinking of the random walk as having two boundaries, one at 0 and another at A.

$p_{i \to A}$: any random walk $\{S(j)\}_{j=0}^{n}$ in the discrete interval $[0, A]$ starting at $i > 0$, terminating at A, and which avoids 0.

$P_{p_{i \to A}}$: the probability that the random walk starting at $S(0) = i$ follows $p_{i \to A}$.

$\mathcal{P}_A(i)$: the probability that a random walk which starts at $S(0) = i$ will achieve state $S = A$ while avoiding the state $S = 0$.

Determination of $\mathcal{P}_A(i)$

$$\begin{aligned}
\mathcal{P}_A(i) &= \sum_{p_{i\to A}} P_{p_{i\to A}} \\
&= \mathbb{P}\left(S(1) = i-1 \,|\, S(0) = i\right) \mathcal{P}_A(i-1) \\
&\quad + \mathbb{P}\left(S(1) = i+1 \,|\, S(0) = i\right) \mathcal{P}_A(i+1) \\
&= \frac{1}{2}\mathcal{P}_A(i-1) + \frac{1}{2}\mathcal{P}_A(i+1)
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This implies

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\end{aligned}$$

This implies

$$\mathcal{P}_A(i-1) - 2\mathcal{P}_A(i) + \mathcal{P}_A(i+1) = 0.$$

Remark: the resemblance of the left-hand side of this equation to the finite difference approximation of a second derivative suggests a form for $\mathcal{P}_A(i)$.

Probability of Exiting

Theorem

Suppose $S(k) = S(0) + \sum_{i=1}^{k} X_i$ *where the* X_i *for* $i = 1, 2, \ldots$ *are independent, identically distributed random variables taking on the values* ± 1*, each with probability* $p = 1/2$*. Suppose further that the boundaries at* 0 *and* A *are absorbing, then if* $0 \leq S(0) = i \leq A$

1. *the probability that the random walk achieves state* A *without achieving state* 0 *is* $\mathcal{P}_A(i) = i/A$*,*
2. *the probability that the random walk achieves state* 0 *without achieving state* A *is* $\mathcal{P}_0(i) = 1 - i/A$*.*

Proof

- Suppose $\mathcal{P}_A(i) = \alpha + \beta i$ where α and β are constants.
- Substituting into the difference equation yields

$$\alpha + \beta(i-1) - 2(\alpha + \beta i) + \alpha + \beta(i+1) = 0$$

  so $\mathcal{P}_A(i)$ solves the difference equation.
- Since $\mathcal{P}_A(0) = 0$, then $\alpha = 0$.
- Since $\mathcal{P}_A(A) = 1$, then $\beta = 1/A$.
- Consequently $\mathcal{P}_A(i) = i/A$.

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Before attempting to answer the second question, we pose the simpler question,

> *What is the expected exit time through either boundary* $A > 0$ *or boundary* 0*?*

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We make the following definitions:

- B: the set of boundary points, $B = \{0, A\}$.
- $\omega_{p_{i \to B}}$: the exit time of the random walk which starts at $S(0) = i$, where $0 \leq i \leq A$ and which follows path $p_{i \to B}$.
- $\Omega_B(i)$: the expected value of the exit time for a random walk which starts at $S(0) = i$, where $0 \leq i \leq A$.

Simple Answer

$$\begin{aligned}\Omega_B(i) &= \sum_{p_{i\to B}} P_{p_{i\to B}} \omega_{p_{i\to B}} \\ &= \frac{1}{2}(1+\Omega_B(i-1)) + \frac{1}{2}(1+\Omega_B(i+1))\end{aligned}$$

Since the path from $i \to B$ can be decomposed into paths from $(i-1) \to B$ and $(i+1) \to B$ with the addition of a single step, the expected value of the exit time of a random walk starting at i is one more than the expected value of a random walk starting at $i \pm 1$.

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Re-write the equation above as

$$\Omega_B(i-1) - 2\Omega_B(i) + \Omega_B(i+1) = -2.$$

Boundary Value Problem

Since the expected exit times for random walks which start at either $S(0) = 0$ or $S(0) = A$ are 0, then we must solve the following boundary value problem.

$$\begin{aligned}\Omega_B(i-1) - 2\Omega_B(i) + \Omega_B(i+1) &= -2 \\ \Omega_B(0) &= 0 \\ \Omega_B(A) &= 0\end{aligned}$$

for $i = 1, 2, \ldots, A-1$.

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for $i = 1, 2, \ldots, A-1$.

Hint: try a solution of the form $\Omega_B(i) = ai^2 + bi + c$ and determine the coefficients a, b, and c.

Mean Exit Time

Theorem

Suppose $S(k) = S(0) + \sum_{i=1}^{k} X_i$ *where the* X_i *for* $i = 1, 2, \ldots$ *are independent, identically distributed random variables taking on the values* ± 1*, each with probability* $p = 1/2$*. Suppose further that the boundaries at* 0 *and* A *are absorbing, then if* $0 \leq S(0) = i \leq A$ *the random walk intersects the boundary (*$S = 0$ *or* $S = A$*) after a mean number of steps given by the formula*

$$\Omega_B(i) = i(A - i).$$

Example

Suppose an unbiased random walk takes place on the discrete interval $\{0, 1, 2, \ldots, 10\}$ for which the boundaries at 0 and 10 are absorbing. As a function of the initial condition *i*, find the expected value of the exit time.

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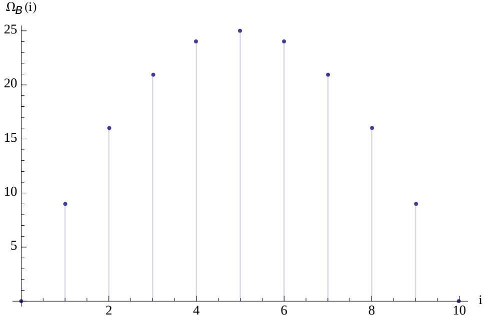
i	0	1	2	3	4	5	6	7	8	9	10
$\Omega_B(i)$	0	9	16	21	24	25	24	21	16	9	0



Main Question: Conditional Exit Time

Remark: now we are in a position to answer the original question of the determining the expected value of the exit time for a random walk which exits through state A while avoiding the absorbing boundary at 0.

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$$\begin{aligned}
\Omega_A(i) &= \frac{\displaystyle\sum_{p_{i\to A}} P_{p_{i\to A}} \omega_{p_{i\to A}}}{\displaystyle\sum_{p_{i\to A}} P_{p_{i\to A}}} \\
&= \frac{\displaystyle\sum_{p_{i\to A}} P_{p_{i\to A}} \omega_{p_{i\to A}}}{\mathcal{P}_A(i)} \\
\Omega_A(i)\mathcal{P}_A(i) &= \sum_{p_{i\to A}} P_{p_{i\to A}} \omega_{p_{i\to A}}
\end{aligned}$$

Decomposing the Walk (1 of 2)

The conditional exit time of a random walk starting in state i will be one more than the conditional exit times of random walks starting in states $i \pm 1$.

$$\begin{aligned}
\Omega_A(i) &= 1 + \frac{\frac{1}{2}\Omega_A(i-1)\mathcal{P}_A(i-1) + \frac{1}{2}\Omega_A(i+1)\mathcal{P}_A(i+1)}{\frac{1}{2}\mathcal{P}_A(i-1) + \frac{1}{2}\mathcal{P}_A(i+1)} \\
&= 1 + \frac{\frac{1}{2}\Omega_A(i-1)\mathcal{P}_A(i-1) + \frac{1}{2}\Omega_A(i+1)\mathcal{P}_A(i+1)}{\mathcal{P}_A(i)} \\
\Omega_A(i)\mathcal{P}_A(i) &= \mathcal{P}_A(i) + \frac{1}{2}\Omega_A(i-1)\mathcal{P}_A(i-1) + \frac{1}{2}\Omega_A(i+1)\mathcal{P}_A(i+1) \\
\Omega_A(i)\frac{i}{A} &= \frac{i}{A} + \frac{i-1}{2A}\Omega_A(i-1) + \frac{i+1}{2A}\Omega_A(i+1) \\
2i\Omega_A(i) &= 2i + (i-1)\Omega_A(i-1) + (i+1)\Omega_A(i+1) \\
-2i &= (i-1)\Omega_A(i-1) - 2i\Omega_A(i) + (i+1)\Omega_A(i+1)
\end{aligned}$$

Decomposing the Walk (2 of 2)

We now have the discrete form of an initial value problem:

$$\begin{aligned}(i-1)\Omega_A(i-1) - 2i\Omega_A(i) + (i+1)\Omega_A(i+1) &= -2i \\ \Omega_A(A) &= 0\end{aligned}$$

for $i = 1, 2, \ldots, A - 1$.

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for $i = 1, 2, \ldots, A-1$.

Assuming $\Omega_A(i) = ai^2 + bi + c$, determine the coefficients a, b, and c.

Question: what is an appropriate value for $\Omega_A(0)$?

Conditional Exit Time

Theorem

Suppose $S(k) = S(0) + \sum_{i=1}^{k} X_i$ *where the* X_i *for* $i = 1, 2, \ldots$ *are independent, identically distributed random variables taking on the values* ± 1*, each with probability* $p = 1/2$*. Suppose further that the boundary at* 0 *is absorbing. The random walk that avoids state* 0 *will stop the first time that* $S(n) = A$*. The expected value of the stopping time is*

$$\Omega_A(i) = \frac{1}{3}\left(A^2 - i^2\right), \quad \textit{for } i = 1, 2, \ldots, A.$$

Remark: If the random walk starts in state 0, since this state is absorbing the expected value of the exit time is infinity.

Example

Suppose an unbiased random walk takes place on the discrete interval $\{0, 1, 2, \dots, 10\}$ for which the boundary at 0 is absorbing. As a function of the initial condition *i*, find the expected value of the conditional exit time through state 10.

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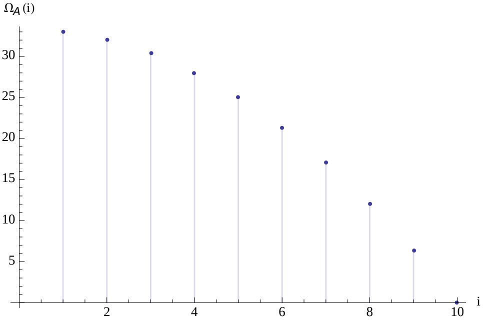
i	1	2	3	4	5	6	7	8	9	10
$\Omega_{10}(i)$	33	32	$\frac{91}{3}$	28	25	$\frac{64}{3}$	17	12	$\frac{19}{3}$	0



Stochastic Processes

Now we begin the development of *continuous* random walks by taking a limit of the previous discrete random walks.

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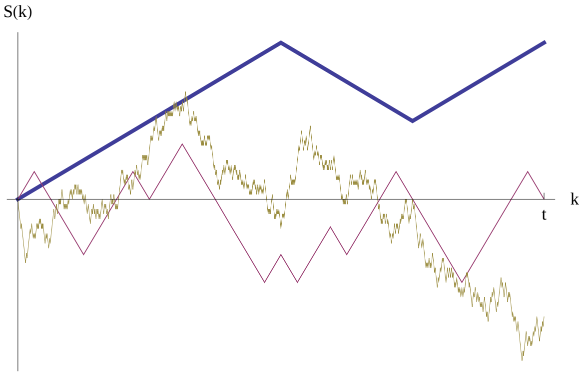
Assumptions:

- $S(0) = 0$.
- n independent steps take place equally spaced in time interval $[0, T]$.
- Probability of a step to the left/right is $1/2$.
- Size of a step is $\sqrt{T/n}$.

The continuous random walk is the limit as $n \to \infty$ of the sequence $\{S(jT/n)\}_{j=0}^{n}$.

Illustration



$$\mathbb{E}\left[S(T)\right] = \mathbb{E}\left[S(0)\right] + \sum_{j=1}^{n} \mathbb{E}\left[\sqrt{T/n}\, X_i\right] = 0$$

$$\mathbb{V}\left(S(T)\right) = \mathbb{V}\left(S(0)\right) + \sum_{j=1}^{n} \mathbb{V}\left(\sqrt{T/n}\, X_i\right) = n(T/n) = T$$

Brownian Motion/Wiener Process

The continuous limit of this random walk is denoted $W(t)$ and is called a **Wiener process.**

Brownian Motion/Wiener Process

The continuous limit of this random walk is denoted $W(t)$ and is called a **Wiener process.**

1. $W(t)$ is a continuous function of t,
2. $W(0) = 0$ with probability one,
3. Spatial homogeneity: if $W_0(t)$ represents a Wiener process for which the initial state is 0 and if $W_x(t)$ represents a Wiener process for which the initial state is x, then $W_x(t) = x + W_0(t)$.
4. Markov property: for $0 < s < t$ the conditional distribution of $W(t)$ depends on the value of $W(s) + W(t - s)$.
5. For each t, $W(t)$ is normally distributed with mean zero and variance t,
6. The changes in W in non-overlapping intervals of t are independent random variables with means of zero and variances equal to the lengths of the time intervals.

Quadratic Variation

Suppose $0 \leq t_1 < t_2$ and define $\Delta W_{[t_1,t_2]} = W(t_2) - W(t_1)$.

$$\begin{aligned}
\mathbb{V}\left(\Delta W_{[t_1,t_2]}\right) &= \mathbb{E}\left[(W(t_2) - W(t_1))^2\right] - \mathbb{E}\left[W(t_2) - W(t_1)\right]^2 \\
&= \mathbb{E}\left[(W(t_2))^2\right] + \mathbb{E}\left[(W(t_1))^2\right] - 2\mathbb{E}\left[W(t_1)W(t_2)\right] \\
&= t_2 + t_1 - 2\mathbb{E}\left[W(t_1)(W(t_2) - W(t_1) + W(t_1))\right] \\
&= t_2 + t_1 - 2\mathbb{E}\left[W(t_1)(W(t_2) - W(t_1))\right] \\
&\quad - 2\mathbb{E}\left[(W(t_1))^2\right] \\
&= t_2 + t_1 - 2t_1 \\
&= t_2 - t_1.
\end{aligned}$$

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&= t_2 + t_1 - 2t_1 \\
&= t_2 - t_1.
\end{aligned}$$

7. For a partition $0 = t_0 < t_1 < \cdots < t_n = t$ of interval $[0, t]$, the **quadratic variation** of $W(t)$ is denoted

$$[W, W](t) = \lim_{n \to \infty} \sum_{k=1}^{n} [W(t_k) - W(t_{k-1})]^2 = t.$$

Differential Wiener Process

We have seen that for $0 \leq t_1 < t_2$,

$$\mathbb{V}\left(\Delta W\right) = \mathbb{E}\left[\left(\Delta W\right)^2\right] = \Delta t.$$

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$$(dW(t))^2 = dt.$$

Theorem

The derivative dW/dt does not exist for any t.

Proof

Recall the limit definition of the derivative from calculus,

$$\frac{df}{dt} = \lim_{h \to 0} \frac{f(t+h) - f(t)}{h}.$$

Suppose $f(t)$ is a Wiener process $W(t)$. Since

$$\mathbb{E}\left[(W(t+h) - W(t))^2\right] = \mathbb{E}\left[|W(t+h) - W(t)|^2\right] = h$$

then on average $|W(t+h) - W(t)| \approx \sqrt{h}$, and therefore

$$\lim_{h \to 0} \frac{W(t+h) - W(t)}{h} \quad \text{does not exist.}$$

Integral of a Wiener Process

The **stochastic integral** of $f(x)$ on the interval $[0, t]$ is defined to be

$$\int_0^t f(\tau)\, dW(\tau) = Z(t) = \lim_{n\to\infty} \sum_{k=1}^{n} f(t_{k-1})\,(W(t_k) - W(t_{k-1}))$$

where $t_k = kt/n$.

Remarks:

- The function f is evaluated at the left-hand endpoint of each subinterval.
- Since $t_{k-1} < t_k$ the future value of $W(t_k)$ is still a random variable.

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- The function f is evaluated at the left-hand endpoint of each subinterval.
- Since $t_{k-1} < t_k$ the future value of $W(t_k)$ is still a random variable.

The stochastic integral can be expressed in its **differential form**:

$$dZ = f(t)\, dW(t).$$

Properties of Stochastic Integrals

Theorem

If f is a deterministic (non-random) function defined on $[0, t]$ for which

$$\int_0^t f(\tau)\, dW(\tau)$$

exists, then

$$\begin{aligned} \mathbb{E}\left[\int_0^t f(\tau)\, dW(\tau)\right] &= 0 \\ \mathbb{V}\left(\int_0^t f(\tau)\, dW(\tau)\right) &= \int_0^t (f(\tau))^2\, d\tau \end{aligned}$$

Proof (1 of 2)

$$\begin{aligned}
\mathbb{E}\left[\int_0^t f(\tau)\,dW(\tau)\right] &= \mathbb{E}\left[\lim_{n\to\infty}\sum_{k=1}^{n} f(t_{k-1})(W(t_k)-W(t_{k-1}))\right] \\
&= \lim_{n\to\infty}\sum_{k=1}^{n} f(t_{k-1})\mathbb{E}\left[(W(t_k)-W(t_{k-1}))\right] \\
&= \lim_{n\to\infty}\sum_{k=1}^{n} f(t_{k-1})\cdot 0 \\
&= 0
\end{aligned}$$

Proof (2 of 2)

$$\begin{aligned}
\mathbb{V}\left(\int_0^t f(\tau)\,dW(\tau)\right) &= \mathbb{V}\left(\lim_{n\to\infty}\sum_{k=1}^{n} f(t_{k-1})(W(t_k)-W(t_{k-1}))\right) \\
&= \lim_{n\to\infty}\sum_{k=1}^{n}(f(t_{k-1}))^2\mathbb{V}\left(W(t_k)-W(t_{k-1})\right) \\
&= \lim_{n\to\infty}\sum_{k=1}^{n}(f(t_{k-1}))^2(t_k-t_{k-1}) \\
&= \int_0^t (f(\tau))^2\,d\tau
\end{aligned}$$

Example

If $f(t) = \sin t$, then find

$$\mathbb{E}\left[\int_0^t \sin\tau \, dW(\tau)\right]$$

$$\mathbb{V}\left(\int_0^t \sin\tau \, dW(\tau)\right)$$

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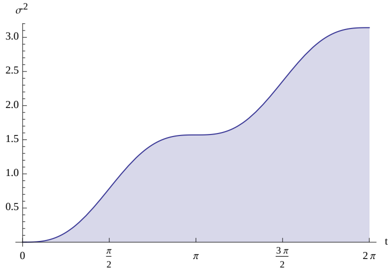
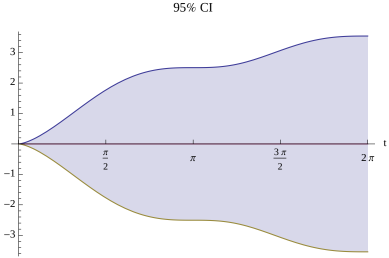
$$\begin{aligned}
\mathbb{E}\left[\int_0^t \sin\tau \, dW(\tau)\right] &= 0 \\
\mathbb{V}\left(\int_0^t \sin\tau \, dW(\tau)\right) &= \int_0^t \sin^2\tau \, d\tau \\
&= \frac{1}{2}\int_0^t (1 - \cos 2\tau) \, d\tau \\
&= \frac{1}{2}t - \frac{1}{4}\sin 2t
\end{aligned}$$

Illustration




95% CI

Riemann *vs.* Stochastic Integrals (1 of 5)

Suppose $f(t)$ is a continuously differentiable function on $[0, t]$ with $f(0) = 0$.

Then

$$\begin{aligned}\int_0^t f(\tau)\,df(\tau) &= \int_0^t f(\tau)\,f'(\tau)\,d\tau \\ &= \int_{f(0)}^{f(t)} u\,du \\ &= \frac{1}{2}u^2\Big|_{f(0)}^{f(t)} \\ &= \frac{1}{2}(f(t))^2\end{aligned}$$

Riemann *vs.* Stochastic Integrals (2 of 5)

Let $W(t)$ be the Wiener process ($W(0) = 0$). Evaluate

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$$\int_0^t W(\tau)\,dW(\tau) = \lim_{n\to\infty} \sum_{k=1}^{n} W(t_{k-1})\,[W(t_k) - W(t_{k-1})]$$

Riemann *vs.* Stochastic Integrals (3 of 5)

$$
\begin{aligned}
&\int_0^t W(\tau)\,dW(\tau) \\
&= \lim_{n\to\infty} \sum_{k=1}^{n} W(t_{k-1})\,[W(t_k) - W(t_{k-1})] \\
&= \lim_{n\to\infty} \sum_{k=1}^{n} \left[W(t_{k-1}) W(t_k) - W^2(t_{k-1}) \right] \\
&= \lim_{n\to\infty} \sum_{k=1}^{n} \left[-\frac{1}{2} W^2(t_{k-1}) + W(t_{k-1}) W(t_k) - \frac{1}{2} W^2(t_{k-1}) \right] \\
&= \lim_{n\to\infty} \left[-\frac{1}{2} \sum_{k=1}^{n} W^2(t_{k-1}) + \sum_{k=1}^{n} W(t_{k-1}) W(t_k) - \frac{1}{2} \sum_{k=1}^{n} W^2(t_{k-1}) \right]
\end{aligned}
$$

Riemann *vs.* Stochastic Integrals (4 of 5)

$$\begin{aligned}
\int_0^t W(\tau)\,dW(\tau) &= \lim_{n\to\infty}\left[-\frac{1}{2}\sum_{k=1}^{n} W^2(t_{k-1}) + \sum_{k=1}^{n} W(t_{k-1})W(t_k) - \frac{1}{2}\sum_{k=1}^{n} W^2(t_{k-1})\right] \\
&= \frac{1}{2}W^2(t) - \frac{1}{2}\lim_{n\to\infty}\left[\sum_{k=2}^{n+1} W^2(t_{k-1}) - 2\sum_{k=1}^{n} W(t_{k-1})W(t_k) + \sum_{k=1}^{n} W^2(t_{k-1})\right] \\
&= \frac{1}{2}W^2(t) - \frac{1}{2}\lim_{n\to\infty}\left[\sum_{k=1}^{n} W^2(t_k) - 2\sum_{k=1}^{n} W(t_{k-1})W(t_k) + \sum_{k=1}^{n} W^2(t_{k-1})\right] \\
&= \frac{1}{2}W^2(t) - \frac{1}{2}\lim_{n\to\infty}\sum_{k=1}^{n}\left[W(t_k) - W(t_{k-1})\right]^2
\end{aligned}$$

Riemann *vs.* Stochastic Integrals (5 of 5)

$$\begin{aligned}
\int_0^t W(\tau)\,dW(\tau) &= \frac{1}{2}W^2(t) - \frac{1}{2}\lim_{n\to\infty}\sum_{k=1}^{n}\left[W(t_k) - W(t_{k-1})\right]^2 \\
&= \frac{1}{2}W^2(t) - \frac{1}{2}\lim_{n\to\infty}\sum_{k=1}^{n}(t_k - t_{k-1}) \\
&= \frac{1}{2}W^2(t) - \frac{1}{2}t
\end{aligned}$$

Remarks:

- The stochastic integral of the Wiener process possesses an extra term ($-t/2$) which is not present in the Riemann integral.
- Integration by substitution (chain rule for derivatives) must operate differently for stochastic integrals.

ODE: Exponential Growth

Consider the familiar mathematical model for exponential growth of quantity P expressed in the form of an initial value problem:

$$\begin{aligned} \frac{dP}{dt} &= \mu P \\ P(0) &= P_0. \end{aligned}$$

If μ is a constant then $P(t) = P_0 e^{\mu t}$.

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If μ is a constant then $P(t) = P_0 e^{\mu t}$.

The ODE above can be written in the equivalent form:

$$\frac{dP}{P} = \mu \, dt.$$

If we let $Z = \ln P$ then the ODE becomes

$$dZ = \mu \, dt.$$

Stochastic Differential Equation (SDE)

- Suppose the deterministic model is disturbed by a random influence.
- Interpret $dW(t)$ as random "noise" with a mean of 0 and variance dt.
- Perturb dZ by adding a random process with mean zero and variance $\sigma^2\,dt$.

$$dZ = \mu\,dt + \sigma\,dW(t)$$

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This is mathematical model is an example of a **stochastic differential equation** of the type called a **generalized Wiener process**. The constant μ is called the **drift** and the constant σ is called the **volatility**. The solution to the SDE is

$$Z(t) = Z(0) + \mu t + \int_0^t \sigma\,dW(\tau) = Z(0) + \mu t + \sigma W(t).$$

Expectation and Variance

$$
\begin{aligned}
\mathbb{E}\left[Z(t) - Z(0)\right] &= \mu t \\
\mathbb{V}\left(Z(t) - Z(0)\right) &= \sigma^2 t
\end{aligned}
$$

Expectation and Variance

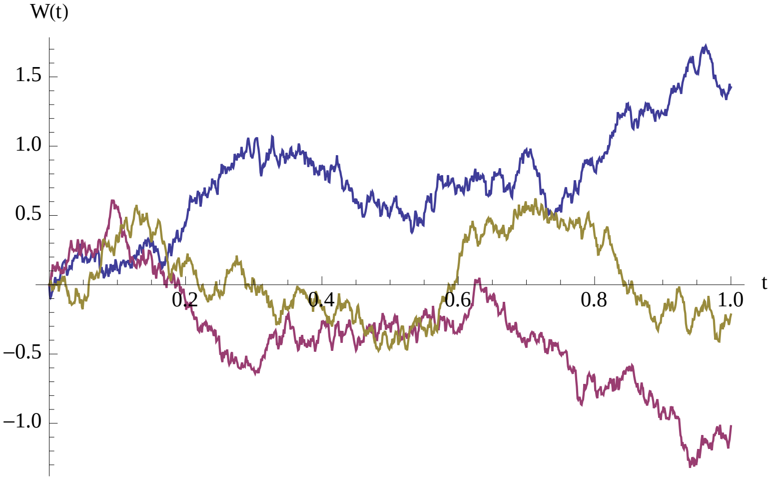
$$\begin{aligned}\mathbb{E}\left[Z(t)-Z(0)\right] &= \mu t \\ \mathbb{V}\left(Z(t)-Z(0)\right) &= \sigma^2 t\end{aligned}$$

In terms of numerical approximation,

$$\int_0^t dW(\tau) = W(t) = \lim_{n\to\infty} \sqrt{\frac{t}{n}} \sum_{j=1}^{n} X_j$$

where X_j is a standard normal random variable with mean 0 and variance 1.

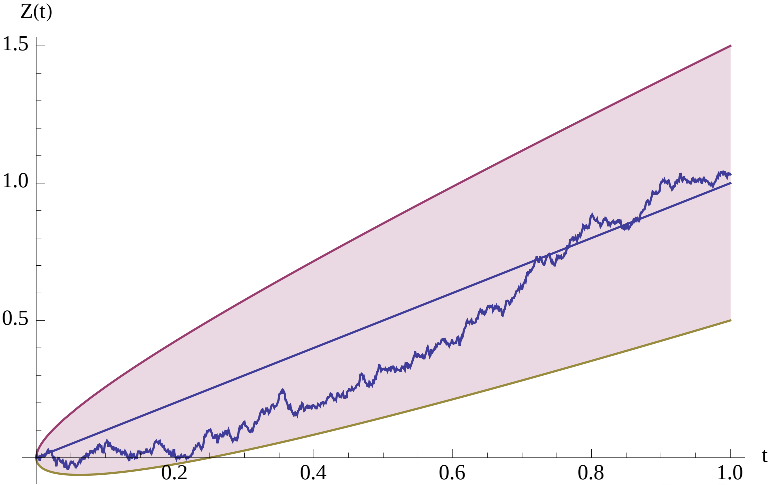
Continuous Random Walks



Several realizations of the continuous Wiener process.

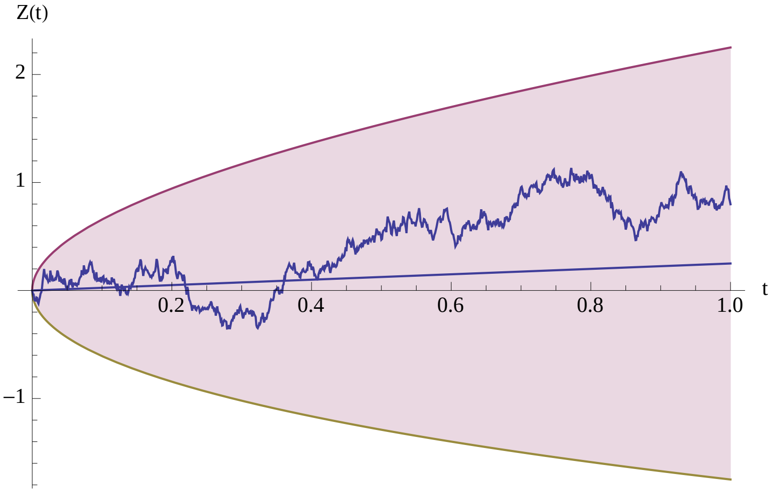
Example

Suppose the drift parameter is $\mu = 1$ and the volatility is $\sigma = 1/4$, then the expected value of the Wiener process is t and the standard deviation is $\sqrt{t}/4$.



Example

Suppose the drift parameter is $\mu = 1/4$ and the volatility is $\sigma = 1$, then the expected value of the Wiener process is $t/4$ and the standard deviation is $\sqrt{t}$.



Stopped Processes

Question: suppose $Z(t) = \mu t + \sigma W(t)$, then what is the expected value of the first time that $Z(t) = z$, a fixed value?

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Answer: define

$$T = \min_{t \geq 0}\{t \,:\, Z(t) = z\} = \min_{t \geq 0}\left\{t \,:\, W(t) = \frac{z - \mu t}{\sigma}\right\},$$

then

$$\begin{aligned}
\mathbb{E}\left[W(T)\right] &= 0 \\
\mathbb{E}\left[\frac{z - \mu T}{\sigma}\right] &= 0 \\
\frac{z - \mu \mathbb{E}\left[T\right]}{\sigma} &= 0 \\
\mathbb{E}\left[T\right] &= \frac{z}{\mu}.
\end{aligned}$$

Exit Probability

Let $m, M > 0$ and define

$$T_B = \min_{t \geq 0}\{(Z(t) = -m) \vee (Z(t) = M)\},$$

the first time the continuous random process $Z(t) = \mu t + \sigma W(t)$ achieves either state $-m$ or state M.

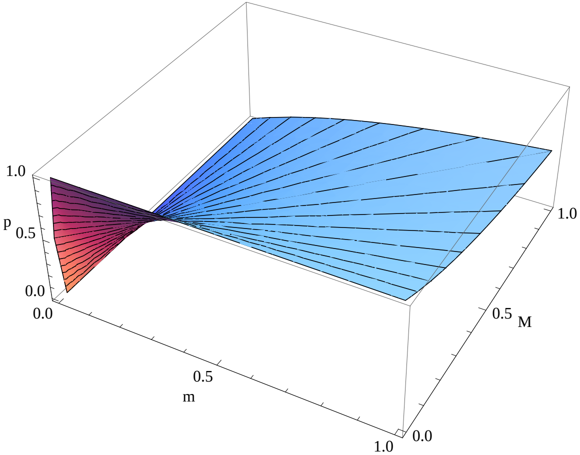
It can be shown that the probability that $Z(t)$ achieves state M before achieving state $-m$ is

$$p = \frac{1 - e^{2\mu m/\sigma^2}}{e^{-2\mu M/\sigma^2} - e^{2\mu m/\sigma^2}}.$$

Unbiased Walk ($\mu = 0$)

If the drift parameter is 0 then

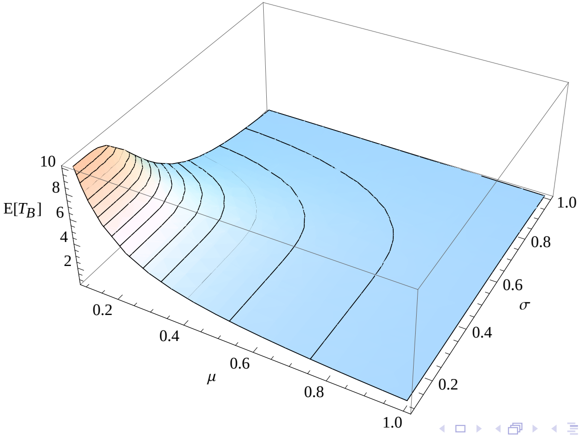
$$\lim_{\mu \to 0} p = \lim_{\mu \to 0} \frac{1 - e^{2\mu m/\sigma^2}}{e^{-2\mu M/\sigma^2} - e^{2\mu m/\sigma^2}} = \frac{m}{m+M}.$$



Expected Exit Time

The mean time for $Z(t)$ to reach either state $-m$ or M is given by

$$\mathbb{E}\left[T_B\right] = \frac{M\left(e^{2\mu m/\sigma^2} - 1\right) + m\left(e^{-2\mu M/\sigma^2} - 1\right)}{\mu\left(e^{2\mu m/\sigma^2} - e^{-2\mu M/\sigma^2}\right)}.$$



Simple Generalization

If the drift and volatility are functions of t then

$$dZ = \mu(t)\,dt + \sigma(t)\,dW(t).$$

and by apply the appropriate integral (Riemann or stochastic, as needed) we have

$$Z(t) = Z(0) + \int_0^t \mu(\tau)\,d\tau + \int_0^t \sigma(\tau)\,dW(\tau).$$

Itô Processes

A stochastic process of the form

$$dS = a(S,t)\,dt + b(S,t)\,dW(t)$$

is called an **Itô process.**

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Suppose $Z = \ln S$, then $dZ = dS/S$ (by the chain rule).

- Are the following two stochastic processes equivalent?

$$\begin{aligned} dS &= \mu S\,dt + \sigma S\,dW(t) \\ dZ &= \mu\,dt + \sigma\,dW(t) \end{aligned}$$

- Which equation is the better model for the price of a security?

Discussion

$$\begin{aligned} dS &= \mu S\,dt + \sigma S\,dW(t) \\ dZ &= \mu\,dt + \sigma\,dW(t) \end{aligned}$$

- As $S \to 0^+$ then $\mu S \to 0$ and $\sigma S \to 0$.

Discussion

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- As $S \to 0^+$ then $\mu S \to 0$ and $\sigma S \to 0$.
- First equation makes a suitable mathematical model for a stock price $S \geq 0$, in second equation Z could go negative.

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- As $S \to 0^+$ then $\mu S \to 0$ and $\sigma S \to 0$.
- First equation makes a suitable mathematical model for a stock price $S \geq 0$, in second equation Z could go negative.
- Second equation can be integrated, first cannot.
- The two equations are not equivalent because the chain rule does not apply to functions of stochastic quantities.

Itô's Lemma

Lemma (Itô's Lemma)

Suppose that the random variable X is described by the Itô process

$$dX = a(X,t)\,dt + b(X,t)\,dW(t)$$

where $dW(t)$ is a normal random variable. Suppose the random variable $Y = F(X,t)$. Then Y is described by the following Itô process.

$$dY = \left(a(X,t)F_X + F_t + \frac{1}{2}(b(X,t))^2 F_{XX} \right) dt + b(X,t)F_X dW(t)$$

Multivariable Form of Taylor's Theorem (1 of 3)

If $f(x)$ is an $(n+1)$-times differentiable function on an open interval containing x_0 then the function may be written as

$$\begin{aligned} f(x) &= f(x_0) + f'(x_0)(x - x_0) + \frac{f''(x_0)}{2!}(x - x_0)^2 \\ &\quad + \cdots + \frac{f^{(n)}(x_0)}{n!}(x - x_0)^n + \frac{f^{(n+1)}(\theta)}{(n+1)!}(x - x_0)^{n+1} \end{aligned} \tag{1}$$

The last term above is usually called the Taylor remainder formula and is denoted by R_{n+1}. The quantity θ lies between x and x_0. The other terms form a polynomial in x of degree at most n and can be used as an approximation for $f(x)$ in a neighborhood of x_0.

Multivariable Form of Taylor's Theorem (2 of 3)

Suppose the function $F(y,z)$ has partial derivatives up to order three on an open disk containing the point with coordinates (y_0, z_0). Define the function $f(x) = F(y_0 + xh, z_0 + xk)$ where h and k are chosen small enough that $(y_0 + h, z_0 + k)$ lie within the disk surrounding (y_0, z_0). Since f is a function of a single variable then we can use the single-variable form of Taylor's formula in Eq. (1) with $x_0 = 0$ and $x = 1$ to write

$$f(1) = f(0) + f'(0) + \frac{1}{2} f''(0) + R_3. \tag{2}$$

Using the multivariable chain rule for derivatives we have, upon differentiating $f(x)$ and setting $x = 0$,

$$\begin{aligned} f'(0) &= hF_y(y_0, z_0) + kF_z(y_0, z_0) && (3) \\ f''(0) &= h^2 F_{yy}(y_0, z_0) + 2hkF_{yz}(y_0, z_0) + k^2 F_{zz}(y_0, z_0). && (4) \end{aligned}$$

Multivariable Form of Taylor's Theorem (3 of 3)

We have made use of the fact that $F_{yz} = F_{zy}$ for this function under the smoothness assumptions. The remainder term R_3 contains only third-order partial derivatives of F evaluated somewhere on the line connecting the points (y_0, z_0) and $(y_0 + h, z_0 + k)$. Thus if we substitute Eqs. (3) and (4) into (2) we obtain

$$\begin{aligned}
\Delta F &= f(1) - f(0) \qquad (5) \\
&= F(y_0 + h, z_0 + k) - F(y_0, z_0) \\
&= R_3 + hF_y(y_0, z_0) + kF_z(y_0, z_0) \\
&\quad + \frac{1}{2}\left(h^2 F_{yy}(y_0, z_0) + 2hkF_{yz}(y_0, z_0) + k^2 F_{zz}(y_0, z_0)\right).
\end{aligned}$$

This last equation can be used to derive Itô's Lemma.

Proof (1 of 3)

Let X be a random variable described by an Itô process of the form

$$dX = a(X,t)\,dt + b(X,t)\,dW(t) \tag{6}$$

where $dW(t)$ is a normal random variable and a and b are functions of X and t. Let $Y = F(X,t)$ be another random variable defined as a function of X and t. Given the Itô process which describes X we will now determine the Itô process which describes Y.

Proof (2 of 3)

Using a Taylor series expansion for Y detailed in (5) we find

$$\begin{aligned}
\Delta Y &= F_X \Delta X + F_t \Delta t + \frac{1}{2} F_{XX} (\Delta X)^2 + F_{Xt} \Delta X \Delta t \\
&\quad + \frac{1}{2} F_{tt} (\Delta t)^2 + R_3 \\
&= F_X (a \Delta t + b\, dW(t)) + F_t \Delta t + \frac{1}{2} F_{XX} (a \Delta t + b\, dW(t))^2 \\
&\quad + F_{Xt} (a \Delta t + b\, dW(t)) \Delta t + \frac{1}{2} F_{tt} (\Delta t)^2 + R_3.
\end{aligned}$$

Proof (3 of 3)

Upon simplifying, the expression ΔX has been replaced by the discrete version of the Itô process. Thus as Δt becomes small

$$\Delta Y \approx F_X(a\,dt + b\,dW(t)) + F_t\,dt + \frac{1}{2!}F_{XX}b^2(dW(t))^2.$$

Using the relationship $(dW(t))^2 = dt$

$$\begin{aligned} \Delta Y &\approx F_X(a\,dt + b\,dW(t)) + F_t\,dt + \frac{1}{2!}F_{XX}b^2\,dt \\ &= (a\,F_X + F_t + \frac{1}{2}b^2F_{XX})\,dt + bF_X\,dW(t). \end{aligned}$$

Examples (1 of 2)

If $Z = \ln S$ and

$$dS = \mu S\,dt + \sigma S\,dW(t),$$

find the stochastic process followed by Z.

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If $Z = \ln S$ then

$$\begin{aligned} dZ &= \left(\mu S \left[\frac{1}{S} \right] + 0 + \frac{1}{2} \sigma^2 S^2 \left[-\frac{1}{S^2} \right] \right) dt + \sigma S \left(\frac{1}{S} \right) dW(t) \\ &= \left(\mu - \frac{\sigma^2}{2} \right) dt + \sigma\, dW(t) \end{aligned}$$

Examples (2 of 2)

If $S = e^Z$ and

$$dZ = \mu\, dt + \sigma\, dW(t),$$

find the stochastic process followed by S.

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If $S = e^Z$ and

$$dZ = \mu\, dt + \sigma\, dW(t),$$

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If $S = e^Z$ then

$$\begin{aligned} dS &= \left(\mu\left[e^Z\right] + 0 + \frac{1}{2}\sigma^2\left[e^Z\right]\right) dt + \sigma\left(e^Z\right) dW(t) \\ &= \left(\mu + \frac{\sigma^2}{2}\right) S\, dt + \sigma S\, dW(t) \end{aligned}$$

Langevin Equation

Consider the stochastic differential equation

$$\begin{aligned} dX(t) &= \mu X(t)\,dt + \sigma\,dW(t) \\ X(0) &= X_0. \end{aligned}$$

- Solve the initial value problem using Itô's lemma and the change of variables $F(X,t) = e^{-\mu t}X(t)$.
- Find the mean and variance of $X(t)$.

Solution (1 of 2)

Note that

$$\begin{aligned} F_X &= e^{-\mu t} \\ F_t &= -\mu e^{-\mu t} X \\ F_{XX} &= 0 \end{aligned}$$

which implies

$$\begin{aligned} dF &= \left(\mu F - \mu F + \frac{1}{2}\sigma^2(0)\right) dt + \sigma e^{-\mu t}\, dW(t) \\ &= \sigma e^{-\mu t}\, dW(t) \\ F(0) &= X_0. \end{aligned}$$

Integrating and using the initial condition yields

$$\begin{aligned} F(t) &= X_0 + \int_0^t \sigma e^{-\mu s}\, dW(s) \\ X(t) &= X_0 e^{\mu t} + \int_0^t \sigma e^{\mu(t-s)}\, dW(s). \end{aligned}$$

Solution (2 of 2)

Recall that

$$X(t) = X_0 e^{\mu t} + \int_0^t \sigma e^{\mu(t-s)}\, dW(s).$$

The mean and variance are respectively

$$\begin{aligned}\mathbb{E}\left[X(t)\right] &= X_0 e^{\mu t} \\ \mathbb{V}\left(X(t)\right) &= \int_0^t \left(\sigma e^{\mu(t-s)}\right)^2 dt = \frac{\sigma^2}{2\mu}\left(e^{2\mu t} - 1\right).\end{aligned}$$

Stock Example (1 of 2)

- Suppose we collect stock prices for $n + 1$ days: $\{S(0), S(1), \ldots, S(n)\}$.

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Stock Example (1 of 2)

- Suppose we collect stock prices for $n + 1$ days: $\{S(0), S(1), \ldots, S(n)\}$.
- Under the lognormal assumption $Z(i) = \ln S(i+1)/S(i)$ is a normal random variable.
- If the mean (drift) and variance (volatility squared) of Z are μ and σ^2 respectively, then

$$dZ = \mu \, dt + \sigma \, dW(t).$$

Stock Example (2 of 2)

Hence

$$Z(t) = Z(0) + \mu t + \int_0^t \sigma \, dW(\tau) = Z(0) + \mu t + \sigma W(t)$$

and

$$S(t) = S(0) e^{\mu t + \int_0^t \sigma \, dW(\tau)} = S(0) e^{\mu t + \sigma W(t)}.$$

Stock Example (2 of 2)

Hence

$$Z(t) = Z(0) + \mu t + \int_0^t \sigma \, dW(\tau) = Z(0) + \mu t + \sigma W(t)$$

and

$$S(t) = S(0) e^{\mu t + \int_0^t \sigma \, dW(\tau)} = S(0) e^{\mu t + \sigma W(t)}.$$

The mean and variance of $S(t)$ are

$$\begin{aligned} \mathbb{E}\left[S(t)\right] &= S(0) e^{(\mu + \sigma^2/2)t} \\ \mathbb{V}\left(S(t)\right) &= (S(0))^2 e^{(2\mu + \sigma^2)t} \left(e^{\sigma^2 t} - 1\right). \end{aligned}$$

Credits

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