Hedging

An Undergraduate Introduction to Financial Mathematics

J. Robert Buchanan

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Definition

Hedging is the practice of making a portfolio of investments less sensitive to changes in market variables.

There are various **hedging strategies.**

During this discussion we will explore, **Delta hedging** which attempts to keep the Δ of a portfolio nearly 0, so that the value of the portfolio is insensitive to changes in the price of a security.

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- The bank must sell the security for the strike price to the investor.

Covered Strategy (1 of 2)

A bank sells 100 European calls on a security where $S(0) = \$50$, $K = \$52$, $r = 2.5\%$, $T = 1/3$, and $\sigma = 22.5\%$. According to the Black-Scholes option pricing formula, $C = \$1.91965$.

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- The financial institution may borrow to purchase 100 shares of the security. This is called a **covered position.**
- At expiry the net cashflow is

$$(\min\{52, S(T)\} - (50 - 1.91965)e^{0.025/3}) \cdot 100.$$

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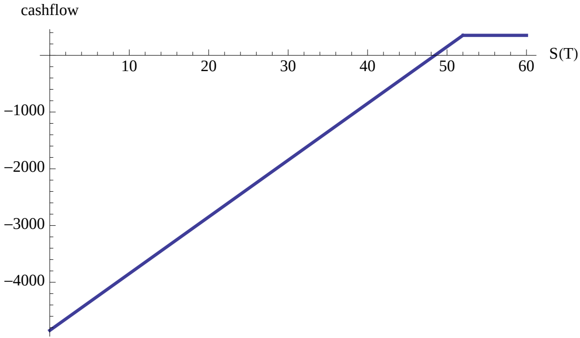
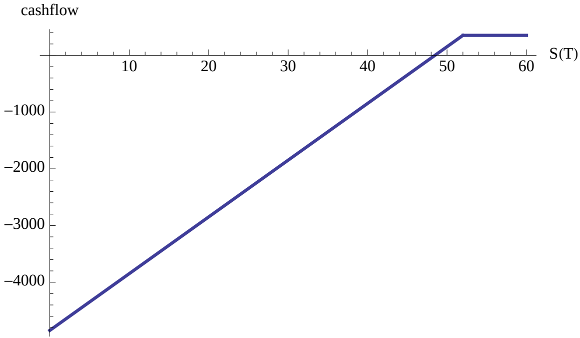
- Explain the meanings of the terms in the expression above.

Covered Strategy (2 of 2)



- If $S(T) \geq 52$ the cashflow is \$351.73.
- If $S(T) \approx 48.4827$ the cashflow is zero.
- If $S(T) = 46$ the cashflow is $-$\$248.27.

Covered Strategy (2 of 2)



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- If $S(T) = 46$ the cashflow is $-$\$248.27.
- In the worst case of $S(T) = 0$ the cashflow is $-$\$4848.27.

Naked Strategy (1 of 2)

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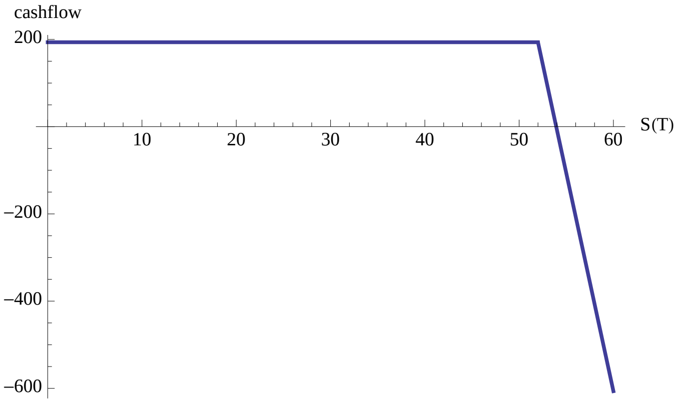
- The profit calculation above assumes the calls will be exercised. More generally the profit will be

$$(1.91965e^{0.025/3} + \min\{0, 52 - S(T)\}) \cdot 100.$$

Naked Strategy (2 of 2)



- If $S(T) \leq 52$ the profit is \$193.71.
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- If $S(T) = \$56$ the profit is $-\$206.43$.

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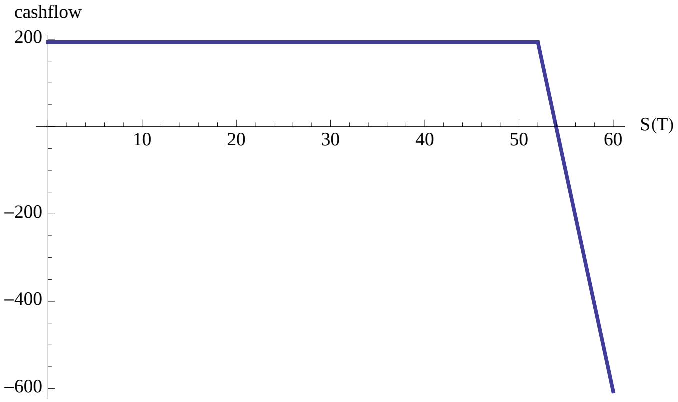

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- If $S(T) = \$56$ the profit is $-\$206.43$.
- The losses to the financial institution are unbounded as $S(T) \to \infty$.

Delta Hedging

Recall: If the value of a solution to the Black-Scholes PDE is F then $\Delta = \frac{\partial F}{\partial S}$ where S is the value of some security underlying F.

- If F is an option then for every unit change in the value of the underlying security, the value of the option changes by approximately Δ.
- A portfolio consisting of securities and options is called **Delta-neutral** if for every call option sold, Δ units of the security are bought.

Example of Delta Hedging

- Suppose $S = \$90$, $r = 10\%$, $\sigma = 50\%$, $K = \$95$, and $T = 1$.
- Under these conditions $w = 0.341866$, the value of a European call option is $C = 19.4603$ and Delta for the option is
$$\Delta = \frac{\partial C}{\partial S} = \Phi\,(w) = 0.633774.$$

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worth of the security.
- The financial institution will finance the security purchase by borrowing
$$\$5703.97 - \$1946.03 = \$3757.94.$$

Rebalancing a Portfolio

- If, after setting up the hedge, a financial institution does nothing else until expiry, this is called a “hedge and forget” strategy.
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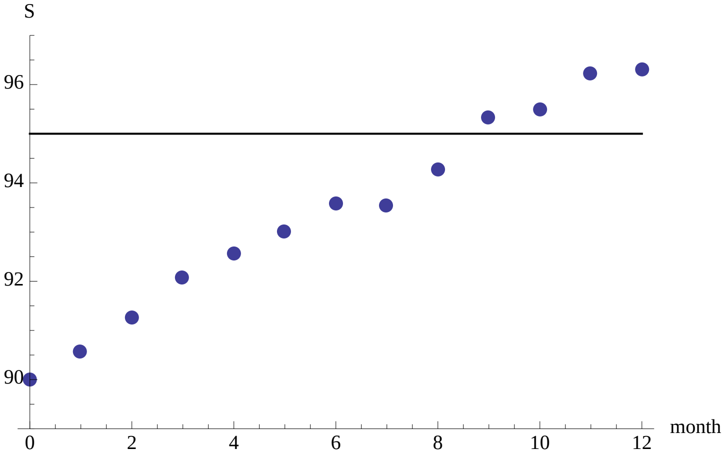
  This activity is known as **rebalancing** the portfolio.

Extended Example

Assume the value of the security follows the random walk shown below.



The European call option will be exercised since $S(1) > 95$.

End of First Month

- Suppose that $S(1/12) = \$90.56$.
- The number of options sold remains constant ($n = 100$), but the value of the options has changed.

$$C(S(1/12), 1/12) = 18.7736$$

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Gain/Loss on Security	$100 \cdot 0.633774(90.56 - 90)$	$=$	\$35.4913
Gain/Loss on Option	$100 \cdot (19.4603 - 18.7736)$	$=$	\$68.6672
Interest	$-3757.94(e^{0.10/12} - 1)$	$=$	$-$\$31.447
Profit			**\$72.7115**

Rebalancing at End of First Month

- Re-compute Δ using $S(1/12)$ and $t = 1/12$.

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$$(0.633774 - 0.629624) \cdot 100 = 0.415$$

  shares of the security at the current price of $S(1/12) = \$90.56$.
- This generates a cashflow of $(0.415)(90.56) = \$37.5824$.

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- For the second month the financial institution owns

$$100 \cdot 0.629624 = 62.9624$$

  shares of the security.

End of Second Month

- Suppose that $S(2/12) = \$91.25$.
- The value of the options has changed.

$$C(S(2/12), 2/12) = 18.1189$$

- Outstanding balance on loan,

$$3757.94 + 31.447 - 37.5824 = \$3751.80.$$

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Gain/Loss on Security	$100 \cdot 0.629624(91.25 - 90.56)$	$=$	\$43.4441
Gain/Loss on Option	$100 \cdot (18.7736 - 18.1189)$	$=$	\$65.47
Interest	$-3751.80(e^{0.10/12} - 1)$	$=$	$-$\$31.3956
Profit			**\$77.5185**

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  shares of the security at the current price of $S(2/12) = \$91.25$.
- This generates $(0.314)(91.25) = \$28.6525$ in cashflow which repays a portion of the loan.

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- For the third month the financial institution owns

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  shares of the security.

Comments on Profit/Loss

- The end-of-the-month profit (or loss) shows the amount of money the financial institution can take from (or must put in to) the portfolio.
- We will assume the financial institution can always borrow up to the value of the shares of the security in the portfolio.
- There are three cashflow streams in to/out of the portfolio:
  - Borrowing/Repaying the loan,
  - Purchasing/Selling the security,
  - Interest charges on outstanding balance of the loan.

End of Month Profit/Loss (1 of 2)

For each month i define the following quantities:

- S_i market price of the security.
- C_i European call option price.
- Δ_i Delta of call option.
- V_i value of portfolio of securities.
- I_i interest charge on loan.

End of Month Profit/Loss (2 of 2)

If n options are sold then

$$\begin{aligned} V_i &= n(\Delta_i S_i - C_i) \\ I_i &= V_{i-1}(e^{r/12} - 1). \end{aligned}$$

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The monthly profit/loss can be calculated as

$$\text{Profit/Loss} = V_i - V_{i-1} - nS_i(\Delta_i - \Delta_{i-1}) - I_i.$$

Profit/Loss Month by Month

i	S_i	C_i	Δ_i	V_i	**Loan**	**Profit/Loss**
0	90.00	19.4603	0.633774	3757.94	3757.94	
1	90.56	18.7736	0.629624	3824.52	3751.80	72.7115
2	91.25	18.1189	0.626484	3904.77	3754.55	77.5185
3	92.07	17.4887	0.624517	4001.06	3767.85	82.9794
4	92.55	16.5817	0.619217	4072.68	3750.34	89.1455
5	93.00	15.5825	0.613317	4145.60	3726.85	96.4049
6	93.59	14.5804	0.608689	4238.68	3714.72	105.209
7	93.54	13.0841	0.595797	4264.68	3625.22	115.496
8	94.28	11.9221	0.592296	4391.96	3622.54	129.956
9	95.32	10.7421	0.594234	4590.03	3671.33	149.284
10	95.50	8.7585	0.582942	4691.25	3594.21	178.336
11	96.21	6.5346	0.586126	4985.66	3654.93	233.697
12	96.32	1.3200	1.000000	9500.00	7671.95	497.323

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- At expiry the financial institution has a cumulative profit of \$1828.05.

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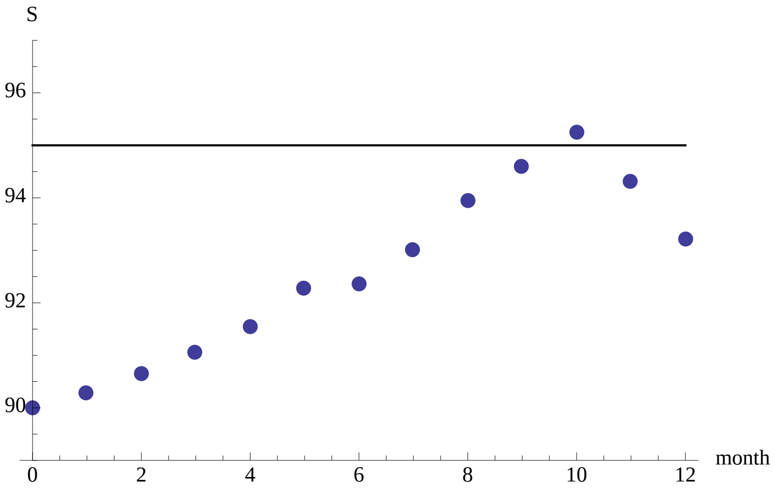
- Note that the firm's profit is the difference between the market value of the securities held and the outstanding balance on the loan.

$$9500.00 - 7671.95 = \$1828.05$$

Second Realization

Suppose the price of the security followed the path shown below.



First 10 Months

i	S_i	C_i	Δ_i	V_i	**Loan**	**Profit/Loss**
0	90.00	19.4603	0.633774	3757.94	3757.94	
1	90.29	18.6039	0.627264	3803.18	3730.61	72.5676
2	90.67	17.7571	0.621181	3856.53	3706.67	77.2959
3	91.04	16.8505	0.614619	3910.44	3677.95	82.6266
4	91.54	15.9615	0.608934	3978.04	3656.69	88.8581
5	92.28	15.1437	0.605507	4073.25	3655.66	96.2398
6	92.35	13.8346	0.59413	4103.33	3581.19	104.558
7	93.00	12.7643	0.588835	4199.74	3561.91	115.682
8	93.93	11.7156	0.587287	4344.82	3577.18	129.816
9	94.58	10.3068	0.582104	4474.85	3558.09	149.121
10	95.26	8.61914	0.578125	4645.31	3549.96	178.577

End of the 11th Month

- Suppose that $S(11/12) = \$94.30$.
- The value of the options has changed.

$$C(S(11/12), 11/12) = 5.46707$$

- Outstanding balance on loan, \$3138.58.

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- **Question**: if the financial institution liquidated their position by selling their stock and re-purchasing the options, would the financial institution make or lose money during the 11th month?

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- Outstanding balance on loan, $3138.58.
- **Question**: if the financial institution liquidated their position by selling their stock and re-purchasing the options, would the financial institution make or lose money during the 11th month?

Gain/Loss on Security	$100 \cdot 0.578125(94.30 - 95.26)$	$=$	$-\$55.50$
Gain/Loss on Option	$100 \cdot (8.61914 - 5.46707)$	$=$	$\$315.207$
Interest	$-3138.58(e^{0.10/12} - 1)$	$=$	$-\$26.2641$
Profit			**$233.443**

Rebalancing at End of 11th Month

- Re-compute Δ using $S(11/12)$ and $t = 11/12$.

$$\Delta = \Phi(w) = 0.53135$$

- The current value of Δ is smaller than the previous value. The financial institution may sell

$$(0.578125 - 0.53135) \cdot 100 = 4.6775$$

  shares of the security at the current price of $S(11/12) = 94.30$.
- This generates $(4.6775)(94.30) = \$441.088$ in cashflow which repays a portion of the loan.
- For the 12th month the financial institution owns

$$100 \cdot 0.53135 = 53.135$$

  shares of the security.

End of the 12th Month (Expiry)

i	S_i	C_i	Δ_i	V_i	**Loan**	**Profit/Loss**
0	90.00	19.4603	0.633774	3757.94	3757.94	
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10	95.26	8.61914	0.578125	4645.31	3549.96	178.577
11	94.30	5.46707	0.531350	4463.93	3138.58	230.000
12	93.20	0.00000	0.000000	0000.00	−1787.34	461.994

Unwinding the Firm's Position

- At expiry the financial institution has a cumulative profit of \$1787.34.
- Note that the firm's profit is the difference between the market value of the securities held and the outstanding balance on the loan.

$$0000.00 - (-1787.34) = \$1787.34$$

Self-Financing Portfolios

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A portfolio consisting of a sold call $C(S, t)$ and a long position in Δ shares of the underlying security is said to be **self-financing** if the profit/loss from a movement in stock price is zero.

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- Consider a sold K-strike European call on a security whose current value is S and purchased Δ shares of the security.
- Suppose the risk-free interest rate is r and the volatility of the security is σ.

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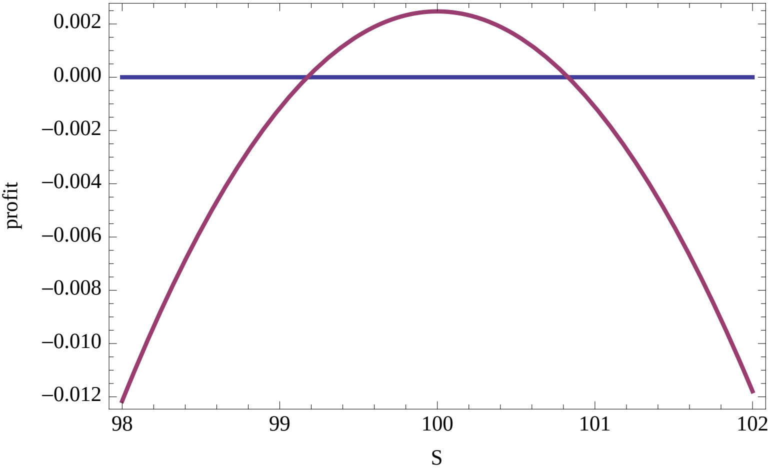
- Consider a sold K-strike European call on a security whose current value is S and purchased Δ shares of the security.
- Suppose the risk-free interest rate is r and the volatility of the security is σ.
- **Question**: what moves in security price result in a self-financing portfolio?

Let $K = 100$, $S = 100$, $r = 0.10$, $\sigma = 0.50$, and $T = 1$. The one-day profit curve resembles that shown below.



Numerical Example (2 of 2)

The self-financing one-day movements in the price of the security are the solutions to the equation:

$$V_1 - V_0 - S_1(\Delta_1 - \Delta_0) - V_0(e^{r/365} - 1) = 0.$$

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$$V_1 - V_0 - S_1(\Delta_1 - \Delta_0) - V_0(e^{r/365} - 1) = 0.$$

Numerically these roots are estimated to be

$$S_1 = 99.1748 \quad \text{and} \quad S_1 = 100.829.$$

Other Solutions to the Black-Scholes PDE

We have already seen that the values of European Call and Put options satisfy the Black-Scholes PDE.

$$rF = F_t + \frac{1}{2}\sigma^2 S^2 F_{SS} + rSF_S$$

Other financial instruments solve the PDE as well (but satisfy different boundary and/or final conditions than the options).

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Hence, the security itself and cash are both solutions to the Black-Scholes PDE.

Delta Neutral Portfolios

A portfolio consists of a short position in a European call option and a long position in the security (Delta hedged). Thus the net value $\mathcal{P}$ of the portfolio is

$$\mathcal{P} = C - (\Delta)S = C - \left.\frac{\partial C}{\partial S}\right|_{S_0} S.$$

$\mathcal{P}$ satisfies the Black-Scholes equation since C and S separately solve it. Thus Delta for the portfolio is

$$\frac{\partial \mathcal{P}}{\partial S} = \frac{\partial C}{\partial S} - \left.\frac{\partial C}{\partial S}\right|_{S_0}.$$

$\frac{\partial \mathcal{P}}{\partial S} \approx 0$ when $S \approx S(0)$.

Taylor Series for $\mathcal{P}$

$$
\begin{aligned}
\mathcal{P} &= \mathcal{P}_0 + \frac{\partial \mathcal{P}}{\partial t}(t - t_0) + \frac{\partial \mathcal{P}}{\partial S}(S - S_0) + \frac{\partial^2 \mathcal{P}}{\partial S^2}\frac{(S - S_0)^2}{2} + \cdots \\
\delta \mathcal{P} &= \Theta \delta t + \Delta \delta S + \frac{1}{2}\Gamma(\delta S)^2 + \cdots \\
\delta \mathcal{P} &\approx \Theta \delta t + \frac{1}{2}\Gamma(\delta S)^2
\end{aligned}
$$

- Θ is not stochastic and thus must be retained.
- What about Γ?

Gamma Neutral Portfolios

Recall: $\Gamma = \dfrac{\partial^2 F}{\partial S^2}$

- Since $\dfrac{\partial^2}{\partial S^2}(S) = 0$ a portfolio cannot be made gamma neutral if it contains only an option and its underlying security.
- Portfolio must include an additional component which depends non-linearly on S.
- Portfolio can include two (or more) different types of option dependent on the same security.

- Suppose a portfolio contains options with two different strike times written on the same security.
- A firm may sell European call options with a strike time three months and buy European call options on the same security with a strike time of six months.
- Let the number of the early options sold be n_e and the number of the later options purchased be n_l.

Example (1 of 5)

- Suppose a portfolio contains options with two different strike times written on the same security.
- A firm may sell European call options with a strike time three months and buy European call options on the same security with a strike time of six months.
- Let the number of the early options sold be n_e and the number of the later options purchased be n_l.

  The Gamma of the portfolio would be

$$\Gamma_{\mathcal{P}} = n_e \Gamma_e - n_l \Gamma_l,$$

  where Γ_e and Γ_l denote the Gammas of the earlier and later options respectively.

Example (2 of 5)

- Choose n_e and n_l so that $\Gamma_{\mathcal{P}} = 0$.
- Introduce the security so as to make the portfolio Delta neutral.
- **Question:** Why does changing the number of shares of the security in the portfolio affect Δ but not Γ?

Example (2 of 5)

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- Introduce the security so as to make the portfolio Delta neutral.
- **Question:** Why does changing the number of shares of the security in the portfolio affect Δ but not Γ?

With the proper values of n_e and n_l then

$$\delta \mathcal{P} \approx \Theta\, \delta t.$$

Example (3 of 5)

- Suppose $S = \$100$, $\sigma = 0.22$, and $r = 2.5\%$.
- An investment firm sells a European call option on this security with $T_e = 1/4$ and $K = \$102$.
- The firm buys European call options on the same security with the same strike price but with $T_l = 1/2$.
- Gamma of the 3-month option is $\Gamma_e = 0.03618$ and Gamma of the 6-month option is $\Gamma_l = 0.02563$.
- The portfolio is Gamma neutral in the first quadrant of $n_e n_l$-space where the equation

$$0.03618 n_e - 0.02563 n_l = 0$$

is satisfied.

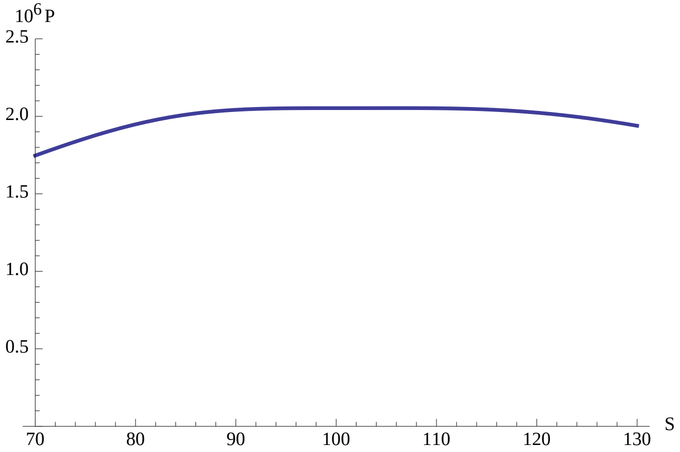
Example (4 of 5)

- Suppose $n_e = 100000$ of the three-month option were sold.
- Portfolio is Gamma neutral if $n_l = 141163$ six-month options are purchased.
- Before including the underlying security in the portfolio, the Delta of the portfolio is

$$\begin{aligned} n_e \Delta_e - n_l \Delta_l &= (100000)(0.4728) - (141163)(0.5123) \\ &= -25038. \end{aligned}$$

- Portfolio can be made Delta neutral if 25,038 shares of the underlying security are sold short.

Over a wide range of values for the underlying security, the value of the portfolio remains nearly constant.



Conclusion

- Rho and Vega can be used to hedge portfolios against changes in the interest rate and volatility respectively.
- We have assumed that the necessary options and securities could be bought or sold so as to form the desired hedge.
- If this is not true then a firm or investor may have to substitute a different, but related security or other financial instrument in order to set up the hedge.

Credits

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