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Question: what happens to the last probability as $n \to \infty$?
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- $X \in \{k/n : k = 0, 1, \ldots, n\}$ and $n \in \mathbb{N}$

**Question:** what happens to the last probability as $n \to \infty$?
Definition

A random variable $X$ has a **continuous distribution** (or **probability distribution function** or **probability density function**) if there exists a non-negative function $f : \mathbb{R} \rightarrow \mathbb{R}$ such that for an interval $[a, b]$ the

$$
P(a \leq X \leq b) = \int_{a}^{b} f(x) \, dx.
$$

The function $f$ must, in addition to satisfying $f(x) \geq 0$, have the following property,

$$
\int_{-\infty}^{\infty} f(x) \, dx = 1.
$$
Remark: the area under the curve may be interpreted as probability.
Definition

A continuous random variable $X$ is **uniformly distributed** in the interval $[a, b]$ (with $b > a$) if the probability that $X$ belongs to any subinterval of $[a, b]$ is equal to the length of the subinterval divided by $b - a$. 
Uniformly Distributed Continuous Random Variables

Definition

A continuous random variable \( X \) is **uniformly distributed** in the interval \([a, b]\) (with \( b > a \)) if the probability that \( X \) belongs to any subinterval of \([a, b]\) is equal to the length of the subinterval divided by \( b - a \).

**Question**: Assuming the PDF vanishes outside of \([a, b]\) and is constant on \([a, b]\), what is the PDF?
Definition

A continuous random variable $X$ is **uniformly distributed** in the interval $[a, b]$ (with $b > a$) if the probability that $X$ belongs to any subinterval of $[a, b]$ is equal to the length of the subinterval divided by $b - a$.

**Question**: Assuming the PDF vanishes outside of $[a, b]$ and is constant on $[a, b]$, what is the PDF?

**Answer**: $f(x) = \begin{cases} \frac{1}{b-a} & \text{if } a \leq x \leq b, \\ 0 & \text{otherwise.} \end{cases}$
Random variable $X$ is continuously and uniformly randomly distributed in the interval $[-5, 5]$. Find the probability that $-1 \leq X \leq 2$. 

$$P(-1 \leq X \leq 2) = \frac{2 - (-1)}{5 - (-5)} = \frac{3}{10}$$
Random variable $X$ is continuously and uniformly randomly distributed in the interval $[-5, 5]$. Find the probability that $-1 \leq X \leq 2$.

$$P(-1 \leq X \leq 2) = \frac{2 - (-1)}{5 - (-5)} = \frac{3}{10}$$
Random variable $X$ is continuously and uniformly randomly distributed in the interval $[-10, 10]$. Find the probability that $-3 \leq X \leq 1$ or $X > 7$. 

$$
\Pr((-3 \leq X \leq 1) \lor (X > 7)) = \Pr(-3 \leq X \leq 1) + \Pr(X > 7) = 
\frac{1}{20} \cdot (10 - (-3)) + \frac{1}{10} - 7 = \frac{7}{20}
$$
Random variable $X$ is continuously and uniformly randomly distributed in the interval $[-10, 10]$. Find the probability that $-3 \leq X \leq 1$ or $X > 7$.

\[
P((-3 \leq X \leq 1) \lor (X > 7)) = P(-3 \leq X \leq 1) + P(X > 7)
\]
\[
= \frac{1 - (-3)}{10 - (-10)} + \frac{10 - 7}{10 - (-10)}
\]
\[
= \frac{7}{20}
\]
Expected Value

Definition

The **expected value** or **mean** of a continuous random variable \( X \) with probability density function \( f(x) \) is

\[
\mathbb{E} [X] = \int_{-\infty}^{\infty} x f(x) \, dx.
\]
Example

Find the expected value of $X$, if $X$ is a continuously uniformly distributed random variable on the interval $[-10, 80]$.

Question: if $X$ is a uniformly distributed but integer-valued RV, what is its expected value?
Example

Find the expected value of \( X \), if \( X \) is a continuously uniformly distributed random variable on the interval \([-10, 80]\).

\[
E[X] = \int_{-\infty}^{\infty} x f(x) \, dx
\]

\[
= \int_{-10}^{80} \frac{x}{90} \, dx
\]

\[
= \left. \frac{x^2}{180} \right|_{-10}^{80}
\]

\[
= \frac{6400}{180} - \frac{100}{180}
\]

\[
= 35
\]
Example

Find the expected value of $X$, if $X$ is a continuously uniformly distributed random variable on the interval $[-10, 80]$.

$$\mathbb{E} [X] = \int_{-\infty}^{\infty} x f(x) \, dx$$

$$= \int_{-10}^{80} x \frac{x}{90} \, dx$$

$$= \left[ \frac{x^2}{180} \right]_{-10}^{80}$$

$$= \frac{6400}{180} - \frac{100}{180}$$

$$= \frac{35}{180}$$

**Question**: if $X$ is a uniformly distributed but integer-valued RV, what is its expected value?
Expected Value of a Function

**Definition**

The expected value of a function $g$ of a continuously distributed random variable $X$ which has probability density function $f$ is defined as

$$E [g(X)] = \int_{-\infty}^{\infty} g(x) f(x) \, dx,$$

provided the improper integral converges absolutely, i.e., $E [g(X)]$ is defined if and only if

$$\int_{-\infty}^{\infty} |g(x)| f(x) \, dx < \infty.$$
Find the expected value of $X^2$ if $X$ is continuously distributed on $[0, \infty)$ with probability density function $f(x) = e^{-x}$. 
Find the expected value of $X^2$ if $X$ is continuously distributed on $[0, \infty)$ with probability density function $f(x) = e^{-x}$.

$$\mathbb{E}[X^2] = \int_0^\infty x^2 e^{-x} \, dx$$

$$= \lim_{M \to \infty} \int_0^M x^2 e^{-x} \, dx$$

$$= \lim_{M \to \infty} \left[ -(x^2 + 2x + 2)e^{-x} \right]_0^M$$

$$= \lim_{M \to \infty} \left[ 2 - (M^2 + 2M + 2)e^{-M} \right]$$

$$= 2$$
A joint probability density for a pair of random variables, $X$ and $Y$, is a non-negative function $f(x, y)$ for which

$$\int_{-\infty}^{\infty} \int_{-\infty}^{\infty} f(x, y) \, dx \, dy = 1.$$
Definition

A joint probability density for a pair of random variables, $X$ and $Y$, is a non-negative function $f(x, y)$ for which

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Definition

If $X$ and $Y$ are continuous random variables with joint distribution $f(x, y)$ then the marginal density for $X$ is defined as the function

$$f_X(x) = \int_{-\infty}^{\infty} f(x, y) \, dy.$$
Joint and Marginal Distributions

**Definition**

A **joint probability density** for a pair of random variables, $X$ and $Y$, is a non-negative function $f(x, y)$ for which

$$
\int_{-\infty}^{\infty} \int_{-\infty}^{\infty} f(x, y) \, dx \, dy = 1.
$$

**Definition**

If $X$ and $Y$ are continuous random variables with joint distribution $f(x, y)$ then the **marginal density** for $X$ is defined as the function

$$
f_X(x) = \int_{-\infty}^{\infty} f(x, y) \, dy.
$$

**Remark:** a similar definition may be stated for the marginal density for $Y$. 
If the joint probability density of \( X \) and \( Y \) is given by

\[
f(x, y) = \begin{cases} 
  \frac{1}{\pi} & \text{if } x^2 + y^2 \leq 1, \\
  0 & \text{otherwise}
\end{cases}
\]

find the marginal probability density of \( X \).
Example

If the joint probability density of $X$ and $Y$ is given by

$$f(x, y) = \begin{cases} \frac{1}{\pi} & \text{if } x^2 + y^2 \leq 1, \\ 0 & \text{otherwise} \end{cases}$$

find the marginal probability density of $X$.

$$f_X(x) = \int_{-\infty}^{\infty} f(x, y) \, dy = \int_{-\sqrt{1-x^2}}^{\sqrt{1-x^2}} \frac{1}{\pi} \, dy = \begin{cases} \frac{2}{\pi} \sqrt{1-x^2} & \text{if } |x| \leq 1, \\ 0 & \text{otherwise} \end{cases}$$
Independence of Jointly Distributed RVs

Definition

Two continuous random variables are **independent** if and only if the joint probability density function factors into the product of the marginal densities of $X$ and $Y$. In other words $X$ and $Y$ are independent if and only if

$$f(x, y) = f_X(x)f_Y(y)$$

for all real numbers $x$ and $y$. 

J. Robert Buchanan

Normal Random Variables and Probability
The joint probability density function of $X$ and $Y$ is

$$f(x, y) = \begin{cases} \frac{1}{2}xy & \text{if } 0 \leq x \leq y \text{ and } 0 \leq y \leq 2 \\ 0 & \text{otherwise.} \end{cases}$$

Are $X$ and $Y$ independent?
The joint probability density function of $X$ and $Y$ is

$$f(x, y) = \begin{cases} \frac{1}{2}xy & \text{if } 0 \leq x \leq y \text{ and } 0 \leq y \leq 2 \\ 0 & \text{otherwise.} \end{cases}$$

Are $X$ and $Y$ independent?

No, since $f_X(x) = x - x^3/4$ if $0 \leq x \leq 2$ and $f_Y(y) = y^3/4$ if $0 \leq y \leq 2$. 

J. Robert Buchanan

Normal Random Variables and Probability
Consider the jointly distributed random variables 
\((X, Y) \in [0, \infty) \times [-2, 2]\) whose density is the function 
\[f(x, y) = \frac{1}{4e^x}.\] Find the mean of \(X + Y\).
$$\mathbb{E}[X + Y] = \int_0^\infty \int_{-2}^2 (x + y) \left( \frac{1}{4e^x} \right) \, dy \, dx$$

$$= \int_0^\infty \frac{1}{4} e^{-x} \int_{-2}^2 (x + y) \, dy \, dx$$

$$= \int_0^\infty \frac{1}{4} e^{-x} (4x) \, dx$$

$$= \int_0^\infty x e^{-x} \, dx$$

$$= \lim_{M \to \infty} \int_0^M x e^{-x} \, dx$$

$$= \lim_{M \to \infty} (1 - M e^{-M} - e^{-M})$$

$$= 1$$
Theorem

If $X_1, X_2, \ldots, X_k$ are continuous random variables with joint probability density $f(x_1, x_2, \ldots, x_k)$ then

$$E[X_1 + X_2 + \cdots + X_k] = E[X_1] + E[X_2] + \cdots + E[X_k].$$
Properties of the Expected Value

Theorem

If $X_1, X_2, \ldots, X_k$ are continuous random variables with joint probability density $f(x_1, x_2, \ldots, x_k)$ then

$$E[X_1 + X_2 + \cdots + X_k] = E[X_1] + E[X_2] + \cdots + E[X_k].$$

Theorem

Let $X_1, X_2, \ldots, X_k$ be pairwise independent random variables with joint density $f(x_1, x_2, \ldots, x_k)$, then

$$E[X_1 X_2 \cdots X_k] = E[X_1] E[X_2] \cdots E[X_k].$$
Variance and Standard Deviation

Definition

If $X$ is a continuously distributed random variable with probability density function $f(x)$, the variance of $X$ is defined as

$$\text{Var}(X) = \mathbb{E}[(X - \mu)^2] = \int_{-\infty}^{\infty} (x - \mu)^2 f(x) \, dx,$$

where $\mu = \mathbb{E}[X]$. The standard deviation of $X$ is

$$\sigma(X) = \sqrt{\text{Var}(X)}.$$
Variance and Standard Deviation

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where $\mu = \mathbb{E}[X]$. The standard deviation of $X$ is

$$\sigma(X) = \sqrt{\text{Var}(X)}.$$

Theorem

Let $X$ be a random variable with probability density $f$ and mean $\mu$, then $\text{Var}(X) = \mathbb{E}[X^2] - \mu^2$. 
Example

Suppose $X$ is continuously distributed on $[0, \infty)$ with probability density function $f(x) = e^{-x}$. Find $\mathbb{V}(X)$. 

\[
\mathbb{V}(X) = \mathbb{E}[X^2] - \left( \mathbb{E}[X] \right)^2 = \int_0^\infty x^2 e^{-x} \, dx - \left( \int_0^\infty x e^{-x} \, dx \right)^2 = 2 - \left( \frac{1}{2} \right)^2 = 1
\]
Example

Suppose $X$ is continuously distributed on $[0, \infty)$ with probability density function $f(x) = e^{-x}$. Find $\mathbb{V}(X)$.

\[
\mathbb{V}(X) = \mathbb{E}[X^2] - (\mathbb{E}[X])^2
\]

\[
= \int_0^\infty x^2 e^{-x} \, dx - \left( \int_0^\infty x e^{-x} \, dx \right)^2
\]

\[
= 2 - \left( \int_0^\infty x e^{-x} \, dx \right)^2
\]

\[
= 2 - (1)^2
\]

\[
= 1
\]
Theorem

Let $X$ be a continuous random variable with probability density $f(x)$ and let $a, b \in \mathbb{R}$, then

$$\text{V}(aX + b) = a^2 \text{V}(X).$$
Properties of Variance

Theorem

Let $X$ be a continuous random variable with probability density $f(x)$ and let $a, b \in \mathbb{R}$, then

$$\mathbb{V}(aX + b) = a^2 \mathbb{V}(X).$$

Theorem

Let $X_1, X_2, \ldots, X_k$ be pairwise independent continuous random variables with joint probability density $f(x_1, x_2, \ldots, x_k)$, then

$$\mathbb{V}(X_1 + X_2 + \cdots + X_k) = \mathbb{V}(X_1) + \mathbb{V}(X_2) + \cdots + \mathbb{V}(X_k).$$
Assumption: any characteristic of an object subject to a large number of independently acting forces typically takes on a normal distribution.
**Assumption:** any characteristic of an object subject to a large number of independently acting forces typically takes on a normal distribution.

- We will develop the normal probability density function from the probability function for the binomial random variable.
Assumption: any characteristic of an object subject to a large number of independently acting forces typically takes on a normal distribution.

- We will develop the normal probability density function from the probability function for the binomial random variable.
- Recall that if $X$ is a binomial random variable of $n$ trials and probability of success on a single trial of $p$, then for $x \in \{0, 1, \ldots, n\}$:

$$
\Pr(X = x) = \frac{n!}{x!(n-x)!} p^x (1 - p)^{n-x}
$$
Thought experiment: Imagine standing at the origin of the number line and for each tick of a clock taking a step to the left or the right. In the long run where will you stand?
Thought experiment: Imagine standing at the origin of the number line and for each tick of a clock taking a step to the left or the right. In the long run where will you stand?

Assumptions:

1. \( n \) steps/ticks,
2. random walk takes place during time interval \([0, t]\), which implies a “tick” lasts \( \Delta t = t/n \),
3. on each tick move a distance \( \Delta x > 0 \),
4. \( n(\Delta x)^2 = 2kt \) or equivalently \( (\Delta x)^2 = 2k(\Delta t) \), for some positive constant \( k \),
5. probability of moving left/right is \( 1/2 \),
6. all steps are independent.
Simulate taking 400 steps, 10 different trials.

Average position on last step: $\bar{x} = 1.4$ with $\sigma(x) \approx 13.5$. 
Take a Few Steps

Suppose $r$ out of $n$ steps ($0 \leq r \leq n$) have been to the right.

**Question:** Where are you?
Suppose $r$ out of $n$ steps ($0 \leq r \leq n$) have been to the right.

**Question:** Where are you?

$$(r - (n - r))\Delta x = (2r - n)\Delta x = m\Delta x$$

**Question:** What is the probability of standing there?
Suppose \( r \) out of \( n \) steps (\( 0 \leq r \leq n \)) have been to the right.

**Question:** Where are you?

\[(r - (n - r)) \Delta x = (2r - n) \Delta x = m \Delta x\]

**Question:** What is the probability of standing there?

\[
P(X = m \Delta x) = P(X = (2r - n) \Delta x)
= \binom{n}{r} \left( \frac{1}{2} \right)^r \left( \frac{1}{2} \right)^{n-r}
= \frac{n!}{r!(n-r)!} \left( \frac{1}{2} \right)^n
= \frac{n!}{\left( \frac{1}{2} \right)^n \binom{n}{r} \left( \frac{1}{2} \right)^{n-r}}
= \frac{n!}{\left( \frac{1}{2} \right)^n \binom{n}{r} \left( \frac{1}{2} \right)^{n-r}}
= \frac{n!}{\left( \frac{1}{2} \right)^n \binom{n}{r} \left( \frac{1}{2} \right)^{n-r}}
= \frac{n! (\frac{1}{2})^n}{(\frac{1}{2})(n+m)! (\frac{1}{2})(n-m)!}
\]
Each step is a Bernoulli experiment with outcomes $\Delta x$ and $-\Delta x$.

Questions:

- What is the expected value of a single step?

- What is the variance in the outcomes?
Each step is a Bernoulli experiment with outcomes $\Delta x$ and $-\Delta x$.

**Questions:**

- What is the expected value of a single step?

$$E[X] = 0$$

- What is the variance in the outcomes?
Each step is a Bernoulli experiment with outcomes $\Delta x$ and $-\Delta x$.

**Questions:**

- What is the expected value of a single step?

$$\mathbb{E} [X] = 0$$

- What is the variance in the outcomes?

$$\mathbb{V} (X) = (\Delta x)^2$$
Questions: after $n$ steps,

- What is the expected value of where you stand?

- What is the variance in final position?
Questions: after $n$ steps,

- What is the expected value of where you stand?
  \[ E \left[ \sum_{i=1}^{n} X \right] = n E \left[ X \right] = 0 \]

- What is the variance in final position?
  \[ V \left( n \sum_{i=1}^{n} X \right) = n V \left( X \right) = n (\Delta x)^2 = 2 \]
Questions: after $n$ steps,

- What is the expected value of where you stand?
  
  $\mathbb{E} \left[ \sum_{i=1}^{n} X \right] = n \mathbb{E} [X] = 0$

- What is the variance in final position?
  
  $\mathbb{V} \left( \sum_{i=1}^{n} X \right) = n \mathbb{V} (X) = n(\Delta x)^2 = 2k t$
Stirling’s formula approximates $n!$ for large $n$.

$$n! \approx \sqrt{2\pi} e^{-n} n^{n+1/2}$$

<table>
<thead>
<tr>
<th>$n$</th>
<th>$n!$</th>
<th>$\sqrt{2\pi} e^{-n} n^{n+1/2}$</th>
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</thead>
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<tr>
<td>5</td>
<td>120</td>
<td>118.019</td>
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<tr>
<td>10</td>
<td>$3.6288 \times 10^6$</td>
<td>$3.5987 \times 10^6$</td>
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<td>15</td>
<td>$1.30767 \times 10^{12}$</td>
<td>$1.30043 \times 10^{12}$</td>
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<td>20</td>
<td>$2.4329 \times 10^{18}$</td>
<td>$2.42279 \times 10^{18}$</td>
</tr>
<tr>
<td>30</td>
<td>$2.65253 \times 10^{32}$</td>
<td>$2.64517 \times 10^{32}$</td>
</tr>
</tbody>
</table>
$n! \approx \sqrt{2\pi}e^{-n}n^{n+1/2}$

Replace all the factorials with Stirling’s Formula.
Stirling’s Formula (3 of 3)

\[ n! \approx \sqrt{2\pi}e^{-n}n^{n+1/2} \]

Replace all the factorials with Stirling’s Formula.

\[
\mathbb{P}(X = m\Delta x) = \frac{n! \left(\frac{1}{2}\right)^n}{\left(\frac{1}{2}(n + m)\right)! \left(\frac{1}{2}(n - m)\right)!} \\
= \frac{\sqrt{2\pi}e^{-n}n^{n+1/2} \left(\frac{1}{2}\right)^n}{\sqrt{2\pi}e^{-\frac{n+m}{2}} \left(\frac{1}{2}(n + m)\right)^{\frac{n+m+1}{2}} \sqrt{2\pi}e^{-\frac{n-m}{2}} \left(\frac{1}{2}(n - m)\right)^{\frac{n-m+1}{2}}} \\
= \frac{2}{\sqrt{2n\pi}} \left(1 + \frac{m}{n}\right)^{-\frac{m}{2}} \left(1 - \frac{m}{n}\right)^{\frac{m}{2}} \left(1 - \frac{m^2}{n^2}\right)^{-\frac{n+1}{2}}
\]
Since \( m = x/\Delta x \) and \( n = t/\Delta t \),

\[
P(X = m\Delta x) = \frac{2}{\sqrt{2n\pi}} \left(1 + \frac{m}{n}\right)^{-m/2} \left(1 - \frac{m}{n}\right)^{m/2} \left(1 - \frac{m^2}{n^2}\right)^{-(n+1)/2}
\]

\[
= \frac{2\sqrt{\Delta t}}{\sqrt{2\pi t}} \left(1 + \frac{x\Delta t}{t\Delta x}\right)^{-\frac{x}{2\Delta x}} \left(1 - \frac{x\Delta t}{t\Delta x}\right)^{\frac{x}{2\Delta x}} \left(1 - \left[\frac{x\Delta t}{t\Delta x}\right]^2\right)^{-\frac{1+t/\Delta t}{t^2}}
\]

\[
= \frac{\Delta x}{\sqrt{k\pi t}} \left[1 + \frac{x\Delta x}{2kt}\right]^{-\frac{x}{2\Delta x}} \left[1 - \frac{x\Delta x}{2kt}\right]^{\frac{x}{2\Delta x}} \left[1 - \left(\frac{x\Delta x}{2kt}\right)^2\right]^{-\frac{kt}{(\Delta x)^2} - \frac{1}{2}}
\]

since \((\Delta x)^2 = 2k\Delta t\).
As \( \Delta x \to 0 \), the probability of standing at exactly one, specific location becomes 0.

Instead we must change our thinking and ask for

\[
P((m - 1)\Delta x < X < (m + 1)\Delta x) \approx 2(\Delta x)f(x, t).
\]
As $\Delta x \to 0$, the probability of standing at exactly one, specific location becomes 0.

Instead we must change our thinking and ask for

$$P \left( (m - 1)\Delta x < X < (m + 1)\Delta x \right) \approx 2(\Delta x)f(x, t).$$

$$f(x, t) = \frac{1}{2\sqrt{k\pi t}} \lim_{\Delta x \to 0} \left[ 1 + \frac{x \Delta x}{2kt} \right]^{\frac{x}{2\Delta x}} \left[ 1 - \frac{x \Delta x}{2kt} \right]^{\frac{x}{2\Delta x}} \left[ 1 - \left( \frac{x \Delta x}{2kt} \right)^2 \right]^{-\frac{kt}{(\Delta x)^2} - \frac{1}{2}}$$

$$= \frac{1}{2\sqrt{k\pi t}} \left( e^{\frac{x}{2kt}} \right)^{-\frac{x}{2}} \left( e^{-\frac{x}{2kt}} \right)^{\frac{x}{2}} \left( e^{-\frac{x^2}{4kt^2}} \right)^{-kt}$$

$$= \frac{1}{2\sqrt{k\pi t}} e^{-\frac{x^2}{4kt}}$$
Suppose \[ S = \int_{-\infty}^{\infty} \frac{1}{2\sqrt{k\pi t}} e^{-\frac{x^2}{4kt}} \, dx \]

Then

\[
S^2 = \int_{-\infty}^{\infty} \frac{1}{2\sqrt{k\pi t}} e^{-\frac{x^2}{4kt}} \, dx \int_{-\infty}^{\infty} \frac{1}{2\sqrt{k\pi t}} e^{-\frac{y^2}{4kt}} \, dy
\]

\[
= \frac{1}{4k\pi t} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} e^{-(x^2+y^2)/4kt} \, dx \, dy
\]

\[
= \frac{1}{4k\pi t} \int_{0}^{2\pi} \int_{0}^{\infty} r \, e^{-r^2/4kt} \, dr \, d\theta
\]

\[
= 1
\]
The graph of the PDF resembles:
For a fixed value of $t$, the graph of the PDF resembles:
Expected Value and Variance

If $X$ is a continuously distributed random variable with PDF:

$$f(x, t) = \frac{1}{2\sqrt{k\pi t}} e^{-\frac{x^2}{4kt}}$$

then
If \( X \) is a continuously distributed random variable with PDF:

\[
f(x, t) = \frac{1}{2\sqrt{k\pi t}} e^{-\frac{x^2}{4kt}}
\]

then

\[
\mathbb{E}[X] = \int_{-\infty}^{\infty} \frac{x}{2\sqrt{k\pi t}} e^{-\frac{x^2}{4kt}} \, dx = 0
\]

and
If $X$ is a continuously distributed random variable with PDF:

$$f(x, t) = \frac{1}{2\sqrt{k\pi t}} e^{-\frac{x^2}{4kt}}$$

then

$$\mathbb{E}[X] = \int_{-\infty}^{\infty} \frac{x}{2\sqrt{k\pi t}} e^{-\frac{x^2}{4kt}} \, dx = 0$$

and

$$\text{Var}(X) = \int_{-\infty}^{\infty} \frac{x^2}{2\sqrt{k\pi t}} e^{-\frac{x^2}{4kt}} \, dx - (\mathbb{E}[X])^2 = 2kt$$

and thus $2kt = \sigma^2$ and we express the PDF for a **normally distributed random variable** with mean $\mu$ and variance $\sigma^2$ as

$$f(x) = \frac{1}{\sigma\sqrt{2\pi}} e^{-\frac{(x-\mu)^2}{2\sigma^2}}.$$
When \( \mu = 0 \) and \( \sigma = 1 \), the PDF \( \phi(x) = \frac{1}{\sqrt{2\pi}} e^{-\frac{x^2}{2}} \) is called the standard normal density.
When $\mu = 0$ and $\sigma = 1$, the PDF $\phi(x) = \frac{1}{\sqrt{2\pi}} e^{-\frac{x^2}{2}}$ is called the standard normal density.

The **cumulative distribution function** (CDF) $\Phi(x)$ is defined as

$$\Phi(x) = \mathbb{P}(X < x) = \int_{-\infty}^{x} \frac{1}{\sqrt{2\pi}} e^{-\frac{u^2}{2}} \, du.$$
When $\mu = 0$ and $\sigma = 1$, the PDF $\phi(x) = \frac{1}{\sqrt{2\pi}} e^{-\frac{x^2}{2}}$ is called the standard normal density.

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Remarks:

- Pay close attention of the case of the symbol used for the probability density and cumulative distribution of $X$.
- The values of $\Phi(x)$ can be produced by many scientific calculators or by looking them up in printed tables.
Theorem

If $X$ is a normally distributed random variable with expected value $\mu$ and variance $\sigma^2$, then $Z = \frac{X - \mu}{\sigma}$ is normally distributed with an expected value of zero and a variance of one.
Suppose the random variables $X_1, X_2, \ldots, X_n$:

1. are pairwise independent but not necessarily identically distributed,
2. have means $\mu_1, \mu_2, \ldots, \mu_n$ and variances $\sigma_1^2, \sigma_2^2, \ldots, \sigma_n^2$,

and we define a new random variable $Y_n$ as

$$Y_n = \frac{\sum_{i=1}^{n} (X_i - \mu_i)}{\sqrt{\sum_{i=1}^{n} \sigma_i^2}}.$$
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A Central Limit Theorem due to Liapounov implies that $Y_n$ has the standard normal distribution.
Suppose that the infinite collection \( \{X_i\}_{i=1}^{\infty} \) of random variables are pairwise independent and that for each \( i \in \mathbb{N} \) we have \( \mathbb{E} [ |X_i - \mu_i|^3] < \infty \). If in addition,

\[
\lim_{n \to \infty} \frac{\sum_{i=1}^{n} \mathbb{E} [ |X_i - \mu_i|^3]}{(\sum_{i=1}^{n} \sigma_i^2)^{3/2}} = 0
\]

then for any \( x \in \mathbb{R} \)

\[
\lim_{n \to \infty} \mathbb{P} (Y_n \leq x) = \Phi (x)
\]

where random variable \( Y_n \) is defined as above.
Suppose the annual snowfall in Millersville, PA is 14.6 inches with a standard deviation of 3.2 inches and is normally distributed. Snowfall amounts in different years are independent. What is the probability that the sum of the snowfall amounts in the next two years will exceed 30 inches?
Solution: If \( X \) represents the random variable standing for the snowfall received in Millersville, PA for one year then \( X + X \) is the random variable representing the snowfall of two years. The random variable \( X + X \) has mean \( \mu = 2(14.6) = 29.2 \) inches and variance \( \sigma^2 = (3.2)^2 + (3.2)^2 = 20.48 \).

\[
\begin{align*}
\mathbb{P}(X + X > 30) &= \mathbb{P}\left(Z > \frac{30 - 29.2}{\sqrt{20.48}}\right) \\
&= 1 - \mathbb{P}(Z \leq 0.176777) \\
&= 1 - \Phi(0.176777) \\
&= 0.429842
\end{align*}
\]
A random variable $X$ is a lognormal random variable with parameters $\mu$ and $\sigma$ if $\ln X$ is a normally distributed random variable with mean $\mu$ and variance $\sigma^2$. 

Remarks: 
- The parameter $\mu$ is sometimes called the drift. 
- The parameter $\sigma$ is sometimes called the volatility.
Definition
A random variable $X$ is a lognormal random variable with parameters $\mu$ and $\sigma$ if $\ln X$ is a normally distributed random variable with mean $\mu$ and variance $\sigma^2$.

Remarks:
- The parameter $\mu$ is sometimes called the drift.
- The parameter $\sigma$ is sometimes called the volatility.
Suppose $X$ is lognormal, then $Y = \ln X$ is normal and

$$
P(X < x) = P(Y < \ln x)
= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\ln x} e^{-(t-\mu)^2/2\sigma^2} \, dt.
$$

If we let $u = e^t$ and $du = e^t \, dt$, then

$$
P(X < x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{x} \frac{1}{u} e^{-(\ln u-\mu)^2/2\sigma^2} \, du,
$$

the cumulative distribution function for the lognormally distributed random variable $X$. 
The probability density function for lognormal $X$ is

$$f(x) = \frac{1}{\sqrt{2\pi} \sigma} e^{-\frac{(\ln x - \mu)^2}{2\sigma^2}}.$$
Lemma

If $X$ is a lognormal random variable with parameters $\mu$ and $\sigma$ then

\[
\mathbb{E}[X] = e^{\mu + \sigma^2 / 2},
\]

\[
\mathbb{V}(X) = e^{2\mu + \sigma^2} \left( e^{\sigma^2} - 1 \right).
\]
Let $X$ be lognormally distributed with parameters $\mu$ and $\sigma$, then

$$
\mathbb{E}[X] = \frac{1}{\sigma \sqrt{2\pi}} \int_0^\infty x \left( \frac{1}{x} e^{-(\ln x - \mu)^2/2\sigma^2} \right) dx
$$

$$
= \frac{1}{\sigma \sqrt{2\pi}} \int_{-\infty}^{\infty} e^t e^{-(t - \mu)^2/2\sigma^2} dt
$$

$$
= e^{\mu + \sigma^2/2} \left[ \frac{1}{\sigma \sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-(t-(\mu+\sigma^2))^2/2\sigma^2} dt \right]
$$

$$
= e^{\mu + \sigma^2/2}.
$$
Let $X$ be lognormally distributed with parameters $\mu$ and $\sigma$, then

\[
\nabla (X) = \mathbb{E}\left[ X^2 \right] - (\mathbb{E} [X])^2
\]

\[
= \frac{1}{\sigma \sqrt{2\pi}} \int_{0}^{\infty} x^2 \left( \frac{1}{x} e^{-\left(\ln x - \mu\right)^2 / 2\sigma^2} \right) \, dx - \left( e^{\mu + \sigma^2 / 2} \right)^2
\]

\[
= \frac{1}{\sigma \sqrt{2\pi}} \int_{-\infty}^{\infty} e^{2t} e^{-(t-\mu)^2 / 2\sigma^2} \, dt - e^{2\mu + \sigma^2}
\]

\[
= e^{2(\mu + \sigma^2)} \left[ \frac{1}{\sigma \sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-(t-(\mu+2\sigma))^2 / 2\sigma^2} \, dt \right] - e^{2\mu + \sigma^2}
\]

\[
= e^{2(\mu + \sigma^2)} - e^{2\mu + \sigma^2}
\]

\[
= e^{2\mu + \sigma^2} \left( e^{\sigma^2} - 1 \right).
\]
Observation:

- Let $S(0)$ denote the price of a security at some starting time arbitrarily chosen to be $t = 0$.
- For $n \geq 1$, let $S(n)$ denote the price of the security on day $n$.
- The random variable $X(n) = \frac{S(n)}{S(n-1)}$ for $n \geq 1$ is lognormally distributed, i.e., $\ln X(n) = \ln S(n) - \ln S(n-1)$ is normally distributed.
Closing prices of Sony Corporation stock (09/20/2010–09/19/2011):
Lognormal behavior of closing prices:

\[ \mu = -0.00177279 \quad \sigma = 0.0181285 \]
What is the probability that the closing price of Sony Corporation stock will be higher today than yesterday?
Example (1 of 2)

What is the probability that the closing price of Sony Corporation stock will be higher today than yesterday?

\[
P \left( \frac{S(n)}{S(n-1)} > 1 \right) = P \left( \frac{\ln S(n)}{\ln S(n-1)} > \ln 1 \right)
\]

\[
= P(X > 0)
\]

\[
= P \left( Z > \frac{0 - (-0.00177279)}{0.01811285} \right)
\]

\[
= 1 - P(Z \leq 0.09779)
\]

\[
= 1 - \Phi (0.09779)
\]

\[
= 0.46105
\]
What is the probability that tomorrow’s closing price will be higher than yesterday’s closing price?
What is the probability that tomorrow’s closing price will be higher than yesterday’s closing price?

\[
\mathbb{P}\left( \frac{S(n+1)}{S(n)} > 1 \right) = \mathbb{P}\left( \frac{S(n+1)}{S(n)} \frac{S(n)}{S(n-1)} > 1 \right)
\]

\[
= \mathbb{P}\left( \ln \frac{S(n+1)}{S(n)} + \ln \frac{S(n)}{S(n-1)} > 0 \right)
\]

\[
= \mathbb{P}(X + X > 0)
\]

\[
= \mathbb{P}\left( Z > \frac{0 - 2(-0.00177279)}{\sqrt{2(0.01811285)^2}} \right)
\]

\[
= 1 - \mathbb{P}(Z \leq 0.138296)
\]

\[
= 1 - \Phi(0.138296)
\]

\[
= 0.445003
\]
If an item is worth $K$ but can only be sold for $X$, a rational investor would sell only if $X \geq K$.

The net payoff of the sale can be expressed as

$$(X - K)^+ = \begin{cases} X - K & \text{if } X \geq K, \\ 0 & \text{if } X < K. \end{cases}$$
Corollary

If $X$ is normal random variable with mean $\mu$ and variance $\sigma^2$ and $K$ is a constant, then

$$E \left[ (X - K)^+ \right] = \frac{\sigma}{\sqrt{2\pi}} e^{-\frac{(\mu-K)^2}{2\sigma^2}} + (\mu - K) \Phi \left( \frac{\mu - K}{\sigma} \right),$$

$$\sqrt{V \left( (X - K)^+ \right)}$$

$$= \left( (\mu - K)^2 + \sigma^2 \right) \Phi \left( \frac{\mu - K}{\sigma} \right) + \frac{(\mu - K)\sigma}{\sqrt{2\pi}} e^{-\frac{(\mu-K)^2}{2\sigma^2}}$$

$$- \left( \frac{\sigma}{\sqrt{2\pi}} e^{-\frac{(\mu-K)^2}{2\sigma^2}} + (\mu - K) \Phi \left( \frac{\mu - K}{\sigma} \right) \right)^2.$$
Corollary

If $X$ is a lognormally distributed random variable with parameters $\mu$ and $\sigma^2$ and $K > 0$ is a constant then

$$\mathbb{E} [(X - K)^+] = e^{\mu + \sigma^2/2} \Phi \left( \frac{\mu - \ln K}{\sigma} + \sigma \right) - K \Phi \left( \frac{\mu - \ln K}{\sigma} \right),$$

$$\mathbb{V} ((X - K)^+) = e^{2(\mu + \sigma^2)} \Phi (w + 2\sigma) + K^2 \Phi (w)$$

$$- 2Ke^{\mu + \sigma^2/2} \Phi (w + \sigma)$$

$$- \left( e^{\mu + \sigma^2/2} \Phi (w + \sigma) - K \Phi (w) \right)^2$$

where $w = (\mu - \ln K)/\sigma$. 
These slides are adapted from the textbook,


author: J. Robert Buchanan


address: 27 Warren St., Suite 401–402, Hackensack, NJ 07601

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