

Optimizing Portfolios

An Undergraduate Introduction to Financial Mathematics

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Investors may wish to adjust the allocation of financial resources including a mixture of cash, bonds, stocks, options, and other financial instruments in such a manner to achieve some time of optimality.

There are various, equally valid notions of optimality.

- Maximum rate of return on a portfolio.
- Minimum deviation from a chosen rate of return.
- Minimum risk of loss of portfolio value.

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The investor must choose the type of optimality they wish their portfolio to have.

Definition

Covariance is a measure of the degree to which two random variables tend to change in the same or opposite direction relative to one another. If X and Y are the random variables then the covariance, denoted $\text{Cov}(X, Y)$, is defined as

$$\text{Cov}(X, Y) = E[(X - E[X])(Y - E[Y])].$$

Theorem

If X and Y are random variables then

$$\text{Cov}(X, Y) = E[XY] - E[X]E[Y].$$

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$$\begin{aligned}\text{Cov}(X, Y) &= E[XY - YE[X] - XE[Y] + E[X]E[Y]] \\ &= E[XY] - E[Y]E[X] - E[X]E[Y] + E[X]E[Y] \\ &= E[XY] - E[X]E[Y]\end{aligned}$$

Example (1 of 2)

Consider the following table listing the heights and armspans of 20 children ([source](#)).

Child	Ht. (cm)	Span (cm)	Child	Ht. (cm)	Span (cm)
1	142	138	11	150	147
2	148	144	12	152	141
3	152	148	13	148	144
4	150	145	14	152	148
5	141	136	15	144	140
6	142	139	16	148	143
7	149	144	17	150	146
8	151	145	18	138	134
9	147	144	19	145	142
10	152	148	20	142	138

Find the covariance of height and armspan.

Example (2 of 2)

If X represents height and Y represents armspan then

$$E[X] = 147.15$$

$$E[Y] = 142.70$$

$$E[XY] = 21013.8$$

which implies

$$\text{Cov}(X, Y) = 21013.8 - (147.15)(142.70) = 15.445.$$

Note: since $\text{Cov}(X, Y) > 0$, then in general as height increases so does arm span.

Properties of Covariance (1 of 4)

Theorem

If X and Y are independent random variables then
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 $\text{Cov}(X, Y) = 0$.

Proof.

$$\text{Cov}(X, Y) = E[XY] - E[X]E[Y] = E[X]E[Y] - E[X]E[Y] = 0$$



Theorem

Suppose X , Y , and Z are random variables, then the following statements are true:

- 1 $\text{Cov}(X, X) = \text{Var}(X)$,
- 2 $\text{Cov}(X, Y) = \text{Cov}(Y, X)$,
- 3 $\text{Cov}(X + Y, Z) = \text{Cov}(X, Z) + \text{Cov}(Y, Z)$.

Let X , Y , and Z be random variables, then

$$\begin{aligned}\text{Cov}(X + Y, Z) &= E[(X + Y)Z] - E[X + Y]E[Z] \\ &= E[XZ] + E[YZ] - E[X]E[Z] - E[Y]E[Z] \\ &= E[XZ] - E[X]E[Z] + E[YZ] - E[Y]E[Z] \\ &= \text{Cov}(X, Z) + \text{Cov}(Y, Z)\end{aligned}$$

Corollary

Suppose $\{X_1, X_2, \dots, X_n\}$ and $\{Y_1, Y_2, \dots, Y_m\}$ are sets of random variables where $n, m \geq 1$.

$$\text{Cov} \left(\sum_{i=1}^n X_i, \sum_{i=1}^m Y_i \right) = \sum_{i=1}^n \sum_{j=1}^m \text{Cov} (X_i, Y_j)$$

Let $\{X_1, X_2, \dots, X_k, X_{k+1}\}$ be random variables.

$$\begin{aligned}\text{Cov}\left(\sum_{i=1}^{k+1} X_i, Y_1\right) &= \text{Cov}\left(\sum_{i=1}^k X_i, Y_1\right) + \text{Cov}(X_{k+1}, Y_1) \\ &= \sum_{i=1}^k \text{Cov}(X_i, Y_1) + \text{Cov}(X_{k+1}, Y_1) \\ &= \sum_{i=1}^{k+1} \text{Cov}(X_i, Y_1)\end{aligned}$$

Therefore by induction we may show that the result is true for any finite, integer value of n and $m = 1$.

When $m > 1$ we can argue that

$$\begin{aligned}\text{Cov} \left(\sum_{i=1}^n X_i, \sum_{j=1}^m Y_j \right) &= \sum_{i=1}^n \text{Cov} \left(X_i, \sum_{j=1}^m Y_j \right) \\ &= \sum_{i=1}^n \text{Cov} \left(\sum_{j=1}^m Y_j, X_i \right) \\ &= \sum_{i=1}^n \sum_{j=1}^m \text{Cov} (Y_j, X_i) \\ &= \sum_{i=1}^n \sum_{j=1}^m \text{Cov} (X_i, Y_j) .\end{aligned}$$

Corollary

If $\{X_1, X_2, \dots, X_n\}$ are random variables then

$$\text{Var} \left(\sum_{i=1}^n X_i \right) = \sum_{i=1}^n \text{Var} (X_i) + \sum_{i=1}^n \sum_{j \neq i}^n \text{Cov} (X_i, X_j) .$$

Let $Y = \sum_{i=1}^n X_i$, then

$$\begin{aligned}\text{Var}(Y) &= \text{Cov}(Y, Y) \\ \text{Var}\left(\sum_{i=1}^n X_i\right) &= \text{Cov}\left(\sum_{i=1}^n X_i, \sum_{j=1}^n X_j\right) \\ &= \sum_{i=1}^n \sum_{j=1}^n \text{Cov}(X_i, X_j) \\ &= \sum_{i=1}^n \text{Cov}(X_i, X_i) + \sum_{i=1}^n \sum_{j \neq i}^n \text{Cov}(X_i, X_j) \\ &= \sum_{i=1}^n \text{Var}(X_i) + \sum_{i=1}^n \sum_{j \neq i}^n \text{Cov}(X_i, X_j)\end{aligned}$$

Correlation

More informative than covariance would be information that large changes in random variable X are accompanied by large changes in Y .

Definition

If X and Y are random variables then the **correlation** of X and Y denoted $\rho(X, Y)$ and defined as

$$\rho(X, Y) = \frac{\text{Cov}(X, Y)}{\sqrt{\text{Var}(X) \text{Var}(Y)}}.$$

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Theorem

*If X and Y are independent random variables then $\rho(X, Y) = 0$. We say that X and Y are **uncorrelated**.*

Properties of the Correlation

Theorem

Suppose X and Y are random variables such that $Y = aX + b$ where $a, b \in \mathbb{R}$ with $a \neq 0$. If $a > 0$ then $\rho(X, Y) = 1$, while if $a < 0$ then $\rho(X, Y) = -1$.

$$\begin{aligned}\text{Cov}(X, Y) &= \text{Cov}(X, aX + b) \\ &= E[X(aX + b)] - E[X]E[aX + b] \\ &= E[aX^2 + bX] - E[X](aE[X] + b) \\ &= aE[X^2] + bE[X] - aE[X]E[X] - bE[X] \\ &= a(E[X^2] - E[X]^2) \\ &= a\text{Var}(X)\end{aligned}$$

$$\rho(X, Y) = \frac{a\text{Var}(X)}{\sqrt{\text{Var}(X) \cdot a^2\text{Var}(X)}} = \frac{a}{|a|}$$

Schwarz Inequality

We would like to prove that the correlation of two random variables is always in the interval $[-1, 1]$. First, we need the following inequality.

Lemma (Schwarz Inequality)

If X and Y are random variables then

$$(E[XY])^2 \leq E[X^2] E[Y^2].$$

Proof (1 of 2)

If a and b are real numbers then the following two inequalities hold:

$$0 \leq E[(aX + bY)^2] = a^2 E[X^2] + 2abE[XY] + b^2 E[Y^2]$$

If we let $a^2 = E[Y^2]$ and $b^2 = E[X^2]$ then the inequality above yields

$$\begin{aligned} 0 &\leq 2E[X^2]E[Y^2] + 2\sqrt{E[Y^2]E[X^2]}E[XY] \\ -2E[X^2]E[Y^2] &\leq 2\sqrt{E[Y^2]E[X^2]}E[XY] \\ -\sqrt{E[X^2]E[Y^2]} &\leq E[XY]. \end{aligned}$$

$$0 \leq E[(aX - bY)^2] = a^2 E[X^2] - 2abE[XY] + b^2 E[Y^2]$$

If we let $a^2 = E[Y^2]$ and $b^2 = E[X^2]$ then

$$E[XY] \leq \sqrt{E[X^2] E[Y^2]}.$$

Therefore, since

$$\begin{aligned} -\sqrt{E[X^2] E[Y^2]} &\leq E[XY] \leq \sqrt{E[X^2] E[Y^2]} \\ (E[XY])^2 &\leq E[X^2] E[Y^2]. \end{aligned}$$

Range of the Correlation

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If X and Y are random variables then $-1 \leq \rho(X, Y) \leq 1$.

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Proof.

$$\begin{aligned}(\text{Cov}(X, Y))^2 &= (\text{E}[(X - \text{E}[X])(Y - \text{E}[Y])])^2 \\ &\leq \text{E}[(X - \text{E}[X])^2] \text{E}[(Y - \text{E}[Y])^2] \\ &= \text{Var}(X) \text{Var}(Y)\end{aligned}$$

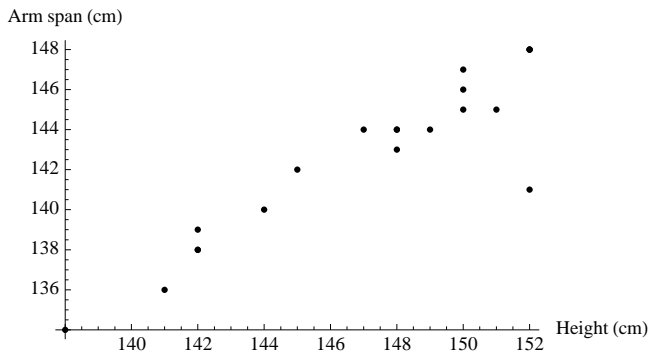
Thus $|\text{Cov}(X, Y)| \leq \sqrt{\text{Var}(X) \text{Var}(Y)}$, which is equivalent to

$$\begin{aligned}-1 &\leq \frac{\text{Cov}(X, Y)}{\sqrt{\text{Var}(X) \text{Var}(Y)}} \leq 1 \\ -1 &\leq \rho(X, Y) \leq 1.\end{aligned}$$



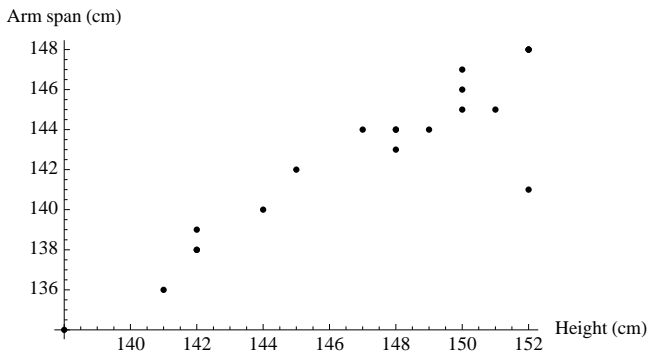
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$$\rho(X, Y) = \frac{\text{Cov}(X, Y)}{\sqrt{\text{Var}(X) \text{Var}(Y)}} = \frac{15.445}{\sqrt{(18.6605)(16.8526)}} \approx 0.870948.$$

Financial Example (1 of 2)

Suppose a portfolio consists of investments $\{w_1, w_2, \dots, w_n\}$ with corresponding random variable rates of return $\{R_1, R_2, \dots, R_n\}$.

If the total accumulated value of the portfolio after one time period is $W = \sum_{i=1}^n w_i(1 + R_i)$ then

$$E[W] = E\left[\sum_{i=1}^n w_i(1 + R_i)\right] = \sum_{i=1}^n w_i + \sum_{i=1}^n w_i E[R_i].$$

Financial Example (2 of 2)

The variance in the portfolio value after one time period is

$$\begin{aligned}\text{Var}(W) &= \text{Var}\left(\sum_{i=1}^n w_i(1 + R_i)\right) \\ &= \text{Var}\left(\sum_{i=1}^n w_i R_i\right) \\ &= \sum_{i=1}^n \text{Var}(w_i R_i) + \sum_{i=1}^n \sum_{j \neq i}^n \text{Cov}(w_i R_i, w_j R_j) \\ &= \sum_{i=1}^n w_i^2 \text{Var}(R_i) + \sum_{i=1}^n \sum_{j \neq i}^n w_i w_j \text{Cov}(R_i, R_j).\end{aligned}$$

Lemma

If X is a lognormal random variable with drift parameter μ and volatility σ^2 and $K > 0$ is a constant then

$$\text{Cov}(X, (X - K)^+) = E[X(X - K)^+] - E[X]E[(X - K)^+],$$

where

$$E[X(X - K)^+] = e^{2(\mu + \sigma^2)}\phi(w + 2\sigma) - Ke^{\mu + \sigma^2/2}\phi(w + \sigma),$$

with $w = (\mu - \ln K)/\sigma$.

One Type of Optimal Portfolio

We will choose as our notion of an optimal portfolio, the portfolio whose value has the **minimum variance**.

Consider a portfolio consisting of a short position on a call option for stock A and a long position of n shares of stock B.

$$\mathcal{P} = C_A - nB.$$

The variance in the value of the portfolio is

$$\begin{aligned}\text{Var}(\mathcal{P}) &= \text{E}[(C_A - nB)^2] - \text{E}[C_A - nB]^2 \\ &= n^2 \text{Var}(B) - 2n \text{Cov}(C_A, B) + \text{Var}(C_A).\end{aligned}$$

Thus the minimum variance is achieved when

$$n = \frac{\text{Cov}(C_A, B)}{\text{Var}(B)} = \rho(C_A, B) \sqrt{\frac{\text{Var}(C_A)}{\text{Var}(B)}}.$$

- 1 If $\rho(C_A, B) = 1$ then $n = \sqrt{\frac{\text{Var}(C_A)}{\text{Var}(B)}}$.
- 2 If $\rho(C_A, B) \approx 0$ then stock B is a poor surrogate for stock A.
- 3 n will decrease as $\text{Var}(B)$ increases.
- 4 If $A = B$ and the time to expiry is very short

$$\begin{aligned}n &= \frac{\text{Cov}(C_A, A)}{\text{Var}(A)} = \frac{\text{Cov}(\Delta C_A, \Delta A)}{\text{Var}(\Delta A)} \\ &= \frac{\text{Cov}\left(\frac{\partial C_A}{\partial A} \Delta A, \Delta A\right)}{\text{Var}(\Delta A)} = \frac{\partial C_A}{\partial A} = \Delta.\end{aligned}$$

An Introduction to Utility Functions

- 1 A **utility function** assigns to the outcomes of an experiment, a value called the outcome's **utility**.
- 2 Individual investors have differing utility functions corresponding to their affinity for taking investment risks.
- 3 The utility function plays a role in making rational investment decisions.

Assigning Utility to Outcomes (1 of 4)

Suppose there are n outcomes to an experiment

$$C_1 \leq C_2 \leq \cdots \leq C_n.$$

An investor can rank the outcomes in order of desirability. Suppose the outcomes have been ranked from least to most desirable as

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Question: How is utility assigned to an outcome?

Answer: Let the utility function be $u(x)$, then assign $u(C_1) = C_1$ and $u(C_n) = C_n$.

Assigning Utility to Outcomes (2 of 4)

For $1 < i < n$ the investor is given a choice:

- 1 participate in a random experiment in which the probability of receiving C_i is 1, or
- 2 participate in a random experiment where they receive C_1 with probability ϕ_i and receive C_n with probability $1 - \phi_i$.

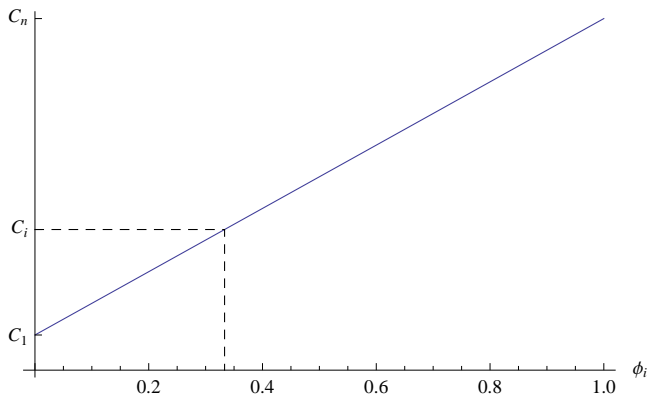
Note:

- The expected value of the first experiment is C_i and the expected value of the second experiment is $\phi_i C_1 + (1 - \phi_i) C_n$.
- By the Intermediate Value Theorem there exists $\phi_i \in [0, 1]$ such that

$$C_i = \phi_i C_1 + (1 - \phi_i) C_n$$

thus define $u(C_i) = \phi_i C_1 + (1 - \phi_i) C_n$.

Assigning Utility to Outcomes (4 of 4)



Remark: when $u(C_i) = \phi_i C_1 + (1 - \phi_i) C_n$ the investor is indifferent to the choice.

Application of the Utility Function

Suppose an investor must choose between two different investment schemes. The union of all outcomes of the two schemes is the set

$$\{C_1, C_2, \dots, C_n\}.$$

The probability of receiving outcome C_i from the first scheme is p_i and the probability of receiving it from the second is q_i .

If the utility of the i th outcome is $u(C_i)$ then the investor will choose the first scheme is

$$\sum_{i=1}^n p_i u(C_i) > \sum_{i=1}^n q_i u(C_i).$$

In other words, the investor will choose the first scheme if the expected value of its utility is greater than that of the second scheme.

A Property of Utility Functions

The amount of extra utility that an investor experiences when x is increased to $x + \Delta x$ is non-increasing.

Concave Functions

Definition

Function $f(t)$ is **concave** on an open interval (a, b) if for every $x, y \in (a, b)$ and every $\lambda \in [0, 1]$ we have

$$\lambda f(x) + (1 - \lambda)f(y) \leq f(\lambda x + (1 - \lambda)y).$$

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$$\lambda f(x) + (1 - \lambda)f(y) \leq f(\lambda x + (1 - \lambda)y).$$

- An investor whose utility function is concave is said to be **risk-averse**.
- An investor with a linear utility function of the form $u(x) = ax + b$ with $a > 0$ is said to be **risk-neutral**.
- An investor whose utility function increases more rapidly as the reward increases is said to be **risk-loving**.

Theorem

If $f \in C^2(a, b)$ then f is concave on (a, b) if and only if $f''(t) \leq 0$ for $a < t < b$.

Proof (f concave implies $f''(t) \leq 0$) (1 of 2)

If f is concave on (a, b) then by definition f satisfies

$$\lambda f(x) + (1 - \lambda)f(y) \leq f(\lambda x + (1 - \lambda)y)$$

for every $x, y \in (a, b)$ and every $\lambda \in [0, 1]$.

- Assume $x < y$.
- If $w = \lambda x + (1 - \lambda)y$ and if $0 < \lambda < 1$ then $a < x < w < y < b$ and

$$(1 - \lambda)[f(y) - f(w)] \leq \lambda[f(w) - f(x)].$$

- By the definition of w ,

$$1 - \lambda = \frac{w - x}{y - x} \quad \text{and} \quad \lambda = \frac{y - w}{y - x}.$$

- Substituting these expressions yields

$$\frac{f(y) - f(w)}{y - w} - \frac{f(w) - f(x)}{w - x} \leq 0.$$

Proof (f concave implies $f''(t) \leq 0$) (2 of 2)

$$\frac{f(y) - f(w)}{y - w} - \frac{f(w) - f(x)}{w - x} \leq 0.$$

- Applying the Mean Value Theorem to each of the difference quotients implies that for some α and β satisfying with $x < \alpha < w < \beta < y$,

$$f'(\beta) - f'(\alpha) \leq 0$$

- Using the Mean Value Theorem once more proves that for some t with $\alpha < t < \beta$

$$f''(t)(\beta - \alpha) \leq 0$$

which implies $f''(t) \leq 0$.

Jensen's Inequality (Discrete Version)

Theorem (Jensen's Inequality (Discrete Version))

Let f be a concave function on the interval (a, b) , suppose $x_i \in (a, b)$ for $i = 1, 2, \dots, n$, and suppose $\lambda_i \in [0, 1]$ for $i = 1, 2, \dots, n$ with $\sum_{i=1}^n \lambda_i = 1$, then

$$\sum_{i=1}^n \lambda_i f(x_i) \leq f\left(\sum_{i=1}^n \lambda_i x_i\right).$$

Jensen's Inequality (Discrete Version) Proof

Let $\mu = \sum_{i=1}^n \lambda_i x_i$ and note that since $\lambda_i \in [0, 1]$ for $i = 1, 2, \dots, n$ and $\sum_{i=1}^n \lambda_i = 1$, then $a < \mu < b$. The equation of the line tangent to the graph of f at $(\mu, f(\mu))$ is $y = f'(\mu)(x - \mu) + f(\mu)$. Since f is concave on (a, b) then

$$f(x_i) \leq f'(\mu)(x_i - \mu) + f(\mu) \quad \text{for } i = 1, 2, \dots, n.$$

Therefore

$$\begin{aligned} \sum_{i=1}^n \lambda_i f(x_i) &\leq \sum_{i=1}^n (\lambda_i [f'(\mu)(x_i - \mu) + f(\mu)]) \\ &= f'(\mu) \sum_{i=1}^n (\lambda_i x_i - \lambda_i \mu) + f(\mu) \sum_{i=1}^n \lambda_i \\ &= f(\mu) = f\left(\sum_{i=1}^n \lambda_i x_i\right). \end{aligned}$$

Jensen's Inequality (Continuous Version)

Theorem (Jensen's Inequality (Continuous Version))

Let $\phi(t)$ be an integrable function on $[0, 1]$ and let f be a concave function, then

$$\int_0^1 f(\phi(t)) dt \leq f\left(\int_0^1 \phi(t) dt\right).$$

Jensen's Inequality (Continuous Version) Proof

Let $\alpha = \int_0^1 \phi(t) dt$ and let $y = f'(\alpha)(x - \alpha) + f(\alpha)$ be the equation of the tangent line at $(\alpha, f(\alpha))$. Since f is concave then

$$f(\phi(t)) \leq f'(\alpha)(\phi(t) - \alpha) + f(\alpha),$$

which implies that

$$\begin{aligned} \int_0^1 f(\phi(t)) dt &\leq \int_0^1 [f'(\alpha)(\phi(t) - \alpha) + f(\alpha)] dt \\ &= f(\alpha) + f'(\alpha) \int_0^1 (\phi(t) - \alpha) dt \\ &= f(\alpha) = f\left(\int_0^1 \phi(t) dt\right). \end{aligned}$$

Definition

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Example

An investor is risk-averse with a utility function $u(x) = x - x^2/25$. The investor must choose between the following two “investments”:

- A: Flip a fair coin, if the coin lands heads up the investor receives \$10, otherwise they receive nothing.
- B: Receive an amount $\$M$ with certainty.

Which do they choose?

Solution

A rational investor will select the investment with the greater expected utility.

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The expected utility for investment A is

$$\frac{1}{2}u(10) + \frac{1}{2}u(0) = \frac{1}{2} \left(10 - \frac{10^2}{25} \right) = 3.$$

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The expected utility for B is $u(M) = M - M^2/25$. Thus the investor will choose the coin flip whenever

$$3 > M - \frac{M^2}{25}$$

Thus investment A is preferable to B whenever $M < \$3.49$.

Certainty Equivalent

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An investor is risk-averse with a utility function $u(x) = x - x^2/25$. The investor wishes to find the certainty equivalent C for the following investment choice:

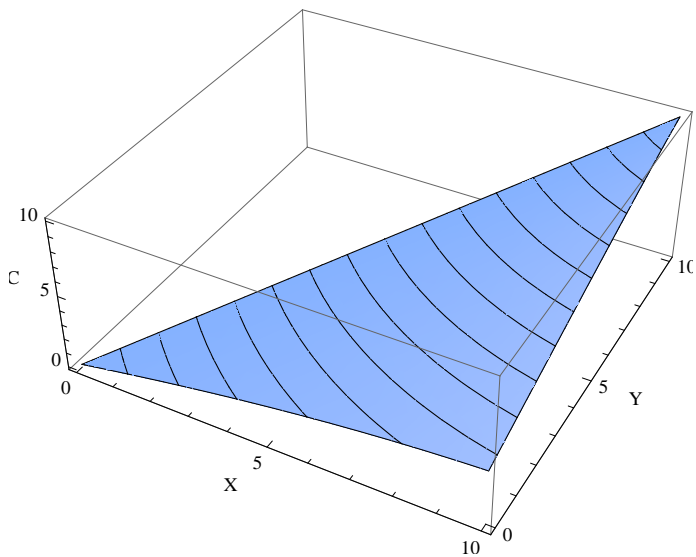
- A: Flip a fair coin, if the coin lands heads up the investor receives $0 < X \leq 10$, otherwise they receive $0 < Y < X$.
- B: Receive an amount C with certainty.

Solution (1 of 2)

The certainty equivalent and payoff of investment A must satisfy the following equation.

$$C - \frac{C^2}{25} = \frac{1}{2} \left(X - \frac{X^2}{25} + Y - \frac{Y^2}{25} \right)$$
$$\left(C - \frac{25}{2} \right)^2 = \frac{1}{2} \left[\left(X - \frac{25}{2} \right)^2 + \left(Y - \frac{25}{2} \right)^2 \right]$$
$$C = \frac{25}{2} - \frac{1}{\sqrt{2}} \sqrt{\left(X - \frac{25}{2} \right)^2 + \left(Y - \frac{25}{2} \right)^2}$$

Solution (2 of 2)



We wish to examine the problem of allocating resources among potentially many different investments in an optimal fashion.

Consider a simple case:

- An investor has x capital to invest.
- The investor allocates fraction $\alpha \in [0, 1]$ to an investment.
- The investment is structured such that an allocation of αx will earn αx with probability p and lose αx with probability $1 - p$.

Portfolio Selection (2 of 2)

The investor's financial position after the investment is

$$X = \begin{cases} x(1 + \alpha) & \text{with probability } p, \\ x(1 - \alpha) & \text{with probability } 1 - p. \end{cases}$$

The allocation fraction α will be optimal when the expected value of the utility is maximized.

$$\begin{aligned} E[u(X)] &= pu(x(1 + \alpha)) + (1 - p)u(x(1 - \alpha)) \\ \frac{d}{d\alpha}E[u(X)] &= pxu'(x(1 + \alpha)) - x(1 - p)u'(x(1 - \alpha)) \\ 0 &= pu'(x(1 + \alpha)) - (1 - p)u'(x(1 - \alpha)) \end{aligned}$$

The appropriate value of α will depend on the utility function $u(t)$.

General Portfolio Selection Problem

- An investor has x resources to allocate among n investments.
- The fraction invested in i will be denoted x_i .
- The return from investment i will be $W_i = 1 + R_i$ where R_i is the rate of return of investment i .
- The general problem will then be that of determining x_i for $i = 1, 2, \dots, n$ such that
 - 1 $0 \leq x_i \leq 1$ for $i = 1, 2, \dots, n$, and
 - 2 $\sum_{i=1}^n x_i = 1$,and which maximizes the investor's total expected utility $E[u(W)]$, where

$$W = \sum_{i=1}^n x_i W_i.$$

- Assume n is large and the W_i 's are not highly correlated. Thus the Central Limit Theorem implies W is a normally distributed random variable.

Example (1 of 3)

Suppose $u(x) = 1 - e^{-bx}$ where $b > 0$.

$$E[u(W)] = E\left[1 - e^{-bW}\right] = 1 - E\left[e^{-bW}\right].$$

The quantity e^{-bW} is a lognormal random variable.

$$E\left[e^{-bW}\right] = e^{-bE[W] + b^2\text{Var}(W)/2}$$

which implies that

$$E[u(W)] = 1 - e^{-b(E[W] - b\text{Var}(W)/2)}.$$

Thus the expected utility is maximized when $E[W] - b\text{Var}(W)/2$ is maximized.

Example (2 of 3)

Suppose $b = 0.005$ and that an investor has \$100 to invest and will allocate y dollars to security A and $100 - y$ dollars to B.

Security	A	B
E [rate of return]	0.16	0.18
$\sqrt{\text{Var}}(\text{rate of return})$	0.20	0.24

Assume that the correlation between the rates of return is $\rho = -0.35$.

$$\begin{aligned}E[W] &= 100 + 0.16y + 0.18(100 - y) = 118 - 0.02y \\ \text{Var}(W) &= y^2(0.20)^2 + (100 - y)^2(0.24)^2 \\ &\quad + 2y(100 - y)(0.20)(0.24)(-0.35) \\ &= 0.04y^2 + 0.0576(100 - y)^2 - 0.0336y(100 - y)\end{aligned}$$

Example (3 of 3)

The optimal portfolio is the one which maximizes

$$\begin{aligned} E[W] - \frac{b}{2} \text{Var}(W) \\ &= 118 - 0.02y - 0.0025(0.04y^2 + 0.0576(100 - y)^2 \\ &\quad - 0.0336y(100 - y)) \\ &= -0.000328y^2 + 0.0172y + 116.56. \end{aligned}$$

This occurs when $y \approx 26.2195$.

$$E[W] \approx 117.476 \quad \text{Var}(W) \approx 276.049 \quad E[u(W)] \approx 0.442296.$$

Minimum Variance Analysis

Theorem

Suppose that $0 \leq \alpha_i \leq 1$ will be invested in security i for $i = 1, 2, \dots, n$ subject to the constraint that $\alpha_1 + \alpha_2 + \dots + \alpha_n = 1$. Suppose the rate of return of security i is a random variable R_i and that all the rates of return are mutually uncorrelated. The optimal, minimum variance portfolio described by the allocation vector $\langle \alpha_1^, \alpha_2^*, \dots, \alpha_n^* \rangle$, is the one for which*

$$\alpha_i^* = \frac{\frac{1}{\sigma_i^2}}{\sum_{j=1}^n \frac{1}{\sigma_j^2}} \quad \text{for } i = 1, 2, \dots, n,$$

where $\sigma_i^2 = \text{Var}(R_i)$.

Proof (1 of 2)

Since the rates of return are uncorrelated, the variance of W is

$$\text{Var}(W) = \sum_{i=1}^n \alpha_i^2 \sigma_i^2,$$

and is subject to the constraint $1 = \sum_{i=1}^n \alpha_i$. Applying the technique of Lagrange Multipliers yields the following equations.

$$\begin{aligned} \nabla \left(\sum_{i=1}^n \alpha_i^2 \sigma_i^2 \right) &= \lambda \nabla \left(\sum_{i=1}^n \alpha_i \right) \\ \sum_{i=1}^n \alpha_i &= 1 \end{aligned}$$

These equations are equivalent to respectively:

$$\begin{aligned} 2\alpha_i \sigma_i^2 &= \lambda \quad \text{for } i = 1, 2, \dots, n, \text{ and} \\ \sum_{i=1}^n \alpha_i &= 1. \end{aligned}$$

Solving for α_j in the first equation and substituting into the second equation determines that

$$\lambda = \frac{2}{\sum_{j=1}^n \frac{1}{\sigma_j^2}}.$$

Substituting this expression for λ into the first equation yields

$$\alpha_i = \frac{\frac{1}{\sigma_i^2}}{\sum_{j=1}^n \frac{1}{\sigma_j^2}} \quad \text{for } i = 1, 2, \dots, n.$$

Including the Effects of Borrowing

Assume:

- $\mathbf{w} = \langle w_1, w_2, \dots, w_n \rangle$ represents a portfolio of investments.
- R_i represents the rate of return on investing w_i in the i^{th} security.
- Investments are purchased by borrowing money which must be paid back in one time unit at simple interest rate r .

Then the net wealth generated by the portfolio is

$$R(\mathbf{w}) = \sum_{i=1}^n w_i(1 + R_i) - (1 + r) \sum_{i=1}^n w_i = \sum_{i=1}^n w_i(R_i - r).$$

The expected value and variance of the net wealth generated by the portfolio are

$$\begin{aligned} r(\mathbf{w}) &= E[R(\mathbf{w})] \\ \sigma^2(\mathbf{w}) &= \text{Var}(R(\mathbf{w})). \end{aligned}$$

Minimizing the Variance

Lemma

Assuming the rates of return on the securities are uncorrelated, the optimal portfolio generating an expected unit amount of net wealth with the minimum variance in the net wealth is

$$\mathbf{w}^* = \left\langle \frac{\frac{r_1 - r}{\sigma_1^2}}{\sum_{j=1}^n \frac{(r_j - r)^2}{\sigma_j^2}}, \frac{\frac{r_2 - r}{\sigma_2^2}}{\sum_{j=1}^n \frac{(r_j - r)^2}{\sigma_j^2}}, \dots, \frac{\frac{r_n - r}{\sigma_n^2}}{\sum_{j=1}^n \frac{(r_j - r)^2}{\sigma_j^2}} \right\rangle$$

where $r_i = E[R_i]$ and $\sigma_i^2 = \text{Var}(R_i)$ for $i = 1, 2, \dots, n$.

Portfolio Separation Theorem

The following result indicates to an investor how portions of a portfolio should be invested so as to minimize the variance in the wealth generated.

Theorem (Portfolio Separation Theorem)

If b is any positive scalar, the variance of all portfolios with expected wealth generated equal to b is minimized by portfolio $b\mathbf{w}^$ where*

$$\mathbf{w}^* = \left\langle \frac{\frac{r_1 - r}{\sigma_1^2}}{\sum_{j=1}^n \frac{(r_j - r)^2}{\sigma_j^2}}, \frac{\frac{r_2 - r}{\sigma_2^2}}{\sum_{j=1}^n \frac{(r_j - r)^2}{\sigma_j^2}}, \dots, \frac{\frac{r_n - r}{\sigma_n^2}}{\sum_{j=1}^n \frac{(r_j - r)^2}{\sigma_j^2}} \right\rangle$$

Suppose \mathbf{x} is a portfolio for which $r(\mathbf{x}) = b$, then

$$\frac{1}{b}r(\mathbf{x}) = r\left(\frac{1}{b}\mathbf{x}\right) = 1.$$

Thus $\frac{1}{b}\mathbf{x}$ is a portfolio with unit expected rate of return. For the portfolio \mathbf{w}^* ,

$$\sigma^2(b\mathbf{w}^*) = b^2\sigma^2(\mathbf{w}^*) \leq b^2\sigma^2\left(\frac{1}{b}\mathbf{x}\right) = \sigma^2(\mathbf{x}).$$

Risky vs. Risk Free Investments

Suppose an investor divides a unit of resources between a risky and a risk-free investment

- 1 $x \in [0, 1]$ is invested in a security whose rate of return is R_S , and
- 2 $1 - x$ earns interest in a safe savings account at rate r_f .

The rate of return on the portfolio is

$$R = (1 - x)r_f + xR_S,$$

and the expected rate of return is

$$E[R] = r = r_f + (r_S - r_f)x$$

where $r_S = E[R_S]$.

$$\text{Var}(R) = \sigma^2 = x^2 \text{Var}(R_S) = x^2 \sigma_S^2$$

Combining the equations for $E[R]$ and $\text{Var}(R)$ and eliminating x produces

$$r = r_f + \frac{r_S - r_f}{\sigma_S} \sigma$$

If instead of a single security x was invested evenly throughout the market then we obtain the equation for the **capital market line**.

$$r = r_f + \frac{r_M - r_f}{\sigma_M} \sigma$$

Capital Market Line

Capital Assets Pricing Model (1 of 4)

Suppose an investor wishes to weigh the return on an investment against the risk associated with the investment.

R_i : return on investment in security i

R_M : return on the entire market

x : fraction of unit wealth invested in security i

$1 - x$: fraction of unit wealth invested in market

The rate of return on the portfolio is

$$R = xR_i + (1 - x)R_M,$$

with

$$E[R] = xE[R_i] + (1 - x)E[R_M]$$

$$\text{Var}(R) = x^2\text{Var}(R_i) + (1 - x)^2\text{Var}(R_M) + 2x(1 - x)\text{Cov}(R_i, R_M).$$

Capital Assets Pricing Model (2 of 4)

The expected value and variance of R can be re-written as

$$\begin{aligned}r &= xr_i + (1 - x)r_M \\ \sigma^2 &= x^2\sigma_i^2 + (1 - x)^2\sigma_M^2 + 2x(1 - x)\rho_{i,M}\sigma_i\sigma_M.\end{aligned}$$

where $r = E[R]$, $r_i = E[R_i]$, $r_M = E[R_M]$, $\sigma^2 = \text{Var}(R)$, $\sigma_i^2 = \text{Var}(R_i)$, and $\sigma_M^2 = \text{Var}(R_M)$.

Capital Assets Pricing Model (3 of 4)

These two equations give us the parametric form of a parabola in the σr -plane.

Using this tangency relationship we can determine that

$$r_i - r_f = \frac{\rho_{i,M}\sigma_i}{\sigma_M}(r_M - r_f) = \beta_i(r_M - r_f).$$

The CAPM can be interpreted as stating that the excess rate of return of security i above the risk-free interest rate is proportional to the excess rate of return of the market above the risk-free interest rate.

Example

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Suppose the risk-free interest rate is 5% per year, the return on the market is 9% per year, and the volatility in the market is 35% per year. Suppose further that the covariance between the return on a particular stock and the return on the market is 0.10. Estimate the return on the stock.

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Using the CAPM,

$$\beta = \frac{\text{Cov}(R, R_M)}{\sigma_M^2} = \frac{0.10}{(0.35)^2} = 0.8153$$

which implies

$$r = r_f + \beta(r_M - r_f) = 0.05 + 0.8153(0.09 - 0.05) = 0.0827$$

or 8.27%.