Student's Solution Manual

1.1.1. The amount due is

$$A = 3500(1 + 0.05(1)) = \$3675.$$

1.1.3. The principal amount is

$$P = \frac{1700}{1 + 0.04(1)} = \$1634.6154.$$

1.1.5. The simple interest rate is found by solving the following equation.

$$\begin{aligned} 20 &= 1500(r)(2) \\ r &= 0.006667 \end{aligned}$$

Thus the simple interest rate is 0.6667% annually.

1.2.1. The amount due is

$$A = 3659 \left(1 + \frac{0.065}{2}\right)^{(2)(5)} \approx \$5038.06.$$

1.2.3. The amount due is

$$A = 3750(1 + 0.00375)^{(4)(8)} \approx \$4227.16.$$

1.2.5. Assuming there are 365 days in a year, the effective annual interest rate paid by Jordan's competitor is

$$r_e = \left(1 + \frac{0.0525}{365}\right)^{365} \approx 0.0538986.$$

Jordan must pay an interest rate compounded monthly of at least

$$\begin{aligned} \left(1 + \frac{r}{12}\right)^{12} &= 1 + 0.0538986 \\ r &\approx 0.0526112 \end{aligned}$$

or 5.2611%.

1.3.1. The amount due is

$$A = 3995e^{(0.045)(3)} \approx \$4572.42.$$

1.3.3. The doubling time T can be calculated as

$$\begin{aligned} 2P &= Pe^{0.075T} \\ T &= \frac{\ln 2}{0.075} \approx 9.24196 \text{ years.} \end{aligned}$$

1.3.5. Using Newton's Method or an equivalent numerical technique to approximate the solution to the equation

$$1000 \left(e^{0.0475t} - \left(1 + \frac{0.0475}{365}\right)^{365t}\right) = 1$$

yields $t \approx 42.6583$ years.

1.4.1. Equate the principal amount borrowed with the present value of all the level payments x.

$$\begin{aligned} 15000 &= x \sum_{k=1}^{72} \left(1 + \frac{0.035}{12}\right)^{-k} = x\left(1 + \frac{0.035}{12}\right)^{-1} \left(\frac{1 - \left(1 + \frac{0.035}{12}\right)^{-72}}{1 - \left(1 + \frac{0.035}{12}\right)}\right) \\ x &\approx \$231.28 \end{aligned}$$

1.4.3. Since the present values are equal then

$$500 + \frac{1000}{(1+r)^5} + \frac{1500}{(1+r)^{10}} = \frac{3500}{(1+r)^8}.$$

Solving for r numerically yields $r \approx 0.0835679$ or 8.3568%.

1.4.5. If the price of the house is \$200,000 and the down payment is 20% then Remi will borrow $P = \$160,000$. If the monthly payment on a $t = 30$ year, fixed-rate mortgage should not exceed \$1500 then using Eq. (1.9) and Newton's Method yields

$$\begin{aligned} 160000 &= 1500\frac{12}{r}\left(1 - \left[1 + \frac{r}{12}\right]^{-(12)(30)}\right) \\ r &\approx 0.108034 \end{aligned}$$

Thus the interest rate must not exceed 10.8034% annually.

1.4.7. At the end of the 7th year Aubrey's account balance is $12000(1+0.03)^7$. At the end of the 8th year the account balance is

$$12000(1.03)^8 - (1.05)T.$$

At the end of the 9th year the account balance is

$$12000(1.03)^9 - (1.05)(1.03)T - (1.05)T = 12000(1.03)^9 - (1.05)T\left[1 + 1.03\right].$$

At the end of the 10th year the account balance is

$$\begin{aligned} &12000(1.03)^{10} - (1.05)(1.03)^2T - (1.05)(1.03)T - (1.05)T \\ &= 12000(1.03)^{10} - (1.05)T\left[1 + (1.03) + (1.03)^2\right]. \end{aligned}$$

At the end of the 11th year Aubrey's account balance is

$$12000(1.03)^{11} - T - (1.05)T\sum_{i=1}^{3}(1.03)^i.$$

Finally at the end of the 15th year Aubrey's account balance is

$$\begin{aligned} 12000 &= 12000(1.03)^{15} - (1.03)^4T\left[1 + (1.05)\sum_{i=1}^{3}(1.03)^i\right] \\ T &= 1369.84. \end{aligned}$$

1.4.9. The present value of the first option is

$$\frac{1600\left(1 + \frac{r}{2}\right)}{1+r},$$

while the present value of the second option is

$$\frac{1600}{\left(1 + \frac{r}{2}\right)^2} + 50.$$

If the present values are equal then

$$\begin{aligned}\frac{1600\left(1+\frac{r}{2}\right)}{1+r} &= \frac{1600}{\left(1+\frac{r}{2}\right)^2}+50 \\ r &\approx 0.0645794.\end{aligned}$$

1.4.11. Let $f(x)=\sum_{k=0}^{\infty} x^k$, then

$$f(x)=\lim_{N\to\infty}\sum_{k=0}^{N} x^k=\lim_{N\to\infty}\frac{1-x^{N+1}}{1-x}=\frac{1}{1-x}$$

when $0<x<1$. Differentiating with respect to x produces,

$$\frac{1}{(1-x)^2}=\sum_{k=1}^{\infty} k\,x^{k-1}$$

for $0<x<1$.
1.5.1. Using Eq. (1.11) setup the equation

$$10000=\frac{2000}{1+r}+\frac{3000}{(1+r)^2}+\frac{4000}{(1+r)^3}+\frac{3000}{(1+r)^4}$$

whose solution is approximated using Newton's Method. The rate of return $r\approx 0.0718$ or equivalently 7.18% per year.
1.5.3. Solve the following equation for r.

$$\begin{aligned}0 &= -50000+\frac{30000}{(1+r)^3}+\frac{30000}{(1+r)^4} \\ r &\approx 0.053575\end{aligned}$$

The annual rate of return is approximately 5.3575%.
1.5.5. The loan is repaid in

$$\begin{aligned}100 &= 50e^{0.18t} \\ t &= 3.85082 \text{ years.}\end{aligned}$$

1.6.3. The amount due can be calculated as

$$A(1)=125e^{\int_0^1 t^2/50\,dt}=125e^{1/150}\approx \$125.84.$$

1.6.5. The total interest earned during the interval $2\le t\le 3$ is

$$\begin{aligned}x &= 500\left(e^{\int_0^3 t^2/50\,dt}-e^{\int_0^2 t^2/50\,dt}\right)+x\left(e^{\int_2^3 t^2/50\,dt}-1\right) \\ x &\approx \$82.3367.\end{aligned}$$

1.6.7. At $t=5$,

$$\begin{aligned}100\left(1+\frac{k}{20}\right)^5 &= 100e^{\int_0^5 \frac{1}{k+t/5}\,dt} \\ \left(1+\frac{k}{20}\right)^5 &= \left(1+\frac{1}{k}\right)^5 \\ k &= 2\sqrt{5}.\end{aligned}$$

1.7.1. The present value of the income stream is

$$\begin{aligned} P &= \int_0^3 (50000)e^{-0.01t}e^{-0.0425t}\,dt = \int_0^3 (50000)e^{-0.0525t}\,dt \\ &= \left[\frac{50000}{-0.0525}e^{-0.0525t}\right]_{t=0}^{t=3} \approx 138,783.99. \end{aligned}$$

1.7.3. Let the continuous rate of investment be S, then since the cumulative future value of the investment must be $\$1,000,000$ then

$$\begin{aligned} 1,000,000 &= e^{0.0249(6)}\int_0^6 Se^{-0.0249t}\,dt \\ &= \left[\frac{Se^{0.0249(6)}}{-0.0249}e^{-0.0249t}\right]_{t=0}^{t=6} \\ &= 6.4713796S \\ S &\approx \$154,526.56/\text{yr}. \end{aligned}$$

1.7.5. Consider the following equation.

$$\begin{aligned} 25000 &= \int_0^{15} k(5+t)e^{\int_t^{15} 1/(5+u)\,du}\,dt = \int_0^{15} k(5+t)\,\frac{20}{5+t}\,dt = 300k \\ k &\approx 83.3333 \end{aligned}$$

2.1.1. Assuming the regular tetrahedron is fair so that it equally likely to land on any of its four faces, the probability of it landing on 3 is $p = 1/4$.

2.1.3. List the $4! = 24$ permutations of the four DVDs and count how many leave at least one DVD in its correct case.

n	1	2	3	4	Any Correct?
1	1	2	3	4	Y
2	1	2	4	3	Y
3	1	3	2	4	Y
4	1	3	4	2	Y
5	1	4	2	3	Y
6	1	4	3	2	Y
7	2	1	3	4	Y
8	2	1	4	3	N
9	2	3	1	4	Y
10	2	3	4	1	N
11	2	4	1	3	N
12	2	4	3	1	Y
13	3	1	2	4	Y
14	3	1	4	2	N
15	3	2	1	4	Y
16	3	2	4	1	Y
17	3	4	1	2	N
18	3	4	2	1	N
19	4	1	2	3	N
20	4	1	3	2	Y
21	4	2	1	3	Y
22	4	2	3	1	Y
23	4	3	1	2	N
24	4	3	2	1	N

Note that 15 of the 24 possible permutations leave at least one DVD in its correct case, thus the desired probability is

$$p = \frac{15}{24} = \frac{5}{8} = 0.625.$$

2.1.5. Out of the 36 possible and equally likely outcomes of this experiment, 6 are doubles. Thus the probability is $p = 6/36 = 1/6$.

2.1.7. Let A denote the event that the card drawn is an ace and Q denote the event that the card drawn is a queen. Since $A \cap Q = \emptyset$ then

$$\mathbb{P}(A \cup Q) = \mathbb{P}(A) + \mathbb{P}(Q) = \frac{4}{52} + \frac{4}{52} = \frac{2}{13}.$$

2.2.1. The number of heads facing up can be any integer number between 0 and 4. Thus the range is the set $\{0, 1, 2, 3, 4\}$.

2.2.3. Consider the sample space,

$$\Omega = \left\{ \begin{array}{cccccccccccc} \text{⚀⚀}, & \text{⚀⚁}, & \text{⚀⚂}, & \text{⚀⚃}, & \text{⚀⚄}, & \text{⚀⚅}, & \text{⚁⚀}, & \text{⚁⚁}, & \text{⚁⚂}, & \text{⚁⚃}, & \text{⚁⚄}, & \text{⚁⚅}, \\ \text{⚂⚀}, & \text{⚂⚁}, & \text{⚂⚂}, & \text{⚂⚃}, & \text{⚂⚄}, & \text{⚂⚅}, & \text{⚃⚀}, & \text{⚃⚁}, & \text{⚃⚂}, & \text{⚃⚃}, & \text{⚃⚄}, & \text{⚃⚅}, \\ \text{⚄⚀}, & \text{⚄⚁}, & \text{⚄⚂}, & \text{⚄⚃}, & \text{⚄⚄}, & \text{⚄⚅}, & \text{⚅⚀}, & \text{⚅⚁}, & \text{⚅⚂}, & \text{⚅⚃}, & \text{⚅⚄}, & \text{⚅⚅} \end{array} \right\}.$$

Since each outcome is equally likely, then $\mathbb{P}(X = 1) = 1/36$, $\mathbb{P}(X = 2) = 3/36$, $\mathbb{P}(X = 3) = 5/36$, $\mathbb{P}(X = 4) = 7/36$, $\mathbb{P}(X = 5) = 9/36$, and $\mathbb{P}(X = 6) = 11/36$.

2.2.5. For each flip of the coin the probability of heads is 1/2. If heads first appears on the nth flip then tails appeared on the preceding $n-1$ flips. Therefore $\mathbb{P}(X = n) = \left(\frac{1}{2}\right)^{n-1} \frac{1}{2} = 1/2^n$ for $n \in \mathbb{N}$.

2.2.7. Using the recurrence relation $\mathbb{P}(X = n + 1) = \frac{1}{n} \mathbb{P}(X = n)$, then

$$\begin{aligned} \mathbb{P}(X = 2) &= \frac{1}{1} \mathbb{P}(X = 1) = \mathbb{P}(X = 0) \\ \mathbb{P}(X = 3) &= \frac{1}{2} \mathbb{P}(X = 2) = \frac{1}{2} \mathbb{P}(X = 0) \\ \mathbb{P}(X = 4) &= \frac{1}{3} \mathbb{P}(X = 3) = \frac{1}{3!} \mathbb{P}(X = 0) \\ &\vdots \\ \mathbb{P}(X = n + 1) &= \frac{1}{n!} \mathbb{P}(X = 0). \end{aligned}$$

Since

$$\begin{aligned} 1 &= \mathbb{P}(X = 0) + \mathbb{P}(X = 1) + \mathbb{P}(X = 2) + \cdots \\ &= \mathbb{P}(X = 0) + \mathbb{P}(X = 0) \sum_{n=0}^{\infty} \frac{1}{n!} \\ &= \mathbb{P}(X = 0)(1 + e) \end{aligned}$$

then $\mathbb{P}(X = 0) = 1/(1 + e)$.

2.3.1. The sums which are divisible by 3 are 3, 6, 9, and 12.

$$\mathbb{P}(3 \cup 6 \cup 9 \cup 12) = \frac{2}{36} + \frac{5}{36} + \frac{4}{36} + \frac{1}{36} = \frac{1}{3}$$

2.3.3. Let J be the event of drawing a jack. Let $\heartsuit$ be the event of drawing a heart. These events are not mutually exclusive.

$$\mathbb{P}(J \cup \heartsuit) = \mathbb{P}(J) + \mathbb{P}(\heartsuit) - \mathbb{P}(J \cap \heartsuit) = \frac{4}{52} + \frac{13}{52} - \frac{1}{52} = \frac{4}{13}$$

2.3.5. Let $\mathbb{P}(A) = 0.75$ and $\mathbb{P}(B) = 0.40$ then by Eq. (2.1),

$$\mathbb{P}(A \cap B) = \mathbb{P}(A) + \mathbb{P}(B) - \mathbb{P}(A \cup B) \geq 0.75 + 0.40 - 1 = 0.15$$

since $\mathbb{P}(A \cup B) \leq 1$.

2.3.7. Using Eq. (2.1) twice,

$$\begin{aligned}\mathbb{P}(A \cup B) &= \mathbb{P}(A) + \mathbb{P}(B) - \mathbb{P}(A \cap B)\\ \mathbb{P}(A \cup B^c) &= \mathbb{P}(A) + \mathbb{P}(B^c) - \mathbb{P}(A \cap B^c).\end{aligned}$$

Adding the two equations produces

$$\begin{aligned}0.75 + 0.65 &= 2\,\mathbb{P}(A) + \mathbb{P}(B) + \mathbb{P}(B^c) - (\mathbb{P}(A \cap B) + \mathbb{P}(A \cap B^c))\\ 0.60 &= 2\,\mathbb{P}(A) - \mathbb{P}((A \cap B) \cup (A \cap B^c)) = 2\,\mathbb{P}(A) - \mathbb{P}(A) = \mathbb{P}(A).\end{aligned}$$

2.3.9. Using the Addition Rule,

$$\mathbb{P}(X \text{ is even}) = \sum_{x=1}^{\infty} f_X(2x) = \sum_{x=1}^{\infty} \frac{1}{2^{2x}} = \sum_{x=1}^{\infty} \frac{1}{4^x} = \frac{1/4}{1 - 1/4} = \frac{1}{3}.$$

2.4.1. The outcomes of different spins of a roulette wheel are assumed to be independent, thus the probability that any spin of the wheel has an outcome of black is $\mathbb{P}(\text{black}) = 9/19$.

2.4.3.

(a) The probability that a five turns up twice,

$$\mathbb{P}(5 \cap 5) = \left(\frac{1}{6}\right)\left(\frac{1}{6}\right) = \frac{1}{36}.$$

(b) The probability that both numbers are even,

$$\mathbb{P}(\text{even} \cap \text{even}) = \left(\frac{1}{2}\right)\left(\frac{1}{2}\right) = \frac{1}{4}.$$

2.4.5.

(a) $\mathbb{P}(+\,|\,\text{ill}) = 0.99$
(b) $\mathbb{P}(\text{ill}) = 1/100000 = 0.00001$
(c) $\mathbb{P}(+\,|\,\text{healthy}) = 0.01$
(d) $\mathbb{P}(\text{healthy}) = 99999/100000 = 0.99999$
(e) The desired probability is calculated as follows.

$$\begin{aligned}\mathbb{P}(\text{ill}\,|\,+) = \frac{\mathbb{P}(\text{ill} \cap +)}{\mathbb{P}(+)} &= \frac{\mathbb{P}(+ \cap \text{ill})}{\mathbb{P}(+\,|\,(\text{ill} \cup \text{healthy}))}\\ &= \frac{\mathbb{P}(+\,|\,\text{ill})\,\mathbb{P}(\text{ill})}{\mathbb{P}(+\,|\,\text{ill})\,\mathbb{P}(\text{ill}) + \mathbb{P}(+\,|\,\text{healthy})\,\mathbb{P}(\text{healthy})}\\ &= \frac{(0.99)(0.00001)}{(0.99)(0.00001) + (0.01)(0.99999)}\\ &= 0.000989031\end{aligned}$$

2.4.9. Since the cards are drawn without replacement the outcome of the second draw is dependent on the outcome of the first draw. Using the Multiplication Rule and conditional probability,

$$\mathbb{P}(2 \text{ aces}) = \mathbb{P}(2\text{nd ace}|1\text{st ace})\,\mathbb{P}(1\text{st ace}) = \left(\frac{3}{51}\right)\left(\frac{4}{52}\right) = \frac{1}{221}.$$

2.4.11. Since the cards are drawn without replacement the outcome of the fourth draw is dependent on the outcome of the third draw which is dependent on the outcome of the second draw which is dependent on the outcome of the first draw. Using the Multiplication Rule and conditional probability,

$$\begin{aligned}\mathbb{P}\left(\text{4th ace}|\text{1st, 2nd, 3rd ace}\right) &= \frac{\mathbb{P}\left(\text{1st, 2nd, 3rd, 4th ace}\right)}{\mathbb{P}\left(\text{1st, 2nd, 3rd ace}\right)} \\ &= \frac{\frac{4}{52}\cdot\frac{3}{51}\cdot\frac{2}{50}\cdot\frac{1}{49}}{\frac{4}{52}\cdot\frac{3}{51}\cdot\frac{2}{50}} = \frac{1}{49}.\end{aligned}$$

2.4.13. For the three random variables described:

(a) $\Omega_Y = \{0, 1\}$,
(b) $\Omega_Z = \{0\}$,
(c) the events $X = x$ and $Y = y$ are dependent, thus

$$\mathbb{P}\left((X = x) \cap (Y = y)\right) = \mathbb{P}\left((Y = y) \,|\, (X = x)\right) \, \mathbb{P}\left(X = x\right).$$

Therefore,

$$\begin{aligned}\mathbb{P}\left((X = 0) \cap (Y = 0)\right) &= \mathbb{P}\left((Y = 0) \,|\, (X = 0)\right) \, \mathbb{P}\left(X = 0\right) \\ &= (0)(1 - p) = 0 \\ \mathbb{P}\left((X = 1) \cap (Y = 0)\right) &= \mathbb{P}\left((Y = 0) \,|\, (X = 1)\right) \, \mathbb{P}\left(X = 1\right) \\ &= (1)(p) = p \\ \mathbb{P}\left((X = 0) \cap (Y = 1)\right) &= \mathbb{P}\left((Y = 1) \,|\, (X = 0)\right) \, \mathbb{P}\left(X = 0\right) \\ &= (1)(1 - p) = 1 - p \\ \mathbb{P}\left((X = 1) \cap (Y = 1)\right) &= \mathbb{P}\left((Y = 1) \,|\, (X = 1)\right) \, \mathbb{P}\left(X = 1\right) \\ &= (0)(p) = 0\end{aligned}$$

(d) The events $X = x$ and $Z = z$ are dependent, thus

$$\mathbb{P}\left((X = x) \cap (Z = z)\right) = \mathbb{P}\left((Z = z) \,|\, (X = x)\right) \, \mathbb{P}\left(X = x\right).$$

Therefore,

$$\begin{aligned}\mathbb{P}\left((X = 0) \cap (Z = 0)\right) &= \mathbb{P}\left((Z = 0) \,|\, (X = 0)\right) \, \mathbb{P}\left(X = 0\right) \\ &= (1)(1 - p) = 1 - p \\ \mathbb{P}\left((X = 1) \cap (Z = 0)\right) &= \mathbb{P}\left((Z = 0) \,|\, (X = 1)\right) \, \mathbb{P}\left(X = 1\right) \\ &= (1)(p) = p\end{aligned}$$

2.4.17. If the number of women exceeds the number of men on the committee, then there are either two women and one man on the committee or three women on the committee. The probability that there are exactly two women on the committee is

$$p_2 = \left(\frac{9}{19}\right)\left(\frac{8}{18}\right)\left(\frac{10}{17}\right) + \left(\frac{9}{19}\right)\left(\frac{10}{18}\right)\left(\frac{8}{17}\right) + \left(\frac{10}{19}\right)\left(\frac{9}{18}\right)\left(\frac{8}{17}\right) = \frac{120}{323}.$$

The probability that there are three women on the committee is

$$p_3 = \left(\frac{9}{19}\right)\left(\frac{8}{18}\right)\left(\frac{7}{17}\right) = \frac{28}{323}.$$

Since these events are mutually exclusive then the probability that the number of women exceeds the number of men on the committee is the sum of the probabilities of these events.

$$p = \frac{120}{323} + \frac{28}{323} = \frac{148}{323} \approx 0.458204$$

2.4.19. Using Eq. (2.4),

$$\mathbb{P}\left(8|\text{different}\right) = \frac{\mathbb{P}\left(8 \cap \text{different}\right)}{\mathbb{P}\left(\text{different}\right)} = \frac{4/36}{30/36} = \frac{2}{15}.$$

2.4.21.

(a) The joint probability distribution is presented in the table below.

$$\begin{array}{llllll}
f_{X,Y}(1,1) = \frac{1}{6} & & & & & \\
f_{X,Y}(2,1) = \frac{1}{36} & f_{X,Y}(2,2) = \frac{5}{36} & & & & \\
f_{X,Y}(3,1) = \frac{1}{36} & f_{X,Y}(3,2) = \frac{1}{36} & f_{X,Y}(3,3) = \frac{1}{9} & & & \\
f_{X,Y}(4,1) = \frac{1}{36} & f_{X,Y}(4,2) = \frac{1}{36} & f_{X,Y}(4,3) = \frac{1}{36} & f_{X,Y}(4,4) = \frac{1}{12} & & \\
f_{X,Y}(5,1) = \frac{1}{36} & f_{X,Y}(5,2) = \frac{1}{36} & f_{X,Y}(5,3) = \frac{1}{36} & f_{X,Y}(5,4) = \frac{1}{36} & f_{X,Y}(5,5) = \frac{1}{18} & \\
f_{X,Y}(6,1) = \frac{1}{36} & f_{X,Y}(6,2) = \frac{1}{36} & f_{X,Y}(6,3) = \frac{1}{36} & f_{X,Y}(6,4) = \frac{1}{36} & f_{X,Y}(6,5) = \frac{1}{36} & f_{X,Y}(6,6) = \frac{1}{36}
\end{array}$$

(b) Since $f_Y(y) = \sum_{x=1}^{6} f_{X,Y}(x,y)$ then

$$f_Y(y) = \begin{cases} 11/36 & \text{if } y = 1 \\ 1/4 & \text{if } y = 2 \\ 7/36 & \text{if } y = 3 \\ 5/36 & \text{if } y = 4 \\ 1/12 & \text{if } y = 5 \\ 1/36 & \text{if } y = 6. \end{cases}$$

2.4.23. Let A and B be the random variables denoting the first defective keyboards Alex and Brent find respectively. The probability distribution for A is $f_A(i) = 0.02(0.98)^{i-1}$. The probability distribution for B is $f_B(j) = 0.01(0.99)^{j-1}$. Hence the joint probability distribution can be expressed as

$$f_{A,B}(i,j) = 0.0002(0.98)^{i-1}(0.99)^{j-1}.$$

The probability that Brent inspects fewer keyboards than Alex is then

$$\begin{aligned}
\mathbb{P}\left(B < A\right) &= \sum_{j=1}^{\infty}\sum_{i=j+1}^{\infty} f_{A,B}(i,j) = 0.0002\sum_{j=1}^{\infty}(0.99)^{j-1}\sum_{i=j+1}^{\infty}(0.98)^{i-1} \\
&= 0.0002\sum_{j=1}^{\infty}(0.99)^{j-1}\frac{(0.98)^j}{1-0.98} = 0.01\sum_{j=1}^{\infty}(0.99)^{j-1}(0.98)^j \\
&= 0.0098\sum_{j=0}^{\infty}(0.99*0.98)^j = \frac{0.0098}{1-0.99*0.98} \approx 0.328859.
\end{aligned}$$

2.5.1. Expressed as a piecewise defined function, the cumulative distribution function is

$$F_X(x) = \begin{cases} 0 & \text{if } x < 3 \\ 0.10 & \text{if } 3 \leq x < 5 \\ 0.25 & \text{if } 5 \leq x < 7 \\ 0.45 & \text{if } 7 \leq x < 9 \\ 0.75 & \text{if } 9 \leq x < 11 \\ 1.00 & \text{if } 11 \leq x. \end{cases}$$

2.5.3. From the cumulative distribution function, random variable X can take on the values $\{-1, 0, 2, 3, 4\}$. The probabilities are given in the table below.

x	-1	0	2	3	4
$f_X(x)$	1/6	1/3	1/4	1/12	1/6

2.5.5. Let the random variable X denote the number of times the outcome of the coin flip is heads, then $X \in \{0, 1, 2, 3, 4, 5\}$. The binomial coefficient $\binom{n}{k} = \frac{n!}{k!(n-k)!}$ is used to express the probability values.

$$\begin{aligned}
f_X(0) &= \binom{5}{0}\left(\frac{3}{5}\right)^0\left(\frac{2}{5}\right)^5 = \frac{32}{3125} \\
f_X(1) &= \binom{5}{1}\left(\frac{3}{5}\right)^1\left(\frac{2}{5}\right)^4 = \frac{48}{625} \\
f_X(2) &= \binom{5}{2}\left(\frac{3}{5}\right)^2\left(\frac{2}{5}\right)^3 = \frac{144}{625} \\
f_X(3) &= \binom{5}{3}\left(\frac{3}{5}\right)^3\left(\frac{2}{5}\right)^2 = \frac{216}{625} \\
f_X(4) &= \binom{5}{4}\left(\frac{3}{5}\right)^4\left(\frac{2}{5}\right)^1 = \frac{162}{625} \\
f_X(5) &= \binom{5}{5}\left(\frac{3}{5}\right)^5\left(\frac{2}{5}\right)^0 = \frac{243}{3125}
\end{aligned}$$

The cumulative distribution function for X is expressed as the following piecewise defined function.

$$F_X(x) = \begin{cases} 0 & \text{if } x < 0 \\ 32/3125 & \text{if } 0 \le x < 1 \\ 272/3125 & \text{if } 1 \le x < 2 \\ 992/3125 & \text{if } 2 \le x < 3 \\ 2072/3125 & \text{if } 3 \le x < 4 \\ 2882/3125 & \text{if } 4 \le x < 5 \\ 1 & \text{if } 5 \le x. \end{cases}$$

2.6.1. The number of elevators out of service is a binomial random variable with $n = 3$ and $p = 1/3$.

$$\begin{aligned}
f_X(0) &= \binom{3}{0}\left(\frac{1}{3}\right)^0\left(\frac{2}{3}\right)^3 = \frac{8}{27} \\
f_X(1) &= \binom{3}{1}\left(\frac{1}{3}\right)^1\left(\frac{2}{3}\right)^2 = \frac{4}{9} \\
f_X(2) &= \binom{3}{2}\left(\frac{1}{3}\right)^2\left(\frac{2}{3}\right)^1 = \frac{2}{9} \\
f_X(3) &= \binom{3}{3}\left(\frac{1}{3}\right)^3\left(\frac{2}{3}\right)^0 = \frac{1}{27}
\end{aligned}$$

2.6.3. Let X be the number of spins of the roulette wheel which result in red. X is a binomial random variable with parameters $n = 20$ and $p = 18/38$. The probability of at least 10 wins is

$$\sum_{k=10}^{20}\binom{20}{k}\left(\frac{18}{38}\right)^k\left(\frac{20}{38}\right)^{20-k} \approx 0.493719.$$

2.6.5. Let the number of subjects dropping out of the study be the binomially distributed random variable X with parameters $n = 30$ and $p = 0.1$. The medical study retains at least 20 subjects with probability

$$F_X(10) = \sum_{k=0}^{10}\binom{30}{k}(0.1)^k(0.9)^{30-k} \approx 0.999911.$$

2.6.7. The 9-seat airplane is oversold if all 10 ticketed passengers show up for the flight. The probability of this event is

$$p = (1 - 0.10)^{10} \approx 0.348678.$$

The 18-seat airplane is oversold if 19 or 20 ticketed passengers show up for the flight. The probability of this event is

$$q = (1 - 0.10)^{20} + 20(0.10)(1 - 0.10)^{19} \approx 0.391747.$$

The 18-seat airplane is more likely to be oversold.

2.6.9. The desired probability is calculated as follows.

$$\mathbb{P}\left(\text{4th black} \,|\, \text{4 black}\right) = \frac{\mathbb{P}\left(\text{4th black} \cap \text{4 black}\right)}{\mathbb{P}\left(\text{4 black}\right)} = \frac{\left(\frac{6}{14}\right) \frac{4!}{3!(4-3)!} \left(\frac{6}{14}\right)^3 \left(\frac{8}{14}\right)^{4-3}}{\frac{5!}{4!(5-4)!} \left(\frac{6}{14}\right)^4 \left(\frac{8}{14}\right)^{5-4}} = \frac{4}{5}$$

2.7.1. Let X be the sum of the dots on the top faces of the dice. Using the definition of expected value,

$$\begin{aligned}
\mathbb{E}\left(X\right) &= \sum_{x=2}^{12} (x \cdot f_X(x)) \\
&= (2)\left(\frac{1}{36}\right) + (3)\left(\frac{1}{18}\right) + (4)\left(\frac{1}{12}\right) + (5)\left(\frac{1}{9}\right) + (6)\left(\frac{5}{36}\right) \\
&\quad + (7)\left(\frac{1}{16}\right) + (8)\left(\frac{5}{36}\right) + (9)\left(\frac{1}{9}\right) + (10)\left(\frac{1}{12}\right) + (11)\left(\frac{1}{18}\right) \\
&\quad + (12)\left(\frac{1}{36}\right) \\
&= 7.
\end{aligned}$$

2.7.7. Using the probabilities calculated in Exercise 2.4.12 and the definition of expected value, the expected position of the first ace,

$$\mathbb{E}\left(X\right) = \sum_{x=1}^{49} (x \cdot f_X(x)) = \frac{53}{5} = 10.6.$$

2.7.9. The conditional probability distribution for Y given $X = 2$ is

$$f_{Y|X}(y, 2) = \begin{cases} 1/5 & \text{if } y = 1 \\ 0 & \text{if } y = 2 \\ (2/15)(5/6)^{y-3} & \text{if } y = 3, 4, \ldots. \end{cases}$$

Thus the conditional expectation is

$$\mathbb{E}\left(Y|X = 2\right) = \frac{1}{5} + \sum_{y=3}^{\infty} \left(\frac{2y}{15}\right)\left(\frac{5}{6}\right)^{y-3} = \frac{33}{5}.$$

2.8.1. The expected value and variance are found as follows.

$$\mathbb{E}\left(X\right) = (-1)\left(\frac{1}{2}\right) + (1)\left(\frac{1}{2}\right) = 0$$

$$\mathbb{V}\mathrm{ar}\left(X\right) = (-1)^2\left(\frac{1}{2}\right) + (1)^2\left(\frac{1}{2}\right) - (0)^2 = 1$$

2.8.3. Using the probabilities calculated in Exercise (2.4.12) and the expected value calculated in Exercise (2.7.7), the variance in the position of the first ace drawn is

$$\sum_{x=1}^{49} \left(x^2 \cdot f_X(x)\right) - \mu^2 = \frac{901}{5} - \frac{2809}{25} = \frac{1696}{25} = 67.84.$$

This implies the standard deviation in the appearance of the first ace is approximately 8.2365.

2.8.5. The expected value and variance are as follows.

$$\begin{aligned}\mathbb{E}(Y) &= \mathbb{E}(3X+4) = 3\,\mathbb{E}(X) + \mathbb{E}(4) = 3(2) + 4 = 10\\ \mathbb{V}\mathrm{ar}(Y) &= \mathbb{V}\mathrm{ar}(3X+4) = (3)^2\,\mathbb{V}\mathrm{ar}(X) = 63\end{aligned}$$

2.10.1. Since $\mathbb{P}(⚅) = 1/6$, the odds in favor of ⚅ are

$$\frac{1}{6} : \frac{5}{6} \iff 1:5.$$

2.10.3. According to odds against of 15 : 2 for each \$2 wagered, the winning gambler will receive a net profit of \$15. Since the gambler wagers \$10, if Nuance wins the net profit will be \$75.

2.10.5. The probability of drawing two ♡'s without replacement is

$$p = \left(\frac{13}{52}\right)\left(\frac{12}{51}\right) = \frac{1}{17}.$$

Thus the odds against drawing two ♡'s are

$$\frac{16}{17} : \frac{1}{17} \iff 16:1.$$

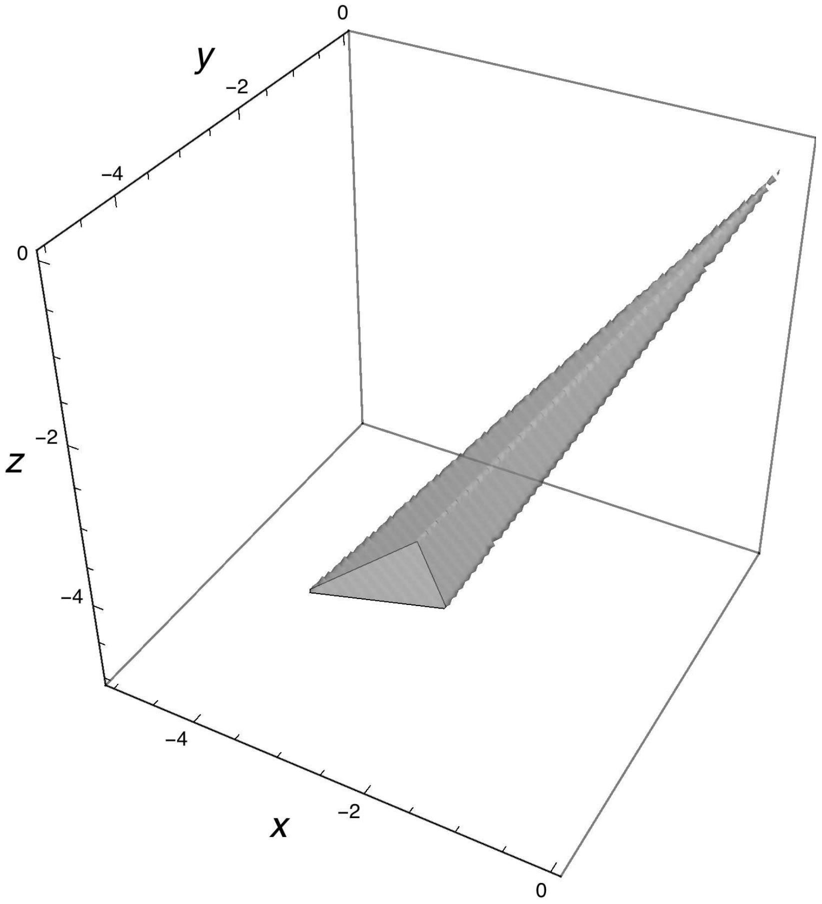
3.1.1. Converting from the odds against to probabilities that the outcomes may occur produces a table of probabilities like that which follows.

Outcome	**Probability**
A	$\frac{1}{3}$
B	$\frac{1}{4}$
C	$\frac{1}{2}$

Since the probabilities sum to $\frac{13}{12} > 1$, the Arbitrage Theorem guarantees the existence of a betting strategy which yields a positive payoff regardless of the outcome. Suppose amounts x, y, and z are wagered on outcomes A, B, and C respectively. The payoffs under different winning scenarios are listed in the next table.

Winning Outcome	**Payoff**
A	$2x - y - z$
B	$-x + 3y - z$
C	$-x - y + z$

Solving this problem will entail finding values for x, y, and z which make the three expressions in the column on the right positive. The graphic below shows the "cone" where all three payoffs are positive. Notice that each amount wagered must be negative (this implies taking the wagers from others). By inspection, $x = -3$, $y = -2.5$, and $z = -5$ is one solution to this set of inequalities.



3.1.7. The feasible set is the simplex in the positive orthant defined by the plane $x_1 + 2x_2 + 3x_3 = 6$. The maximum of the cost function will occur at one of the points $(x_1, x_2, x_3) = (6, 0, 0)$, $(x_1, x_2, x_3) = (0, 3, 0)$, or $(x_1, x_2, x_3) = (0, 0, 2)$. Checking the value of the cost function at each of these points reveals

$$\max\{2x_1 + 7x_2 + 5x_3\} = 21 \text{ at } (x_1, x_2, x_3) = (0, 3, 0).$$

3.1.9. Maximize $\mathbf{c}^T\mathbf{x} = -7x_1 - 9x_2 - 16x_3$ subject to $\mathbf{x} \geq \mathbf{0}$ and

$$\begin{bmatrix} -1 & -2 & -9 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} \leq -2.$$

3.1.11. Introduce the slack variables

$$\begin{aligned} \hat{y}_4 &= 21y_1 + 13y_2 + 4y_3 - 11 \geq 0 \\ \hat{y}_5 &= 17y_1 + 4y_2 + 8y_3 - 23 \geq 0 \\ \hat{y}_6 &= 14y_1 + 2y_2 + 27y_3 - 15 \geq 0 \\ \hat{y}_7 &= 11y_1 + 8y_2 + 18y_3 - 29 \geq 0. \end{aligned}$$

The linear program can then be stated as follows.

Maximize $-7y_1 - 2y_2 - 11y_3 + 0\hat{y}_4 + 0\hat{y}_6 + 0\hat{y}_6 + 0\hat{y}_7 = \overline{\mathbf{c}}^T\overline{\mathbf{y}}$ subject to $\overline{\mathbf{y}} \geq \mathbf{0}$ and

$$\overline{A}\overline{\mathbf{y}} = \left[\begin{array}{rrr|rrrr} 21 & 13 & 4 & -1 & 0 & 0 & 0 \\ 17 & 4 & 8 & 0 & -1 & 0 & 0 \\ 14 & 2 & 27 & 0 & 0 & -1 & 0 \\ 11 & 8 & 18 & 0 & 0 & 0 & -1 \end{array}\right] \left[\begin{array}{c} y_1 \\ y_2 \\ y_3 \\ \hline \hat{y}_4 \\ \hat{y}_5 \\ \hat{y}_6 \\ \hat{y}_7 \end{array}\right] = \left[\begin{array}{c} 11 \\ 23 \\ 15 \\ 29 \end{array}\right].$$

3.2.7. Minimize $2y_1 + y_2 + 5y_3$ subject to $\langle y_1, y_2, y_3\rangle \geq \mathbf{0}$ and

$$\begin{aligned} -4y_1 + y_2 + 2y_3 &\geq 2 \\ y_1 - y_2 + y_3 &\geq -1. \end{aligned}$$

3.2.9. Introduce a slack variable x_4 to convert the inequality constraint in the problem to an equality constraint. Think of the linear problem as: minimize $x_1 + x_2 + x_3$ subject to $2x_1 + x_2 = 4$ and $x_3 + x_4 = 6$ with $x_i \geq 0$ for $i = 1, 2, 3, 4$. Define the following vectors and matrix

$$\mathbf{c} = \langle 1, 1, 1, 0\rangle,\ \mathbf{x} = \langle x_1, x_2, x_3, x_4\rangle,\ A = \left[\begin{array}{cccc} 2 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 \end{array}\right],\ \text{and } \mathbf{b} = \langle 4, 6\rangle,$$

then the linear problem is the one of minimizing $\mathbf{c}^T\mathbf{x}$ subject to $A\mathbf{x} = \mathbf{b}$ and $\mathbf{x} \geq \mathbf{0}$. The dual to this problem is the one of maximizing $\mathbf{b}^T\mathbf{y}$ subject to $A^T\mathbf{y} \leq \mathbf{c}$ where $\mathbf{y} = \langle y_1, y_2\rangle$. This is equivalent to maximizing $4y_1 + 6y_2$ subject to the system of inequalities

$$\begin{aligned} 2y_1 &\leq 1 \\ y_1 &\leq 1 \\ y_2 &\leq 1 \\ y_2 &\leq 0. \end{aligned}$$

3.2.11. Written in matrix/vector form this linear problem becomes: maximize $\mathbf{b}^T\mathbf{y} = \langle 2, 0, 4\rangle \cdot \langle y_1, y_2, y_3\rangle$ subject to

$$A^T\mathbf{y} = \left[\begin{array}{ccc} 1 & 1 & 0 \\ 0 & 1 & 2 \end{array}\right] \left[\begin{array}{c} y_1 \\ y_2 \\ y_3 \end{array}\right] \leq \left[\begin{array}{c} 1 \\ 1 \end{array}\right] = \mathbf{c}.$$

Thus the dual problem is minimize $\mathbf{c}^T\mathbf{x} = x_1 + x_2$ subject to

$$A\mathbf{x} = \left[\begin{array}{cc} 1 & 0 \\ 1 & 1 \\ 0 & 2 \end{array}\right] \left[\begin{array}{c} x_1 \\ x_2 \end{array}\right] = \left[\begin{array}{c} 2 \\ 0 \\ 4 \end{array}\right] = \mathbf{b}.$$

3.2.13. Let vector $\mathbf{b} = \langle 1, 1, 2\rangle$ and $\mathbf{y} = \langle y_1, y_2, y_3\rangle$. The problem as stated can be thought of as a dual problem: minimize $\mathbf{b}^T\mathbf{y}$ subject to

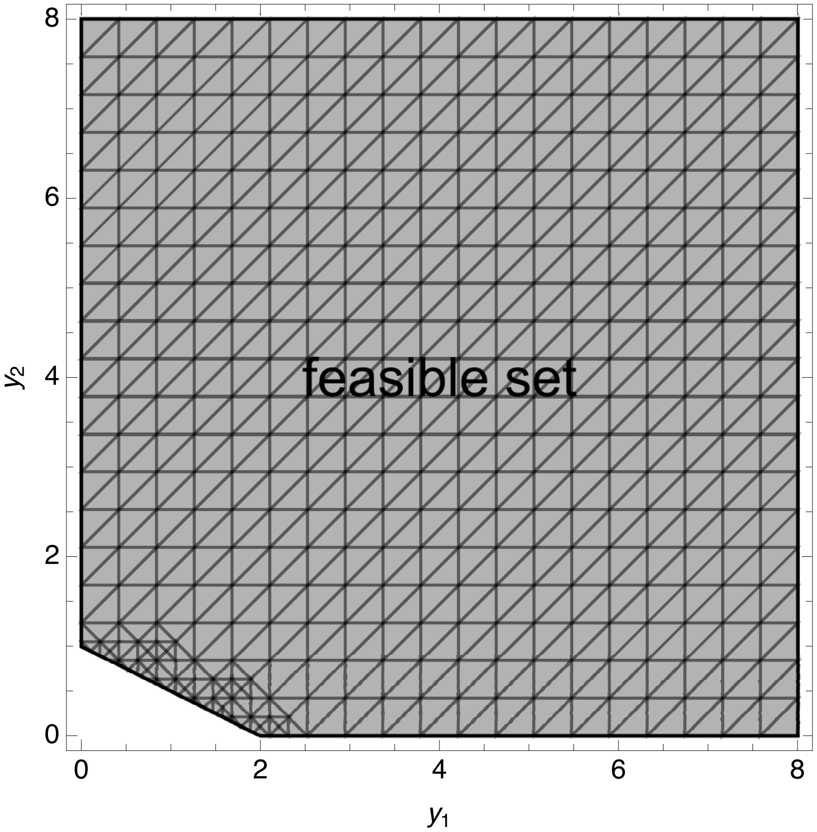
$$A^T\mathbf{y} = \left[\begin{array}{ccc} 1 & 2 & 3 \end{array}\right]\mathbf{y} \geq [15] = \mathbf{c}.$$

This is equivalent to the primal problem: maximize $\mathbf{c}^T\mathbf{x} = 15\mathbf{x}$ subject to $A\mathbf{x} \leq \mathbf{b}$, where $\mathbf{x} = \langle x_1\rangle$. From the inequality constraint in the primal problem $x_1 \leq 1/2$. Thus the maximum value of $15x_1$ is $15/2$. Thus the minimum value of $\mathbf{b}^T\mathbf{y} = 15/2$. According to Thm. 3.4 this minimum occurs when $\mathbf{y} = \langle 0, \frac{15}{2}, 0\rangle$.

3.2.15. The dual problem is one of minimizing $3y_1 + 7y_2$ subject to $\langle y_1, y_2\rangle \geq \mathbf{0}$ and

$$\begin{aligned} y_1 + 3y_2 &\geq 1 \\ y_1 + 2y_2 &\geq 2 \\ 2y_1 + 2y_2 &\geq 1 \\ y_1 + 3y_2 &\geq 1. \end{aligned}$$

The feasible region is shown below.



The minimum of 6 for the dual problem is found at $\langle y_1, y_2\rangle = \langle 2, 0\rangle$. Since the first, third, and fourth constraints in the dual are strict at the optimal point, $x_1 = x_3 = x_4 = 0$ in the optimal solution for the primal. Thus the primal linear program can be simplified to maximizing $2x_2$ subject to $\langle 0, x_2, 0, 0\rangle \geq \mathbf{0}$ and

$$\begin{aligned} x_2 &\leq 3 \\ 2x_2 &\leq 7. \end{aligned}$$

Hence the optimal value for $x_2 = 3$ and the optimal solution to the primal linear program is $\langle 0, 3, 0, 0\rangle$ with a maximum of 6.

3.2.17. Let x be the number of acres of corn planted and y be the number of acres of saw grass planted. The problem requires the maximization of $100x + 80y$ subject to the constraints

$$\begin{aligned} 0 &\geq x \\ 0 &\geq y \\ 440 &\geq x + y \\ 50000 &\geq 75x + 50y \\ 30000 &\geq 110x + 30y. \end{aligned}$$

The level sets of the function $L(x, y) = 100x + 80y$ intersect a vertex of the feasible region at $(x, y) = (210, 230)$. Thus the maximum profit is \$39,400.

3.3.5. The price of the asset is the present value of its expected value in two months.

$$\begin{aligned} S &= e^{-0.035(2/12)} \left(90(0.25) + 100(0.35) + 105(0.20) + 110(0.20)\right) \\ &\approx 99.9155 \end{aligned}$$

4.1.1. The monthly returns expressed as percentages are as follows.

$$R = \{-9.8979, 15.0639, -6.4237, 12.4287, 1.6331, -0.5634, 9.2602, 6.6500, 1.1025, -6.5025, -18.6493\}$$

The annualized average return is

$$12\left(\frac{1}{11}\sum_{R} R_i\right) = \frac{12}{11}(4.1017) = 4.4746\%.$$

The annualized volatility is

$$\sqrt{12\left(\frac{1}{11-1}\sum_{R}(R_i - \mathbb{E}\left(R\right))^2\right)} = \sqrt{\frac{12}{10}(1043.35)} = 35.3839\%.$$

4.1.3. If the investment is made at the risk-free rate for two years,

$$\mathbb{E}\left(F\right) = 1000(1+0.075)^2 = \$1,155.6250.$$

If investment is made at the risk-free rate for the first year and in the stock index for the second year,

$$\mathbb{E}\left(F\right) = 1000(1+0.075)\left[(1+0.37)(0.5) + (1-0.15)(0.5)\right] = \$1,193.25.$$

If investment is made at the in the stock index for the two years, the stock index could return 37% both years (with probability 1/4), could return 37% in one of the years and -15% in the other year (with probability 1/2), or could return -15% in both years (with probability 1/4).

$$\mathbb{E}\left(F\right) = 1000\left[\frac{1}{4}(1+0.37)^2 + \frac{1}{2}(1+0.37)(1-0.15) + \frac{1}{4}(1-0.15)^2\right] = \$1,232.10.$$

4.1.5. Consider the mean returns and variances of the three individual stocks.

$$\begin{aligned}
r_A &= (0.55)(0.06) + (0.45)(0.14) = 0.096\\
\mathbb{Var}\left(R_A\right) &= 0.001584\\
r_B &= (0.55)(0.17) + (0.45)(-0.01) = 0.089\\
\mathbb{Var}\left(R_B\right) &= 0.008019\\
r_C &= (0.55)(0.35) + (0.45)(-0.09) = 0.152\\
\mathbb{Var}\left(R_C\right) &= 0.047916
\end{aligned}$$

Next find the pairwise covariances.

$$\begin{aligned}
\mathbb{Cov}\left(R_A, R_B\right) &= (0.55)(0.06)(0.17) + (0.45)(0.14)(-0.01) - (0.096)(0.089) = -0.003564\\
\mathbb{Cov}\left(R_A, R_C\right) &= (0.55)(0.06)(0.35) + (0.45)(0.14)(-0.09) - (0.096)(0.152) = -0.008712\\
\mathbb{Cov}\left(R_B, R_C\right) &= (0.55)(0.17)(0.35) + (0.45)(-0.01)(-0.09) - (0.089)(152) = 0.019602
\end{aligned}$$

The variance in the portfolio return is

$$\begin{aligned}
\mathbb{Var}\left(R_P\right) &= \left[\begin{array}{ccc} 0.25 & 0.25 & 0.50 \end{array}\right] \left[\begin{array}{ccc} 0.001584 & -0.003564 & -0.008712\\ -0.003564 & 0.008019 & 0.019602\\ -0.008712 & 0.019602 & 0.047916 \end{array}\right] \left[\begin{array}{c} 0.25\\ 0.25\\ 0.50 \end{array}\right] = 0.0148562\\
\sigma_{R_P} &= 0.121886 = 12.1886\%.
\end{aligned}$$

4.2.7. The weight of stock A in the portfolio is

$$x = \frac{-3,000,000}{10,000,000} = -0.30$$

while the weight of stock B is $1 - x = 1.30$. Thus the expected return and volatility are respectively,

$$r_P = (-0.30)0.145 + (1.30)(0.1675) = 0.17425 = 17.425\%$$
$$\sigma_p = \sqrt{(-0.3)^2(0.42)^2 + (1.3)^2(0.38)^2 + 2(-0.3)(1.3)(0.63)(0.42)(0.38)} = 0.775611 = 77.5611\%.$$

4.2.9. The system of linear equations for the stocks can be written as

$$\begin{aligned} 0.019w_1 + 0.007w_2 + 0.001w_3 &= 0.15 - 0.06 \\ 0.007w_1 + 0.038w_2 - 0.002w_3 &= 0.10 - 0.06 \\ 0.001w_1 - 0.002w_2 + 0.022w_3 &= 0.08 - 0.06 \end{aligned}$$

The solution is $(w_1, w_2, w_3) = (4.60993, 0.241408, 0.721495)$ (with sum $\eta = 5.57283$). After rescaling the solution becomes $(x_1, x_2, x_3) = (0.827215, 0.0433186, 0.129466)$. Thus 82.7215% of an investor's wealth should be placed in stock A, 4.33186% in stock B, and 12.9466% in stock C, if the investor wishes to hold only risky assets.

4.3.1. Between a good economy and a bad economy the market portfolio changes in value by $35 - (-25) = 60\%$. For the specific asset the change in value is also $40 - (-20) = 60\%$. Thus the beta of the asset is

$$\beta = \frac{60}{60} = 1.$$

4.3.3. The systematic risk is $\beta\sigma_M = (1.36)(0.15) = 0.204$. An efficient portfolio will have the same Sharpe ratio as the market portfolio which is

$$\frac{0.12 - 0.05}{0.15} = \frac{7}{15}.$$

The expected return of the efficient portfolio is thus

$$0.05 + \frac{7}{15}(0.204) = 0.1452 = 14.52\%,$$

the same as the return on the asset.

4.3.5.

(a) Volatilities:

$$\begin{aligned} \mathbb{E}(R_M) &= \frac{1}{4}(-0.12 - 0.06 + 0.06 + 0.13) = 0.0025 \\ \mathbb{E}(R_M^2) &= \frac{1}{4}((-0.12)^2 + (-0.06)^2 + (0.06)^2 + (0.13)^2) = 0.009625 \\ \mathbb{E}(R_A) &= \frac{1}{4}(-0.07 - 0.04 + 0.08 + 0.17) = 0.035 \\ \mathbb{E}(R_A^2) &= \frac{1}{4}((-0.07)^2 + (-0.04)^2 + (0.08)^2 + (0.17)^2) = 0.01045 \\ \mathbb{E}(R_B) &= \frac{1}{4}(-0.18 - 0.10 + 0.04 + 0.08) = -0.04 \\ \mathbb{E}(R_B^2) &= \frac{1}{4}((-0.18)^2 + (-0.10)^2 + (0.04)^2 + (0.08)^2) = 0.0126 \\ \sigma_M &= \sqrt{0.009625 - (0.0025)^2} = 0.0980752 \\ \sigma_A &= \sqrt{0.01045 - (0.035)^2} = 0.0960469 \\ \sigma_B &= \sqrt{0.0126 - (-0.04)^2} = 0.104881 \end{aligned}$$

(b) Covariances:

$$\begin{aligned}
\mathbb{E}\left(R_M R_A\right) &= \frac{1}{4}((-0.12)(-0.07) + (-0.06)(-0.04) + (0.06)(0.08) + (0.13)(0.17)) = 0.009425 \\
\mathbb{E}\left(R_M R_B\right) &= \frac{1}{4}((-0.12)(-0.18) + (-0.06)(-0.10) + (0.06)(0.04) + (0.13)(0.08)) = 0.0101 \\
\mathbb{Cov}\left(R_M, R_A\right) &= 0.009425 - (0.0025)(0.035) = 0.0093375 \\
\mathbb{Cov}\left(R_M, R_B\right) &= 0.0101 - (0.0025)(-0.04) = 0.0102
\end{aligned}$$

(c) Betas:

$$\begin{aligned}
\beta_A &= \frac{0.0093375}{\sqrt{0.0980752}} = 0.97076 \\
\beta_B &= \frac{0.0102}{\sqrt{0.0980752}} = 1.06043
\end{aligned}$$

4.3.7. First find the beta for the portfolio.

$$\mathbb{E}\left(R_P\right) = 0.22 = 0.05 + \beta_P(0.17 - 0.05) \iff \beta_P = 1.41667$$

The beta of the portfolio is the weighted average of the betas for the stocks.

$$1.41667 = 1.45x + 0.90(1 - x) \iff x = 0.939394$$

Thus the portfolio consists of 93.9394$ stock A and 6.0606% stock B.

4.3.9. According to the capital asset pricing model,

$$\mathbb{E}\left(R\right) = 0.03 + 0.95(0.10 - 0.03) = 0.0965 = 9.65\% < 15\%.$$

The expected return is not that predicted by the CAPM. The stock is overpriced.

4.3.11. Substituting $r_f = 0.0475$, $r_M = 0.0765$, $\sigma_M = 0.22$, and $\mathbb{Cov}\left(R, R_M\right) = 0.15$ into Eq. (4.10) produces

$$r = 0.0475 + \frac{0.15}{(0.22)^2}(0.0765 - 0.0475) = 0.137376.$$

4.4.1. According to Thm. 4.1

$$\begin{aligned}
\alpha_A &= \frac{\frac{1}{0.24}}{\frac{1}{0.24} + \frac{1}{0.41} + \frac{1}{0.27} + \frac{1}{0.16} + \frac{1}{0.33}} = 0.212697 \\
\alpha_B &= \frac{\frac{1}{0.41}}{\frac{1}{0.24} + \frac{1}{0.41} + \frac{1}{0.27} + \frac{1}{0.16} + \frac{1}{0.33}} = 0.124505 \\
\alpha_C &= \frac{\frac{1}{0.27}}{\frac{1}{0.24} + \frac{1}{0.41} + \frac{1}{0.27} + \frac{1}{0.16} + \frac{1}{0.33}} = 0.189064 \\
\alpha_D &= \frac{\frac{1}{0.16}}{\frac{1}{0.24} + \frac{1}{0.41} + \frac{1}{0.27} + \frac{1}{0.16} + \frac{1}{0.33}} = 0.319045 \\
\alpha_E &= \frac{\frac{1}{0.33}}{\frac{1}{0.24} + \frac{1}{0.41} + \frac{1}{0.27} + \frac{1}{0.16} + \frac{1}{0.33}} = 0.154689
\end{aligned}$$

4.4.3. According to Lemma 4.2

$$\begin{aligned}
\alpha_A &= 4.44549 \\
\alpha_B &= 1.30112 \\
\alpha_C &= 7.9031 \\
\alpha_D &= 0.0 \\
\alpha_E &= 9.69925.
\end{aligned}$$

4.5.1. According to Thm. 4.4, a twice differentiable function $u(x)$, is concave when $u''(x) \leq 0$.

(a) $u(x) = \ln x$

$$\begin{aligned} u'(x) &= \frac{1}{x} \\ u''(x) &= -\frac{1}{x^2} < 0 \end{aligned}$$

for $0 < x < \infty$.

(b) $u(x) = (\ln x)^2$

$$\begin{aligned} u'(x) &= \frac{2 \ln x}{x} \\ u''(x) &= \frac{2(1 - \ln x)}{x^2} \end{aligned}$$

This function is not concave on its whole domain. It is concave only for x such that $\ln x \geq 1$, in other words only for $x \geq e$.

(c) $u(x) = \tan^{-1} x$

$$\begin{aligned} u'(x) &= \frac{1}{1+x^2} \\ u''(x) &= -\frac{2x}{(1+x^2)^2} \end{aligned}$$

This function is not concave on its whole domain. It is concave only for $x \geq 0$.

4.5.5. The certainty equivalent is the solution C of the equation

$$\begin{aligned} \frac{1}{2} f(15) + \frac{1}{2} f(-15) &= f(C) \\ -\frac{225}{2} &= C - \frac{C^2}{2} \\ C &= 1 \pm \sqrt{226}. \end{aligned}$$

The certainty equivalent will be the smaller of the two roots of the quadratic equation, *i.e.*, $C = 1 - \sqrt{226} \approx -14.0333$.

5.1.1. At settlement the farmer will receive

$$(15000)(3.75) = \$56,250.$$

5.1.3. Let the bid price be B, then the ask price is $A = B + 0.25$. The net cost to the investor is

$$1000B - 1000A = -250.$$

The round trip cost is \$250.

5.1.5. Devon is able to purchase Drew's house for \$100,000 and sell it for \$105,000, earning a profit of \$5,000.

5.2.3. The delivery price of the forward contract is

$$F_{0,3/12} = 23e^{0.0475(3/12)} \approx 23.2748.$$

5.2.5. Using Eq. (5.1),

$$68 = 65e^{r(6/12)} \iff r \approx 0.090241$$

or $r = 9.0241\%$.

5.2.7. The no-arbitrage delivery price of the forward contract is

$$F_{0,1} = 1200e^{0.0375} \approx \$1245.85.$$

The quoted price for the forward contract is \$1250, higher than the no-arbitrage price. An investor could sell the forward for \$1250 and buy the stock for \$1200 with borrowed funds. At maturity the investor sells the stock for \$1250 and repays the loan. The final cash flow is

$$1250 - 1200e^{0.0375} \approx \$4.1456 > 0.$$

5.3.1. The delivery price of the prepaid forward is

$$F^P_{0,1} = 97 - 2.50e^{-0.0365(6/12)} - 2.75e^{-0.0365(12/12)} \approx \$91.8938.$$

The delivery price of a forward contract on the dividend paying stock is

$$F_{0,1} = 91.8938e^{0.0365(12/12)} \approx \$95.3099.$$

5.3.3.

(a) The delivery price of a prepaid forward contract is

$$F^P_{0,1} = S(0) - \sum_{i=1}^{4} D_i e^{-0.0375(3i/12)} = 75 - \sum_{i=1}^{4} 2e^{-0.0375(i/4)} \approx \$67.1849.$$

(b) The delivery price of a forward contract is

$$F_{0,1} = S(0)e^{0.0375} - \sum_{i=1}^{4} D_i e^{0.0375-0.0375(3i/12)} = 75e^{0.0375} - \sum_{i=1}^{4} 2e^{0.0375(1-i/4)} \approx \$69.7522.$$

5.3.5. Let $S(0) = 75$, $r = 0.0475$, and $\delta = 0.08$.

(a) Prepaid forward contract:

$$F^P_{0,1} = 75e^{-0.08} \approx \$69.2337.$$

(b) Forward contract:

$$F_{0,1} = 75e^{0.0475-0.08} \approx \$72.6017.$$

5.3.7.

(a) The delivery price of a prepaid forward contract is

$$F^P_{0,1} = S(0)e^{-\delta T} = 115e^{-0.02} \approx \$112.723.$$

(b) The delivery price of a forward contract is

$$F_{0,1} = S(0)e^{(r-\delta)T} = 115e^{0.035-0.02} \approx \$116.738.$$

5.3.9. Let $S(0) = 85$, $r = 0.0625$, and $F_{0,1} = 89$.

$$\begin{aligned} F_{0,1} &= S(0)e^{(r-\delta)T} \\ 89 &= 85e^{0.0625-\delta} \\ \delta &= 0.0625 - \ln\frac{89}{85} \approx 0.0165149 \end{aligned}$$

5.3.11.

(a) The delivery price of a six-month forward contract for the non-dividend-paying stock is

$$F_{0,6/12} = S(0)e^{rT} = 45e^{0.0425(6/12)} \approx \$45.97.$$

(b) The continuous dividend yield for the stock is

$$\begin{aligned} F_{0,T} &= S(0)e^{(r-\delta)T} \\ 44 &= 45e^{(0.0425-\delta)(6/12)} \\ \frac{44}{45} &= e^{(0.0425-\delta)/2} \\ \ln\frac{44}{45} &= (0.0425-\delta)/2 \\ 2\ln\frac{44}{45} &= 0.0425-\delta \\ \delta &= 0.0425 - 2\ln\frac{44}{45} \approx 0.087446. \end{aligned}$$

5.3.15.

$$\begin{aligned} F^{P}_{0,3/12} &= \frac{10^{10}}{107.68}e^{-0.03(3/12)} = \$92,173,853.53 \\ F_{0,3/12} &= \frac{10^{9}}{107.68}e^{(0.04-0.03)(3/12)} = \$93,100,216.16 \end{aligned}$$

5.4.1. The interval of arbitrage-free prices is

$$\begin{aligned} (97-2(0.50))e^{0.035(4/12)} \leq F &\leq (97.50+2(0.50))e^{0.045(4/12)} \\ \$97.1266 \leq F &\leq \$99.9886. \end{aligned}$$

5.4.3. The delivery price is

$$F_{0,1} = 1000\left(14.83 + \sum_{i=1}^{4} 0.25e^{-0.042(3i/12)}\right)e^{0.042} \approx \$16,482.07.$$

5.4.5. The delivery price is

$$F_{0,1} = 1000\left(14.83 + \sum_{i=0}^{3} 0.25e^{-0.042(3i/12)}\right)e^{0.042} \approx \$16,492.79.$$

5.4.7. If the storage costs are a negative dividend then

$$F_{0,T} = S(0)e^{(r+\gamma)T}.$$

5.4.9. The present value is

$$\begin{aligned} P &= 1500(4.0995-1.25)e^{-0.03(2)} + 1575(4.0945-1.25)e^{-0.03(3)} \\ &\quad + 1625(4.1215-1.25)e^{-0.03(4)} + 1500(4.1235-1.25)e^{-0.03(7)} \\ &\approx \$15,752.20. \end{aligned}$$

5.5.3. The arbitrage-free, 3-month forward should be priced at

$$F_{0,3/12} = 1200e^{(0.04-0.015)(3/12)} = \$1207.5235$$

and thus the market price of the forward is too low. An arbitrage opportunity can be created by taking a long position in the forward contract, shorting $e^{-0.015(3/12)}$ shares of the stock and paying the dividends at the continuously compounded rate of $\delta = 0.015$ for $0 \leq t \leq 3/12$, and lending $S(0)e^{-0.015(3/12)} = 1200e^{-0.015(3/12)} = 1195.5084$ at the risk-free rate.

	Cash flow $t = 0$	Cash flow $t = 3/12$
Long forward	0	$S(T) - 1175$
Short stock	1195.5084	$-S(T)$
Lend	-1195.5084	1207.5235
Aggregate	0	32.5235

Thus a risk-less profit of \$32.5235 can be made.

5.5.5. The arbitrage-free price of the 6-month forward contract should be

$$F_{0,6/12} = 100e^{0.05(6/12)} - 2e^{0.05(6/12-4/12)} = 100.5148$$

and thus the market price for the forward contract is too low. An arbitrage opportunity can be created by taking a long position in the forward contract, shorting the stock, and lending \$100 at the risk-free rate. The dividend will be paid by borrowing at the risk-free rate.

	Cash flow $t = 0$	Cash flow $t = 4/12$	Cash flow $t = 6/12$
Long forward	0	0	$S(T) - 100$
Short stock	100	-2	$-S(T)$
Lend	-100	0	102.5315
Borrow	0	2	-2.0167
Aggregate	0	0	0.5148

Thus a risk-less profit of \$0.5148 can be made.

5.6.1. A margin call is issued if the value of margin drops \$500. Thus if the futures price drops to \$5.15 the lost value in the margin is

$$5000(5.25 - 5.15) = 5000(0.10) = 500.$$

5.6.3. The initial margin is in the amount of

$$(100)(950)(0.125) = \$11,875,$$

which after one month of interest compounded continuously at 6% per annum will be \$11,934.52. A margin call will be issued only if the price of the security at the one-month mark satisfies

$$\begin{aligned} 11934.52 + (S(1) - 950)(100) &\le S(1)(100)(0.125) \\ S(1) &\le \$949.32. \end{aligned}$$

5.6.5. The results are presented in the table below.

Futures Price	**Daily Change**	**Cumulative Change**	**Margin Balance**	**Margin Call**
775.00			150,000.00	
774.67	−495	−495.00	149,546.10	0
779.39	7,080	6,584.86	156,667.08	0
778.42	−1,455	5,131.67	155,255.01	0
749.56	−43,290	−38,156.93	112,007.55	22,992.45
742.87	−10,035	−48,202.38	125,001.99	9,998.01
735.64	−10,845	−59,060.59	124,191.99	10,808.01
741.59	8,925	−50,151.77	143,961.99	0
759.88	27,435	−22,730.51	171,436.44	0
766.25	9,555	−13,181.74	181,038.41	0
805.36	58,665	45,479.65	239,753.02	0

The final profit to the investor is \$45,479.65.

6.1.5. If the option is European with strike time T, then the payoff can be expressed as

$$(K - S(T))^+ = \max\{K - S(T), 0\} = \begin{cases} K - S(T) & \text{if } K > S(T), \\ 0 & \text{if } K \leq S(T). \end{cases}$$

6.2.1. Here $S = 31$, $T = 1/4$, $C^e = 3$, $K = 31$, $\delta = 0$, and $r = 0.10$ then according to the Put-Call Parity formula in Eq. (6.9):

$$\begin{aligned} P^e + S &= C^e + Ke^{-rT} \\ P^e + 31 &= 3 + 31e^{-(0.10)(0.25)} \\ P^e &= 2.23. \end{aligned}$$

6.2.3. Here $T = 1/6$, $K = 14$, $S = 11$, and $r = 0.07$ then according to the Put-Call Parity formula in Eq. (6.1):

$$\begin{aligned} P^e &= C^e - S + Ke^{-rT} \\ &\geq -S + Ke^{-rT} \\ &= -11 + 14e^{-(0.07)/6} \\ &= 2.84. \end{aligned}$$

6.2.5. Using the Put-Call Parity formula Eq. (6.9),

$$\begin{aligned} P + 36 &= 2.25 + 38e^{-0.0475(4/12)} \\ P &\approx 3.65307. \end{aligned}$$

6.2.7. Using the Put-Call Parity formula Eq. (6.9), we may write the following two equations.

$$\begin{aligned} C^e(50, 0; 45, 6/12) - P^e(50, 0; 45, 6/12) &= 50 - 45e^{-0.06(6/12)} \\ 4.35 + C^e(50, 0; 45, 6/12) - P^e(50, 0; 50, 6/12) &= 50 - 50e^{-0.06(6/12)} \end{aligned}$$

Subtracting the second equation from the first produces,

$$\begin{aligned} P^e(50, 0; 50, 6/12) - P^e(50, 0; 45, 6/12) + 4.35 &= -45e^{-0.03} + 50e^{-0.03} \\ P^e(50, 0; 50, 6/12) - P^e(50, 0; 45, 6/12) &= -45e^{-0.03} + 50e^{-0.03} - 4.35 \\ &= \$0.5022 \end{aligned}$$

6.2.9. The market price of the put is too low.

Transaction	CF $t = 0$	CF $t = 3/12$
Short call	\$7	$-(S(3/12) - 70)^+$
Borrow PV (K)	\$69.1304	$-\$70$
Long put	$-\$6$	$(70 - S(3/12)^+$
Long prepaid forward	$-\$70.0042$	$S(3/12)$
Aggregate	\$0.1262	\$0

Thus the investor makes a positive profit at $t = 0$ with no risk of loss. This is Type A arbitrage.

6.2.11.

$$\begin{aligned} C^e + F^P_{0,4/12} &= 0.075 + 0.95e^{-0.05(4/12)} \\ C^e &= 0.075 + 0.95e^{-0.05(4/12)} - \frac{1}{1.1}e^{(0.05-0.02)(4/12)} \\ &= \$0.0911 \end{aligned}$$

6.3.1. If $r = 0.06$, $S = 50$, $K = 51$, $T = 3/12$, and $P^a = 9.75$ then according to inequality (6.12)

$$\begin{aligned} 50 - 51 \leq C^a - 9.75 &\leq 50 - 51e^{-0.06(3/12)} \\ 8.75 \leq C^a &\leq 9.51. \end{aligned}$$

6.3.3. If the dividend paid is 10% of $S = 115$ then the present value of the dividend due in three months assuming $r = 0.0375$ is

$$\text{PV(div)} = (0.10)(115)e^{-(0.0375)(3/12)} \approx 11.3927.$$

Substituting r, S, $K = 110$, and $T = 6/12$ into the inequality in Eq. (6.14) yields

$$\begin{aligned} 115 - 110 - 11.3927 \leq C^a - P^a &\leq 115 - 110e^{-(0.0375)(6/12)} \\ -6.3927 \leq C^a - P^a &\leq 7.04328. \end{aligned}$$

6.3.5. Multiplying Eq. (6.14) by -1 produces the following inequality.

$$\begin{aligned} Ke^{-r(T-t)} - S(t) \leq P^a - C^a &\leq K - F^P_{t,T}(S) \\ 36e^{-0.05(9/12)} - 35 + 2.25 \leq P^a &\leq 36 - 35e^{-0.03(9/12)} + 2.25 \\ \$1.925 \leq P^a &\leq \$4.02871 \end{aligned}$$

6.4.5. Let $r = 0.0325$, $S(0) = 500$, $P = 40$, $K = 495$, and $t = 2/12$.

(a) $\max\{450, 495\} - (500 + 40)e^{0.0325(2/12)} \approx -47.93$
(b) $\max\{550, 495\} - (500 + 40)e^{0.0325(2/12)} \approx 7.07$

6.4.7. Let $r = 0.0295$, $S(0) = 525$, $C = 50$, $K = 530$, and $t = 2/12$.

(a) $(525 - 50)e^{0.0295(2/12)} - \min\{530, 500\} \approx -22.66$
(b) $(525 - 50)e^{0.0295(2/12)} - \min\{530, 555\} \approx -52.66$

6.4.9. Let $r = 0.0375$, $C(K_1) = 7.57$, $K_1 = 100$, $C(K_2) = 4.75$, $K_2 = 110$, and $T = 2/12$.

(a) When $S(2) = 98$, the profit is $(4.75 - 7.57)e^{0.0375(2/12)} \approx -2.84$.
(b) When $S(2) = 107$, the profit is $(4.75 - 7.57)e^{0.0375(2/12)} + 107 - 100 \approx 4.16$.
(c) When $S(2) = 115$, the profit is $(4.75 - 7.57)e^{0.0375(2/12)} + 110 - 100 \approx 7.16$.

6.4.11. Given the premiums charged for the options:

(a) The premium to set up the straddle is the sum of the premiums of the put and call with the common strike price of \$50.

$$\$3.64 + \$2.72 = \$6.36$$

(b) The premium to set up the strangle is the sum of the premiums of the put with a strike price of \$45 and the call with the strike price of \$55.

$$\$0.94 + \$1.70 = \$2.64$$

(c) The straddle generates more profit when

$$|S(t) - 50| \geq 2.64 \iff S(t) \leq 47.36 \text{ or } S(t) \geq 52.64.$$

7.1.1. Let $\delta = 0$, $T = 3/12$, $V_u = (42 - 41)^+ = 1$, $V_d = (37 - 41)^+ = 0$, $u = 42/40$, $d = 37/40$, and $r = 0.03$. Substituting these values in Eq. (7.1) yields

$$C^e = e^{(0)(3/12)} \frac{1 - 0}{\frac{42}{40} - \frac{37}{40}} - e^{-0.03(3/12)} \frac{\frac{37}{40}(1) - \frac{42}{40}(0)}{\frac{42}{40} - \frac{37}{40}} = \$0.6553.$$

7.1.3. Let $\delta = 0.01$, $T = 3/12$, $u = 130/120$, $d = 115/120$, $K = 125$, and $r = 0.05$.

(a) For the European call we may use Eq. (7.1).

$$C^e = e^{-0.01(3/12)}\frac{5-0}{\frac{130}{120}-\frac{115}{120}} - e^{-0.05(3/12)}\frac{\frac{115}{120}(0)-\frac{130}{120}(5)}{\frac{130}{120}-\frac{115}{120}} = \$1.8532$$

(b) For the European put we may use Eq. (7.1).

$$P^e = e^{-0.01(3/12)}\frac{0-10}{\frac{130}{120}-\frac{115}{120}} - e^{-0.05(3/12)}\frac{\frac{115}{120}(10)-\frac{130}{120}(0)}{\frac{130}{120}-\frac{115}{120}} = \$6.2189$$

(c) According to Eq. (6.9),

$$\begin{aligned} P^e + F^P_{0,3/12} &= C^e + 125e^{-0.05(3/12)} \\ 6.2189 + 120e^{-0.01(3/12)} &= 1.8532 + 125e^{-0.05(3/12)} \\ 125.919 &= 125.919. \end{aligned}$$

The Put-Call Parity formula holds.

7.1.5. The payoff of the butterfly spread will either be $V_d = 0$ or $V_u = 0$. Since the payoff is zero, the price of the butterfly spread is \$0.

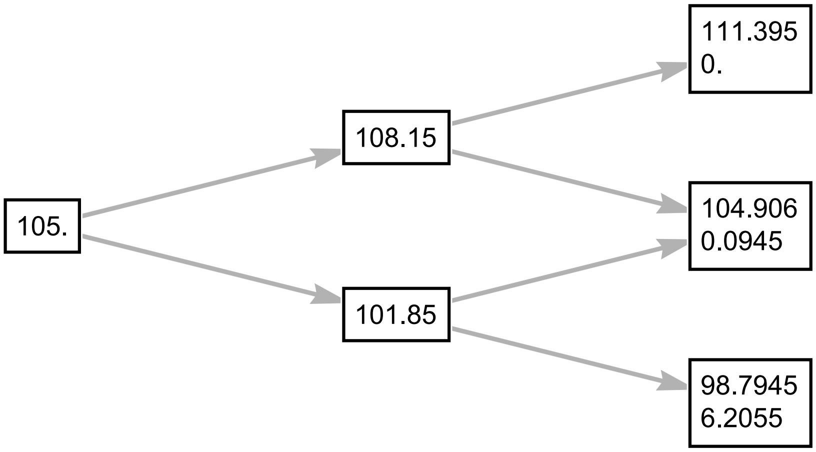
7.1.7. Let $\delta = 0.03$, $T = 5/12$, $u = 120/100$, $d = 90/100$, and $r = 0.06$. The payoff of the option will either be $V_d = 0$ or $V_u = 14,400$. Substituting these values into Eq. (7.1) produces,

$$V = e^{-0.03(5/12)}\frac{14400-0}{\frac{120}{100}-\frac{90}{100}} - e^{-0.06(5/12)}\frac{\frac{90}{100}(14400)-\frac{120}{100}(0)}{\frac{120}{100}-\frac{90}{100}} = \$5,270.35.$$

7.1.9. Since the European put is at-the-money then $V_u = (K - uF_{0,T})^+ = 0$ and $V_d = (K - dF_{0,T})^+ = 250(1-d)$. Substituting these values into Eq. (7.1) produces,

$$\begin{aligned} 17.0738 &= e^{-0.05}\frac{0-250(1-d)}{u-d} - e^{-0.05}\frac{0-250u(1-d)}{u-d} \\ 0.0717968 &= \frac{u(1-d)}{u-d} - \frac{1-d}{u-d} \\ &= \frac{1-d}{1-3/4} - \frac{1-d}{(4/3-1)d} \\ &= 4(1-d) - \frac{3(1-d)}{d} \\ d &= 0.866025 \end{aligned}$$

7.2.3. Using $S = K = 105$, $\delta = 0.02$, $r = 0.07$, $u = 1.03$, and $d = 0.97$, the following binomial lattice may be constructed.



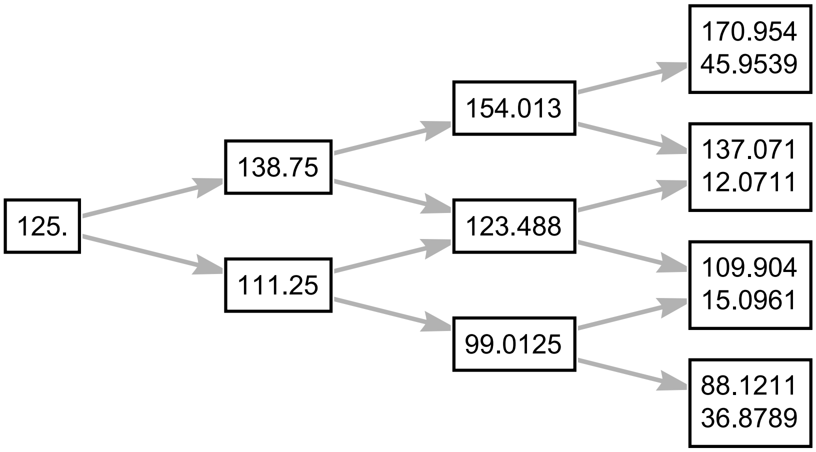
The second value in each terminating leaf of the lattice is the payoff of the European put. The risk-neutral probability of an up step is

$$p^* = \frac{e^{(0.07-0.02)(1/12)} - 0.97}{1.03 - 0.97} = 0.569589.$$

The price of the option is approximately

$$\begin{aligned} P^e &= e^{-0.07(2/12)} \left[2(0.569589)(0.430411)(0.0945) + (0.430411)^2(6.2055)\right] \\ &= \$1.1821. \end{aligned}$$

7.2.5. Using $S = 125$, $K = 125$, $\delta = 0$, $r = 0.07$, $u = 1.11$, and $d = 0.89$, the following binomial lattice may be constructed.



The second value in each terminating leaf of the lattice is the payoff of the European straddle. The risk-neutral probability of an up step is

$$p^* = \frac{e^{(0.07-0)(1/12)} - 0.89}{1.11 - 0.89} = 0.526593.$$

The price of the straddle is approximately

$$\begin{aligned} C^e &= e^{-0.07(3/12)} \left[(0.526594)^3(45.9539) + 3(0.526594)^2(0.473407)(12.0711)\right. \\ &\quad \left. + 3(0.526594)(0.473407)^2(15.0961) + (0.473407)^3(36.8789)\right] \\ &= \$20.3624. \end{aligned}$$

7.3.3.

Month	$R_k = \ln \frac{S(k+1)}{S(k)}$
1	
2	0.33068
3	−0.12745
4	0.40955
5	−0.18891
6	−0.39670
7	0.22391
8	0.03231
9	−0.42894
10	0.81114
11	−0.74273
12	0.46028

Thus the sample mean is $\overline{r} = 0.034832$ and

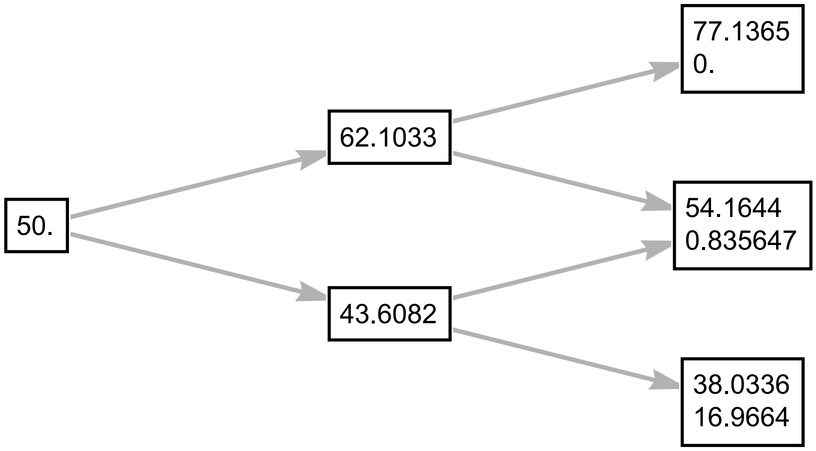
$$
\begin{aligned}
s_R^2 &= \frac{1}{11-1}\sum_{k=1}^{11}(0.034832 - R_k)^2 = 0.212966 \\
s_R &= 0.461482.
\end{aligned}
$$

The monthly volatility is 0.461482. The annualized volatility is

$$\sigma \approx s_R\sqrt{12} = 1.59862 = 159.862\%.$$

7.3.5. Let $h = 1/2$, then the increase factor, decrease factor, and risk-neutral probability are respectively,

$$
\begin{aligned}
u &= e^{(0.08-0)(1/2)+0.25\sqrt{1/2}} = 1.24207 \\
d &= e^{(0.08-0)(1/2)-0.25\sqrt{1/2}} = 0.87217 \\
p^* &= \frac{1}{1+e^{0.25\sqrt{1/2}}} = 0.45592.
\end{aligned}
$$



The second value in each terminating leaf of the lattice is the payoff of the European put.

The time $t = 0$ price of the put option is

$$
\begin{aligned}
P^e &= e^{-0.08(1)}\left[2(0.45592)(0.54408)(0.835647) + (0.54408)^2(16.9664)\right] \\
&= \$5.019.
\end{aligned}
$$

7.4.1. Let $h = 6/12$ then the upward and downward movement factors and the risk-neutral probability of an upward movement are respectively,

$$
\begin{aligned}
u &= e^{(0.06-0.01)(6/12)+0.50\sqrt{6/12}} = 1.46017 \\
d &= e^{(0.06-0.01)(6/12)-0.50\sqrt{6/12}} = 0.71996 \\
p^* &= \frac{1}{1+e^{0.50\sqrt{6/12}}} = 0.412521.
\end{aligned}
$$

At the vertex where $S(6/12) = 153.318$ the intrinsic value of the put is 0 and likewise the present value of the risk-neutral expected value of the premium for the option at $T = 1$ is 0. Thus the premium for the option at this vertex is 0. At the vertex where $S(6/12) = 75.5963$ the intrinsic value of the put is 32.4037 while the present value of the expected value of the premium of the option at $t = 1$ is

$$e^{-0.06(6/12)}\left[(0.412521)(0) + (0.587479)(53.5734)\right] = 30.5431.$$

The premium for the put at this vertex is then

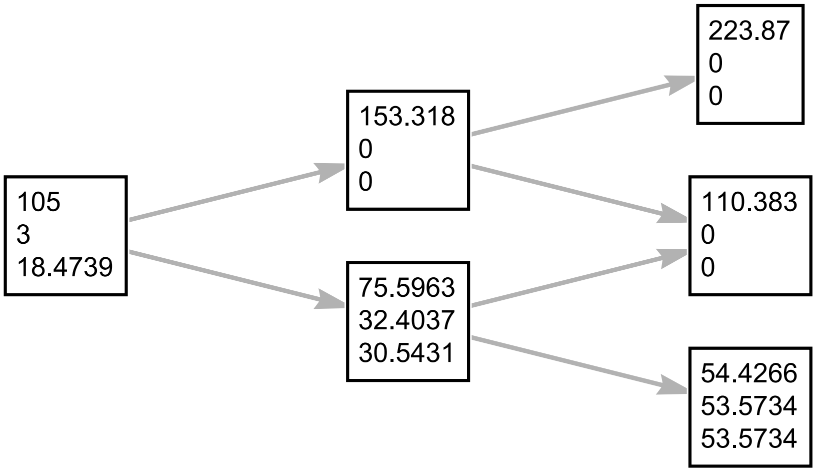
$$\max\{32.4037, 30.5431\} = 32.4037.$$

At this vertex early exercise of the American put would be optimal. At the initial vertex where $S(0) = 105$ the intrinsic value of the option is 3 and the present value of the expected value of the premium of the option $t = 6/12$ is

$$e^{-0.06(6/12)}\left[(0.412521)(0) + (0.587479)(32.4037)\right] = 18.4739.$$

The premium for the American put is then

$$P^a(105, 0; 108, 1) = \max\{3, 18.4739\} = \$18.4739.$$



7.4.3. Let $h = 4/12$ then the upward and downward movement factors and the risk-neutral probability of an upward movement are respectively,

$$\begin{aligned} u &= e^{(0.07-0.03)(4/12)+0.45\sqrt{4/12}} = 1.31409 \\ d &= e^{(0.07-0.03)(4/12)-0.45\sqrt{4/12}} = 0.78155 \\ p^* &= \frac{1}{1 + e^{0.45\sqrt{4/12}}} = 0.435411. \end{aligned}$$

At the vertex where $S(8/12) = 259.023$ the intrinsic value of the option is 109.023 and the time $t = 8/12$ present value of the risk-neutral expected value of the option premium at time $t = 1$ is

$$e^{-0.07(4/12)}\left[(0.435411)(190.378) + (0.564589)(52.4399)\right] = 109.905.$$

Thus the option premium at this vertex is

$$C^a(259.023, 8/12; 150, 1) = \max\{109.023, 109.905\} = 109.905.$$

At the vertex where $S(8/12) = 154.054$ the intrinsic value of the option is 4.05381 and the time $t = 8/12$ present value of the risk-neutral expected value of the option premium at time $t = 1$ is

$$e^{-0.07(4/12)}\left[(0.435411)(52.4399) + (0.564589)(0)\right] = 22.3063.$$

Thus the option premium at this vertex is

$$C^a(154.054, 8/12; 150, 1) = \max\{4.05381, 22.3063\} = 22.3063.$$

At the vertex where $S(8/12) = 91.6234$ the intrinsic value of the option is 0 and the time $t = 8/12$ present value of the risk-neutral expected value of the option premium at time $t = 1$ is also 0. Thus the option premium at this vertex is

$$C^a(91.6234, 8/12; 150, 1) = \max\{0, 0\} = 0.$$

At the vertex where $S(4/12) = 197.113$ the intrinsic value of the option is 47.1128 and the time $t = 4/12$ present value of the risk-neutral expected value of the option premium at time $t = 8/12$ is

$$e^{-0.07(4/12)}\left[(0.435411)(109.905) + (0.564589)(22.3063)\right] = 59.0536.$$

Thus the option premium at this vertex is

$$C^a(197.113, 4/12; 150, 1) = \max\{47.1128, 59.0536\} = 59.0536.$$

At the vertex where $S(4/12) = 117.233$ the intrinsic value of the option is 0 and the time $t = 4/12$ present value of the risk-neutral expected value of the option premium at time $t = 8/12$ is

$$e^{-0.07(4/12)}\left[(0.435411)(22.3063) + (0.564589)(0)\right] = 9.48841.$$

Thus the option premium at this vertex is

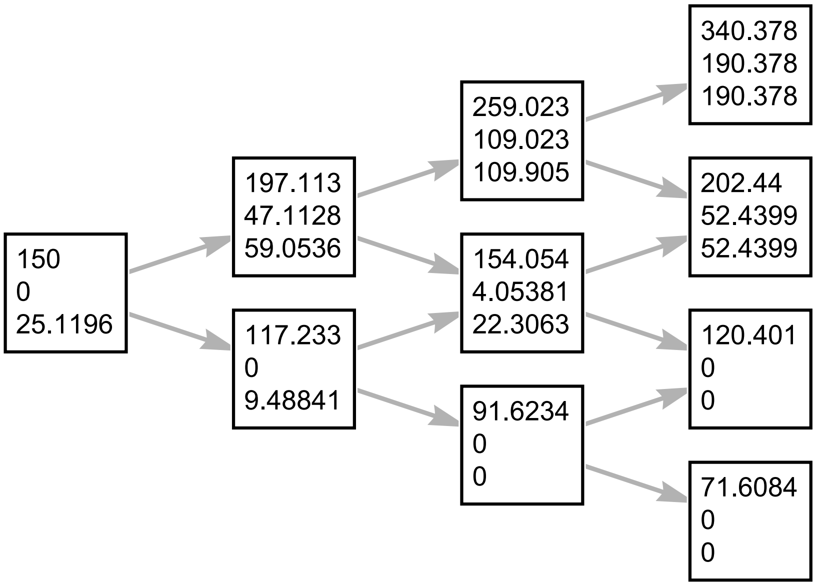
$$C^a(117.2333, 4/12; 150, 1) = \max\{0, 9.48841\} = 9.48841.$$

At the vertex where $S(0) = 150$ the intrinsic value of the option is 0 and the time $t = 0$ present value of the risk-neutral expected value of the option premium at time $t = 4/12$ is

$$e^{-0.07(4/12)}\left[(0.435411)(59.0536) + (0.564589)(0)\right] = 25.1196.$$

Thus the option premium at this vertex is

$$C^a(150, 0; 150, 1) = \max\{0, 25.1196\} = \$25.1196.$$



7.4.5. Let $h = 6/12$ then the upward and downward movement factors and the risk-neutral probability of an upward movement are respectively,

$$u = e^{(0.05-0.05)(6/12)+0.50\sqrt{6/12}} = 1.42412$$

$$d = e^{(0.05-0.05)(6/12)-0.50\sqrt{6/12}} = 0.70219$$

$$p^* = \frac{1}{1 + e^{0.50\sqrt{6/12}}} = 0.412521.$$

At the vertex where $S(6/12) = 85.4471$ the intrinsic value of the call is 20.4471 and the present value of the risk-neutral expected value of the premium for the option at $t = 1$ is

$$e^{-0.05(6/12)}\left[(0.412521)(56.6869) + (0.587479)(0)\right] = 22.8072.$$

Thus the premium for the option at this vertex is

$$\max\{20.4471, 22.8072\} = 22.8072.$$

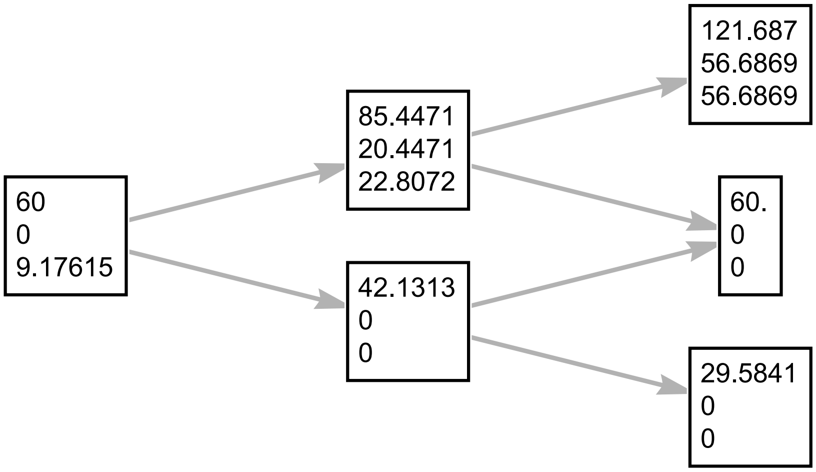
At the vertex where $S(6/12) = 42.1313$ the intrinsic value of the call is 0 while the present value of the expected value of the premium of the option at $t = 1$ is also 0. The premium for the option at this vertex is then 0.

At the initial vertex where $S(0) = 60$ the intrinsic value of the option is 0 and the present value of the expected value of the premium of the option $t = 6/12$ is

$$e^{-0.05(6/12)}\left[(0.412521)(22.8072) + (0.587479)(0)\right] = 9.17615.$$

The premium for the American call is then

$$C^a(60, 0; 65, 1) = \max\{0, 9.17615\} = \$9.17615.$$



8.1.1. The probability distribution function for X is given by the piecewise defined function

$$f_X(x) = \begin{cases} \frac{1}{5} & \text{if } -4 \le x \le 1 \\ 0 & \text{otherwise.} \end{cases}$$

Therefore

$$\mathbb{P}\left(X \ge 0\right) = \int_0^\infty f_X(x)\,dx = \int_0^1 \frac{1}{5}\,dx = \frac{1}{5}.$$

8.1.3. We know that for any random variable X with probability distribution function $f_X(x)$ we must have $\int_{-\infty}^{\infty} f_X(x)\,dx = 1$, thus

$$\begin{aligned}\int_{-\infty}^{\infty} f_X(x)\,dx = \int_1^\infty \frac{C}{x^3}\,dx = C \lim_{M\to\infty} \int_1^M \frac{1}{x^3}\,dx = C \lim_{M\to\infty}\left[\frac{-1}{2x^2}\right]_{x=1}^{x=M} &= C \lim_{M\to\infty}\left(\frac{-1}{2M^2} + \frac{1}{2}\right) \\ 1 &= \frac{C}{2} \\ C &= 2.\end{aligned}$$

8.1.7.

$$f_X(x) = F'(x) = \begin{cases} 0 & \text{if } x \le 3, \\ 18/x^3 & \text{if } x > 3 \end{cases}$$

8.1.9. Let $r(x) = e^x$ then $r^{-1}(y) = \ln y$. According to Eq. (8.4)

$$g_Y(y) = \frac{1}{e^{\ln y}} f_X(\ln y) = \frac{1}{y} e^{-\ln y} = \frac{1}{y^2}.$$

The PDF is defined for the interval $[e^0, \infty) = [1, \infty)$.

8.1.11. Let A and B be the arrival times of Alice and Bob respectively. A and B are independent and each uniformly distributed on the interval $[0, 1]$. Thus the ordered pair of arrival times (A, B) is uniformly distributed in the square $S = \{(x, y) \,|\, 0 \le x \le 1,\ 0 \le y \le 1\}$. The pair will meet if $|A - B| \le 1/4$. The region in S corresponding to Alice and Bob meeting is the region where

- $0 \le A \le 1$,
- $0 \le B \le 1$, and
- $|A - B| \le 1/4$.

The area of this region is the probability they meet.

$$\mathbb{P}\,(\text{meeting}) = 1 - 2\left[\left(\frac{1}{2}\right)\left(\frac{3}{4}\right)\left(\frac{3}{4}\right)\right] = 1 - \frac{9}{16} = \frac{7}{16}$$

8.2.3. Use integration by parts.

$$\begin{aligned}\mathbb{E}\,(X) &= \int_{-\infty}^{\infty} x f_X(x)\,dx = \int_0^{\infty} \lambda x e^{-\lambda x}\,dx = \left[-xe^{-\lambda x}\right]_{x=0}^{x\to\infty} + \int_0^{\infty} e^{-\lambda x}\,dx \\ &= 0 + \left[\frac{-1}{\lambda} e^{-\lambda x}\right]_{x=0}^{x\to\infty} = \frac{1}{\lambda}\end{aligned}$$

8.2.7. The conditional probability distribution $f_{Y|X}(y|x)$ is calculated as

$$f_{Y|X}(y|x) = \frac{f_{X,Y}(x,y)}{\int_{-3}^{x} f_{X,Y}(x,y)\,dy} = \frac{\frac{x^2y^2}{162}}{\int_{-3}^{x} \frac{x^2y^2}{162}\,dy} = \frac{y^2}{\int_{-3}^{x} y^2\,dy} = \frac{3y^2}{x^3 + 27}.$$

Consequently the conditional expected value is

$$\mathbb{E}\,(Y|X = x) = \int_{-3}^{x} y\left(\frac{3y^2}{x^3+27}\right)\,dy = \left[\frac{3y^4}{4(x^3+27)}\right]_{y=-3}^{y=x} = \frac{3(x^4 - 81)}{4(x^3+27)},$$

for $-3 < x < 3$.

8.2.9. Using the joint probability distribution we find the following.

(a)

$$\begin{aligned}\mathbb{E}\,(XY) &= \int_0^2 \int_0^x (xy) f_{X,Y}(x,y)\,dy\,dx = \int_0^2 \int_0^x (xy)\left(\frac{3}{8}xy^3\right)\,dy\,dx = \frac{3}{8}\int_0^2 \int_0^x x^2y^4\,dy\,dx \\ &= \frac{3}{40}\int_0^2 x^7\,dx \\ &= \frac{3}{320}\left[x^8\right]_{x=0}^{x=2} = \frac{12}{5} = 2.4\end{aligned}$$

(b)

$$\begin{aligned}\mathbb{E}\left(X^2Y\right) &= \int_0^2\int_0^x (x^2y)f_{X,Y}(x,y)\,dy\,dx = \int_0^2\int_0^x (x^2y)\left(\frac{3}{8}xy^3\right)\,dy\,dx = \frac{3}{8}\int_0^2\int_0^x x^3y^4\,dy\,dx\\ &= \frac{3}{40}\int_0^2 x^8\,dx\\ &= \frac{3}{360}\left[x^9\right]_{x=0}^{x=2} = \frac{64}{15} \approx 4.26667\end{aligned}$$

(c)

$$\begin{aligned}\mathbb{E}\left(X\sqrt{Y}\right) &= \int_0^2\int_0^x (x\sqrt{y})f_{X,Y}(x,y)\,dy\,dx = \int_0^2\int_0^x (xy^{1/2})\left(\frac{3}{8}xy^3\right)\,dy\,dx = \frac{3}{8}\int_0^2\int_0^x x^2y^{7/2}\,dy\,dx\\ &= \frac{1}{12}\int_0^2 x^{13/2}\,dx\\ &= \frac{1}{90}\left[x^{15/2}\right]_{x=0}^{x=2} = \frac{64\sqrt{2}}{45} \approx 2.01133\end{aligned}$$

8.3.1. The expected value of X is calculated as

$$\begin{aligned}\mathbb{E}(X) &= \int_{-\infty}^{\infty} xf_X(x)\,dx = \int_{-1}^{2}\frac{2}{5}x|x|\,dx\\ &= \frac{2}{5}\left(\int_{-1}^{0}(-x^2)\,dx + \int_0^2 x^2\,dx\right)\\ \mu &= \frac{14}{15}.\end{aligned}$$

The variance is calculated as

$$\begin{aligned}\mathbb{Var}(X) &= \int_{-\infty}^{\infty} x^2f_X(x)\,dx - \mu^2 = \int_{-1}^{2}\frac{2}{5}x^2|x|\,dx - \frac{196}{225}\\ &= \frac{2}{5}\left(\int_{-1}^{0}(-x^3)\,dx + \int_0^2 x^3\,dx\right) - \frac{196}{225}\\ &= \frac{373}{450}.\end{aligned}$$

8.3.7. Let $Y = X^2$ and find the probability density function for Y.

$$\mathbb{P}(Y \le y) = \mathbb{P}\left(X^2 \le y\right) = \mathbb{P}(-\sqrt{y} \le X \le \sqrt{y}) = \mathbb{P}(X \le \sqrt{y}) = \sqrt{y}$$

for $0 \le \sqrt{y} \le 1$. Thus $f_Y(y) = 1/(2\sqrt{y})$ for $0 \le y \le 1$ and 0 otherwise.

$$\begin{aligned}\mathbb{E}(Y) &= \int_0^1 y\,\frac{1}{2\sqrt{y}}\,dy = \left[\frac{1}{3}y^{3/2}\right]_{y=0}^{y=1} = \frac{1}{3}\\ \mathbb{E}\left(Y^2\right) &= \int_0^1 y^2\frac{1}{2\sqrt{y}}\,dy = \left[\frac{1}{5}y^{5/2}\right]_{y=0}^{y=1} = \frac{1}{5}\\ \mathbb{Var}(Y) &= \frac{1}{5} - \left(\frac{1}{3}\right)^2 = \frac{4}{45}\end{aligned}$$

8.3.11. Let $Y = X/(1-X)$ then

$$\mathbb{P}(Y \le y) = \mathbb{P}\left(\frac{X}{1-X} \le y\right) = \mathbb{P}\left(X \le \frac{y}{1+y}\right).$$

Using the PDF for random variable X then

$$\begin{aligned}\mathbb{P}\left(Y \leq y\right) &= \int_{-1}^{y/(1+y)} 30x(1-x)^4\,dx = \left[15x^2 - 40x^3 + 45x^4 - 24x^5 + 5x^6\right]_{x=-1}^{x=y/(1+y)} \\ &= 1 + \frac{5}{(1+y)^6} - \frac{6}{(1+y)^5} \\ f_Y(y) &= \frac{30}{(1+y)^6} - \frac{30}{(1+y)^7} = \frac{30y}{(1+y)^7}\end{aligned}$$

for $0 \leq y < \infty$ and 0 otherwise.

$$\begin{aligned}\mathbb{E}\left(Y\right) &= \int_0^\infty y\,\frac{30y}{(1+y)^7}\,dy \\ &= \int_0^\infty \left(\frac{30}{(1+y)^7} - \frac{60}{(1+y)^6} + \frac{30}{(1+y)^5}\right) dy \\ &= \frac{1}{2} \\ \mathbb{E}\left(Y^2\right) &= \int_0^\infty y^2\,\frac{30y}{(1+y)^7}\,dy \\ &= \int_0^\infty \left(\frac{-30}{(1+y)^7} + \frac{90}{(1+y)^6} - \frac{90}{(1+y)^5} + \frac{30}{(1+y)^4}\right) dy \\ &= \frac{1}{2} \\ \mathbb{Var}\left(Y\right) &= \frac{1}{2} - \left(\frac{1}{2}\right)^2 = \frac{1}{4}\end{aligned}$$

8.3.13. Since $X \sim \mathcal{U}(0,1)$, then

$$f_{Y,Z}(y,z) = \begin{cases} 1 & \text{if } 0 \leq y \leq 1 \text{ and } y^2 \leq z \leq y^2+1, \\ 0 & \text{otherwise.}\end{cases}$$

$$\begin{aligned}f_Y(y) &= \int_{y^2}^{y^2+1} 1\,dz = 1 \\ \mathbb{E}\left(Y\right) &= \frac{1}{2} \\ \mathbb{Var}\left(Y\right) &= \frac{1}{12} \\ \mathbb{E}\left(Z\right) &= \int_0^1\int_0^1 (x+y^2)\,dx\,dy = \int_0^1 \left(\frac{1}{2} + y^2\right) dy = \frac{5}{6} \\ \mathbb{Var}\left(Z\right) &= \int_0^1\int_0^1 (x+y^2)^2\,dx\,dy - \left(\frac{5}{6}\right)^2 = \int_0^1 \left(y^4 + y^2 + \frac{1}{3}\right) dy - \frac{25}{36} = \frac{31}{180} \\ \mathbb{Cov}\left(Y,Z\right) = \mathbb{E}\left(YZ\right) - \mathbb{E}\left(Y\right)\mathbb{E}\left(Z\right) &= \int_0^1\int_0^1 y(x+y^2)\,dx\,dy - \left(\frac{1}{2}\right)\left(\frac{5}{6}\right) = \int_0^1 \left(y^3 + \frac{y^2}{2}\right) dy - \frac{5}{12} = \frac{1}{12} \\ \mathbb{Cor}\left(Y,Z\right) &= \frac{1/12}{(1/12)(31/180)} = \sqrt{\frac{15}{31}}\end{aligned}$$

8.4.7.

$$\begin{aligned}\mathbb{P}\left(|X-2| \geq 1\right) &= \mathbb{P}\left(-1 \leq X-2 \leq 1\right) = \mathbb{P}\left(1 \leq X \leq 3\right) \\ &= \Phi\left(3\right) - \Phi\left(1\right) = 0.99865 - 0.84134 = 0.15731\end{aligned}$$

8.4.9.

$$\begin{aligned}0.15 &= \mathbb{P}\left(X \leq 100\right) = \mathbb{P}\left(Z \leq \frac{100-\mu}{5}\right)\\ -1.03643 &= \frac{100-\mu}{5}\\ \mu &= 105.1822\end{aligned}$$

8.4.13. Using the technique of integration by parts with

$$\begin{aligned}u &= \frac{x}{\sqrt{k\pi t}} & v &= -kte^{-\frac{x^2}{4kt}}\\ du &= \frac{1}{\sqrt{k\pi t}}\,dx & dv &= \frac{x}{2}e^{-\frac{x^2}{4kt}}\,dx\end{aligned}$$

we obtain

$$\begin{aligned}\int \frac{x^2}{2\sqrt{k\pi t}}e^{-\frac{x^2}{4kt}}\,dx &= -\frac{ktx}{\sqrt{k\pi t}}e^{-\frac{x^2}{4kt}} + \int \frac{kt}{\sqrt{k\pi t}}e^{-\frac{x^2}{4kt}}\,dx\\ &= \sqrt{\frac{kt}{\pi}}\left(-xe^{-\frac{x^2}{4kt}} + \int e^{-\frac{x^2}{4kt}}\,dx\right).\end{aligned}$$

8.4.15.

$$\begin{aligned}\mathbb{P}\left(-1 < X < 1\right) &= \Phi\left(1\right) - \Phi\left(-1\right) \approx 0.682689\\ \mathbb{P}\left(-2 < X < 2\right) &= \Phi\left(2\right) - \Phi\left(-2\right) \approx 0.9545\\ \mathbb{P}\left(-3 < X < 3\right) &= \Phi\left(3\right) - \Phi\left(-3\right) \approx 0.9973\\ \mathbb{P}\left(1 < X < 3\right) &= \Phi\left(3\right) - \Phi\left(1\right) \approx 0.157305\end{aligned}$$

8.5.5. Since $Y = e^X$ then

$$\begin{aligned}\mathbb{P}\left(0.15 < Y \leq 1.75\right) &= \mathbb{P}\left(\ln 0.15 < X \leq \ln 1.75\right)\\ &= \mathbb{P}\left(\frac{\ln 0.15 - 0.1}{\sqrt{0.5}} < Z \leq \frac{\ln 1.75 - 0.1}{\sqrt{0.5}}\right)\\ &= \mathbb{P}\left(-2.8244 < Z \leq 0.6500\right) = \Phi\left(0.6500\right) - \Phi\left(-2.8244\right) = 0.7398\end{aligned}$$

8.5.7. Let $x_{0.50}$ be the median then

$$0.50 = \int_0^{x_{0.50}} \frac{1}{t\sqrt{2\pi\sigma^2}}e^{-(\ln t-\mu)^2/(2\sigma^2)}\,dt$$

Use the substitution

$$\begin{aligned}u &= \frac{\ln x - \mu}{\sigma} \iff x = e^{\sigma u + \mu}\\ du &= \frac{1}{\sigma x}\,dx \iff dx = \sigma e^{\sigma u+\mu}\,du\end{aligned}$$

to rewrite the improper integral as

$$\begin{aligned}0.50 &= \int_{-\infty}^{(\ln x_{0.50}-\mu)/\sigma} \frac{1}{\sqrt{2\pi}}e^{-u^2/2}\,du\\ &= \Phi\left(\frac{\ln x_{0.50}-\mu}{\sigma}\right)\\ 0 &= \frac{\ln x_{0.50}-\mu}{\sigma}\\ x_{0.50} &= e^{\mu}.\end{aligned}$$

8.6.1. Both portfolios have the same expected value. Portfolio A has a smaller standard deviation than portfolio B and thus may be perceived as less risky. Thus portfolio A is preferable to portfolio B.
8.6.3. Let c be the certainty equivalent return. We must solve the equation:

$$\begin{aligned} 1 - e^{-b\,c} &= \frac{1}{2}(1 - e^{-b(0.20)}) + \frac{1}{2}(1 - e^{-b(-0.05)} \\ e^{-b\,c} &= \frac{1}{2}e^{-0.20b} + \frac{1}{2}e^{0.05b} \\ c &= \frac{-1}{b}\ln\left(\frac{1}{2}e^{-0.20b} + \frac{1}{2}e^{0.05b}\right). \end{aligned}$$

8.6.5.

$$\begin{aligned} \mathbb{E}\left(u(X)\right) &= \int_{-\infty}^{\infty}(1 - e^{-b\,x})\frac{1}{\sqrt{2\pi\sigma^2}}e^{-\frac{(x-\mu)^2}{2\sigma^2}}\,dx \\ &= 1 - \int_{-\infty}^{\infty}\frac{1}{\sqrt{2\pi\sigma^2}}e^{-b\,x}e^{-\frac{(x-\mu)^2}{2\sigma^2}}\,dx \\ &= 1 - e^{\frac{b^2\sigma^2-2b\mu}{2}}\int_{-\infty}^{\infty}\frac{1}{\sqrt{2\pi\sigma^2}}e^{-\frac{(x-\mu+b\sigma^2)^2}{2\sigma^2}}\,dx \\ &= 1 - e^{\frac{b^2\sigma^2-2b\mu}{2}} \end{aligned}$$

8.7.5. Since X is uniformly distributed on $[a, b]$ its probability distribution function is

$$f(x) = \begin{cases} \frac{1}{b-a} & \text{if } a \leq x \leq b, \\ 0 & \text{otherwise.} \end{cases}$$

Note that $(X - K)^+ = \max\{K, X\} - K$. There are three cases to consider. First, if $K \leq a$ then

$$\mathbb{E}\left((X-K)^+\right) = \int_a^b \frac{1}{b-a}\max\{K, x\}\,dx - K = \int_a^b \frac{x}{b-a}\,dx - K = \frac{a+b}{2} - K.$$

Next, if $K \geq b$ then

$$\mathbb{E}\left((X-K)^+\right) = \int_a^b \frac{1}{b-a}\max\{K, x\}\,dx - K = \int_a^b \frac{K}{b-a}\,dx - K = 0.$$

Finally if $a < K < b$ then

$$\begin{aligned} \mathbb{E}\left((X-K)^+\right) &= \int_a^b \frac{1}{b-a}\max\{K, x\}\,dx - K \\ &= \int_a^K \frac{K}{b-a}\,dx + \int_K^b \frac{x}{b-a}\,dx - K \\ &= \frac{K(K-a)}{b-a} + \frac{b^2 - K^2}{2(b-a)} - K \\ &= \frac{(b-K)^2}{2(b-a)}. \end{aligned}$$

8.7.9. Let $\mu = 0.45$, $\sigma = 1$, and $K = 0.5$, then

$$\begin{aligned} v &= \frac{1 + 0.45 - \ln 0.5}{\sqrt{1}} = 2.14315 \\ v - \sigma &= 1.14315 \\ \Phi\left(-v\right) &= 0.016051 \\ \Phi\left(\sigma - v\right) &= 0.126489 \\ \mathbb{E}\left(X | X \leq 0.5\right) &= e^{0.45+1/2}\frac{0.016051}{0.126489} = 0.328111 \end{aligned}$$

8.8.5. Equation (8.29) can be solved for Z_1 immediately.

$$Z_1 = \frac{X - \mu_X}{\sigma_X}$$

Substituting this into Eq. (8.30) produces,

$$\begin{aligned}
Y &= \mu_Y + \sigma_Y \left(\frac{\rho(X - \mu_X)}{\sigma_X} + \sqrt{1 - \rho^2} Z_2 \right) \\
\frac{Y - \mu_Y}{\sigma_Y} &= \frac{\rho(X - \mu_X}{\sigma_X} + \sqrt{1 - rho^2} Z_2 \\
Z_2 &= \frac{Y - \mu_Y}{\sqrt{1 - \rho^2}\sigma_Y} - \frac{\rho(X - \mu_X)}{\sqrt{1 - \rho^2}\sigma_X}.
\end{aligned}$$

8.8.7. By definition the conditional density is $f_{X|Y}(x|y) = f_{X,Y}(x,y)/f_Y(y)$ where $f_Y(y)$ is the marginal probability density for random variable Y.

$$\begin{aligned}
f_Y(y) &= \int_{-\infty}^{\infty} f_{X,Y}(x,y)\,dx \\
&= \frac{1}{2\pi\sqrt{1-\rho^2}\sigma_X\sigma_Y} e^{\frac{-1}{2(1-\rho^2)} \frac{(y-\mu_Y)^2}{\sigma_Y^2}} \int_{-\infty}^{\infty} e^{\frac{-1}{2(1-\rho^2)} \left[\frac{(x-\mu_X)^2}{\sigma_X^2} - \frac{2\rho(x-\mu_X)(y-\mu_Y)}{\sigma_X\sigma_Y} \right]}\,dx \\
&= \frac{1}{2\pi\sqrt{1-\rho^2}\sigma_X\sigma_Y} e^{\frac{-(y-\mu_Y)^2}{2\sigma_Y^2}} \int_{-\infty}^{\infty} e^{\frac{-1}{2(1-\rho^2)} \left[\frac{x-\mu_X}{\sigma_X} - \frac{\rho(y-\mu_Y)}{\sigma_Y} \right]^2}\,dx \\
&= \frac{1}{\sqrt{2\pi}\sigma_Y} e^{\frac{-(y-\mu_Y)^2}{2\sigma_Y^2}} \int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi}} e^{-u^2/2}\,du \\
&= \frac{1}{\sqrt{2\pi}\sigma_Y} e^{\frac{-(y-\mu_Y)^2}{2\sigma_Y^2}}
\end{aligned}$$

Using this marginal distribution, then

$$\begin{aligned}
f_{X|Y}(x|y) &= \frac{\frac{1}{2\pi\sqrt{1-\rho^2}\sigma_X\sigma_Y} e^{\frac{-1}{2(1-\rho^2)} \left[\frac{(x-\mu_X)^2}{\sigma_X^2} - \frac{2\rho(x-\mu_X)(y-\mu_Y)}{\sigma_X\sigma_Y} + \frac{(y-\mu_Y)^2}{\sigma_Y^2} \right]}}{\frac{1}{\sqrt{2\pi}\sigma_Y} e^{\frac{-(y-\mu_Y)^2}{2\sigma_Y^2}}} \\
&= \frac{1}{\sqrt{2\pi(1-\rho^2)\sigma_X^2}} e^{\frac{-1}{2(1-\rho^2)} \left[\frac{(x-\mu_X)^2}{\sigma_X^2} - \frac{2\rho(x-\mu_X)(y-\mu_Y)}{\sigma_X\sigma_Y} + \frac{\rho^2(y-\mu_Y)^2}{\sigma_Y^2} \right]} \\
&= \frac{1}{\sqrt{2\pi(1-\rho^2)\sigma_X^2}} e^{\frac{-1}{2(1-\rho^2)} \left[\frac{x-\mu_X}{\sigma_X} - \frac{\rho(y-\mu_Y}{\sigma_Y} \right]^2} \\
&= \frac{1}{\sqrt{2\pi(1-\rho^2)\sigma_X^2}} e^{\frac{-1}{2(1-\rho^2)\sigma_X^2} \left[x - \left(\mu_X + \rho\frac{\sigma_X}{\sigma_Y}(y-\mu_Y) \right) \right]^2}.
\end{aligned}$$

Hence random variable X is conditionally distributed on Y as

$$\mathcal{N}\left(\mu_X + \rho\frac{\sigma_X}{\sigma_Y}(y - \mu_Y, (1-\rho^2)\sigma_X^2 \right).$$

9.1.1. Since $R(t_1, t_2) = \ln(S(t_2)/S(t_1))$ then

$$R(t_1, t_2) \sim \mathcal{N}((\alpha - \delta - \sigma^2/2)(t_2 - t_1), \sigma^2(t_2 - t_1)).$$

9.1.3. Note that $S(1) = 100e^{0.1+0.4Z}$ where $Z \sim \mathcal{N}(0,1)$. The random variable Z lies in the interval $(-z_{0.025}, z_{0.025})$ with probability 0.95 when $z_{0.025} = 1.95996$. Thus the 95% prediction interval is

$$\begin{aligned} 100e^{0.1-0.4(1.95996)} &< S(1) < 100e^{0.1+0.4(1.95996)} \\ 50.4603 &< S(1) < 242.0524. \end{aligned}$$

9.1.7. Note that

$$\frac{S(4)}{S(2)} \sim \mathcal{LN}((0.10 - 0 - (0.25)^2/2)(4-2), (0.25)^2(4-2)) \sim \mathcal{LN}(0.1375, 0.125).$$

Thus

$$\mathbb{E}\left(S(4) | S(2) = 85\right) = 85e^{0.1375+0.125/2} = \$103.819$$

9.1.9. Using the stock's parameters,

$$\begin{aligned} \hat{d}_1 &= \frac{\ln \frac{68}{65} + (0.10 - 0 + (0.30)^2/2)(2)}{0.30\sqrt{2}} = 0.789886 \\ \hat{d}_2 &= \frac{\ln \frac{68}{65} + (0.10 - 0 - (0.30)^2/2)(2)}{0.30\sqrt{2}} = 0.365622 \\ \mathbb{E}\left(S(2) | S(2) \leq 65\right) = \mathbb{E}\left(S(2)\right) &\frac{\Phi\left(-0.789886\right)}{\Phi\left(-0.365622\right)} = 68e^{0.10(2)}\frac{0.214797}{0.357323} = \$49.9269. \end{aligned}$$

9.2.9. A significant amount of work goes into expressing the exponential portion of the integrand in a form that allows the properties of the normal random variable to be used. The reader can verify algebraically that

$$\frac{-(x-y-\mu\sqrt{t})^2}{4kt} - y^2 = \frac{-(1+4kt)}{4kt}\left(y - \frac{x-\mu\sqrt{t}}{1+4kt}\right)^2 - \frac{(x-\mu\sqrt{t})^2}{1+4kt}.$$

Making this replacement in the solution integral produces

$$\begin{aligned} u(x,t) &= e^{\frac{-(x-\mu\sqrt{t})^2}{1+4kt}} \int_{-\infty}^{\infty} \frac{1}{\sqrt{4\pi kt}} e^{\frac{-\left(y-(x-\mu\sqrt{t})/(1+4kt)\right)^2}{4kt/(1+4kt)}}\, dy \\ &= e^{\frac{-(x-\mu\sqrt{t})^2}{1+4kt}} \int_{-\infty}^{\infty} \frac{\sqrt{2kt/(1+4kt)}}{\sqrt{4\pi kt}} e^{-z^2/2}\, dz \\ &= \sqrt{\frac{1}{1+4kt}} e^{\frac{-(x-\mu\sqrt{t})^2}{1+4kt}} \int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi}} e^{-z^2/2}\, dz \\ &= \sqrt{\frac{1}{1+4kt}} e^{\frac{-(x-\mu\sqrt{t})^2}{1+4kt}}. \end{aligned}$$

9.4.3. Using the integrated Brownian motion derived in Example 9.5,

$$\int_0^t \tau\, dW(\tau) = tW(t) - \int_0^t W(\tau)\, d\tau.$$

The mean is zero and the variance is

$$\int_0^t \tau^2\, d\tau = \frac{t^3}{3}.$$

9.4.5. This is an Itô integral of a non-random function, thus the integral is normally distributed with a mean of 0 and a variance of

$$\int_0^t (e^{-\tau/2})^2\, d\tau = \int_0^t e^{-\tau}\, d\tau = \left[-e^{-\tau}\right]_{\tau=0}^{\tau=t} = 1 - e^{-t}.$$

9.5.1. Applying Itô's formula to $f(W(t),t) = \frac{1}{3}(W(t))^3$ produces

$$d\left(\frac{1}{3}(W(t))^3\right) = (W(t))^2\,dW(t) + \left(0 + \frac{1}{2}\cdot 2W(t)\right)\,dt.$$

Therefore,

$$\int_0^t (W(s))^2\,dW(s) = \int_0^t d\left(\frac{1}{3}(W(s))^3\right) - \int_0^t W(s)\,ds = \frac{1}{3}(W(t))^3 - \int_0^t W(s)\,ds.$$

The mean of the integral is 0. Calculation of the variance requires more effort.

$$\begin{aligned}
\mathbb{V}\mathrm{ar}\left(\frac{1}{3}(W(t))^3 - \int_0^t W(s)\,ds\right) &= \mathbb{E}\left(\left(\frac{1}{3}(W(t))^3 - \int_0^t W(s)\,ds\right)^2\right) \\
&= \frac{1}{9}\,\mathbb{E}\left((W(t))^6\right) - \frac{2}{3}\,\mathbb{E}\left((W(t))^3\int_0^t W(s)\,ds\right) + \mathbb{E}\left(\left(\int_0^t W(s)\,ds\right)^2\right)
\end{aligned}$$

Since $W(t) \sim \mathcal{N}(0,t)$ then first expectation above is

$$\begin{aligned}
\mathbb{E}\left((W(t))^6\right) &= \int_{-\infty}^{\infty} \frac{x^6}{\sqrt{2\pi t}} e^{-x^2/(2t)}\,dx = t^3 \int_{-\infty}^{\infty} \frac{z^6}{\sqrt{2\pi}} e^{-x^2/2}\,dz \\
&= 5t^3 \int_{-\infty}^{\infty} \frac{z^4}{\sqrt{2\pi}} e^{-x^2/2}\,dz = 15t^3 \int_{-\infty}^{\infty} \frac{z^2}{\sqrt{2\pi}} e^{-x^2/2}\,dz \\
&= 15t^3 \int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi}} e^{-x^2/2}\,dz = 15t^3.
\end{aligned}$$

The second expectation is found as

$$\begin{aligned}
\mathbb{E}\left((W(t))^3 \int_0^t W(s)\,ds\right) &= \mathbb{E}\left(\int_0^t (W(t))^3 W(s)\,ds\right) = \int_0^t \mathbb{E}\left((W(t))^3 W(s)\right)\,ds \\
&= 3\int_0^t \mathbb{E}\left((W(t))^2\right)\mathbb{E}\left(W(t)W(s)\right)\,ds = 3t\int_0^t s\,ds = \frac{3}{2}t^3.
\end{aligned}$$

Note that the hint was used here as well as the fact that when $0 \le s \le t$, $\mathbb{C}\mathrm{ov}\left(W(t), W(s)\right) = s$. As a result of the work done in Example 9.5,

$$\mathbb{E}\left(\left(\int_0^t W(s)\,ds\right)^2\right) = \frac{t^3}{3}$$

thus

$$\mathbb{V}\mathrm{ar}\left(\frac{1}{3}(W(t))^3 - \int_0^t W(s)\,ds\right) = \frac{1}{9}(15t^3) - \frac{2}{3}\left(\frac{3}{2}t^3\right) + \frac{t^3}{3} = t^3.$$

9.5.5. If $F \equiv F(y,z)$ and $f(x) = F(y_0 + xh, z_0 + xk)$ then

$$\begin{aligned}
f'(x) &= F_y(y_0 + xh, z_0 + xk)\frac{dy}{dx} + F_z(y_0 + xh, z_0 + xk)\frac{dz}{dx} \\
&= F_y(y_0 + xh, z_0 + xk)\frac{d}{dx}\left[y_0 + xh\right] + F_z(y_0 + xh, z_0 + xk)\frac{d}{dx}\left[z_0 + xk\right] \\
&= F_y(y_0 + xh, z_0 + xk)h + F_z(y_0 + xh, z_0 + xk)k \\
f'(0) &= F_y(y_0, z_0)h + F_z(y_0, z_0)k.
\end{aligned}$$

Differentiating once again produces

$$\begin{aligned}
f''(x) &= F_{yy}(y_0+xh, z_0+xk)h\frac{dy}{dx} + F_{yz}(y_0+xh, z_0+xk)h\frac{dz}{dx} \\
&\quad + F_{zy}(y_0+xh, z_0+xk)k\frac{dy}{dx} + F_{zz}(y_0+xh, z_0+xk)k\frac{dz}{dx} \\
&= hF_{yy}(y_0+xh, z_0+xk)\frac{d}{dx}\left[y_0+xh\right] + hF_{yz}(y_0+xh, z_0+xk)\frac{d}{dx}\left[z_0+xk\right] \\
&\quad + kF_{zy}(y_0+xh, z_0+xk)\frac{d}{dx}\left[y_0+xh\right] + kF_{zz}(y_0+xh, z_0+xk)\frac{d}{dx}\left[z_0+xk\right] \\
&= h^2F_{yy}(y_0+xh, z_0+xk) + 2hkF_{yz}(y_0+xh, z_0+xk) + k^2F_{zz}(y_0+xh, z_0+xk) \\
f''(0) &= h^2F_{yy}(y_0, z_0) + 2hkF_{yz}(y_0, z_0) + k^2F_{zz}(y_0, z_0).
\end{aligned}$$

By definition the Taylor remainder of order 3 for $f(x)$ is $R_3(x) = \dfrac{f^{(3)}(\theta)}{3!}x^3$ for some θ between 0 and x. If we notice the binomial coefficient pattern among the partial derivatives in the first and second derivatives of $f(x)$ then we can readily see that

$$\begin{aligned}
f^{(3)}(x) &= h^3F_{yyy}(y_0+xh, z_0+xk) + 3h^2kF_{yyz}(y_0+xh, z_0+xk) \\
&\quad + 3hk^2F_{yzz}(y_0+xh, z_0+xk) + k^3F_{zzz}(y_0+xh, z_0+xk).
\end{aligned}$$

From this expression we may directly determine the Taylor remainder of order 3.

9.5.7. If we apply Itô's Lemma with $a(P,t) = \mu P$ and $b(P,t) = \sigma P$ and $Y = \ln P$, then

$$\begin{aligned}
dY &= \left(\mu P\left[\frac{1}{P}\right] + \frac{1}{2}(\sigma P)^2\left[\frac{-1}{P^2}\right]\right)dt + \sigma P\left[\frac{1}{P}\right]dW(t) \\
&= \left(\mu - \frac{\sigma^2}{2}\right)dt + \sigma\, dW(t).
\end{aligned}$$

9.5.9. Mean reverting Ornstein-Uhlenbeck equation:

(a) Let $a(X,t) = k(\mu - X(t))$ and $b(X,t) = \sigma$ and note that

$$\begin{aligned}
\frac{\partial Z}{\partial X} &= -e^{kt} \\
\frac{\partial^2 Z}{\partial X^2} &= 0 \\
\frac{\partial Z}{\partial t} &= ke^{kt}(\mu - X(t)).
\end{aligned}$$

Applying Itô's Lemma yields

$$\begin{aligned}
dZ(t) &= \left[k(\mu - X(t))(-e^{kt}) + ke^{kt}(\mu - X(t)) + 0\right]\,dt + \sigma(-e^{kt})\,dW(t) \\
&= -\sigma e^{kt}\,dW(t).
\end{aligned}$$

(b) Integrating both sides of the last equation yields

$$\begin{aligned}
\int_0^t dZ(s) &= -\sigma\int_0^t e^{ks}\,dW(s) \\
Z(t) - Z(0) &= -\sigma\int_0^t e^{ks}\,dW(s) \\
Z(t) &= Z(0) - \sigma\int_0^t e^{ks}\,dW(s) \\
&= \mu - X(0) - \sigma\int_0^t e^{ks}\,dW(s).
\end{aligned}$$

(c) Since $X(t) = \mu - e^{-kt} Z(t)$ then

$$\begin{aligned}
\mathbb{E}\left(X(t)\right) &= \mathbb{E}\left(\mu - e^{-kt} Z(t)\right) \\
&= \mathbb{E}\left(\mu\right) - e^{-kt}\, \mathbb{E}\left(Z(t)\right) \\
&= \mu - e^{-kt}(\mu - X(0)) \\
\mathbb{Var}\left(X(t)\right) &= \mathbb{Var}\left(\mu - e^{-kt} Z(t)\right) \\
&= e^{-2kt}\, \mathbb{Var}\left(Z(t)\right) \\
&= \frac{\sigma^2}{2k}\left(1 - e^{-2kt}\right).
\end{aligned}$$

9.5.11. Using the stochastic form of the product rule,

$$\begin{aligned}
dY(t) &= X_2\, dX_1 + X_1\, dX_2 + dX_1\, dX_2 \\
&= X_2(a_1(t)X_1\, dt + b_1(t)X_1\, dW(t)) + X_1(a_2(t)X_2\, dt + b_2(t)X_2\, dW(t)) \\
&\quad + (a_1(t)X_1\, dt + b_1(t)X_1\, dW(t))(a_2(t)X_2\, dt + b_2(t)X_2\, dW(t)) \\
&= (a_1(t) + a_2(t) + b_1(t)b_2(t))Y\, dt + (b_1(t) + b_2(t))Y\, dW(t),
\end{aligned}$$

since $(dt)^2 = dt\, dW(t) = 0$ and $(dW(t))^2 = dt$.

9.6.1. Using Eq. (9.40), for $a \geq 0$,

$$\begin{aligned}
\mathbb{P}\left(\overline{M}(t) \leq a\right) &= 1 - \frac{2}{\sqrt{2\pi\sigma^2 t}} \int_a^\infty e^{-\frac{x^2}{2\sigma^2 t}}\, dx \\
f_{\overline{M}(t)}(a) &= \frac{2}{\sqrt{2\pi\sigma^2 t}} e^{-\frac{a^2}{2\sigma^2 t}} \\
f_{\overline{M}(t)}(x) &= \begin{cases} 0 & \text{if } x < 0, \\ \dfrac{2}{\sqrt{2\pi\sigma^2 t}} e^{-\frac{x^2}{2\sigma^2 t}} & \text{if } x \geq 0 \end{cases}
\end{aligned}$$

9.6.5. The probability density function is

$$\begin{aligned}
\hat{f}_{\overline{M}(t)}(m) &= \int_{-\infty}^{m} \frac{2(2m - w)}{\sigma^2 t\sqrt{2\pi\sigma^2 t}} e^{\frac{-(2m-w)^2 + 2\mu w t - \mu^2 t^2}{2\sigma^2 t}}\, dw \\
&= \int_{-\infty}^{m} \frac{2(2m - w)}{\sigma^2 t\sqrt{2\pi\sigma^2 t}} e^{\frac{-(w-2m-\mu t)^2}{2\sigma^2 t} + \frac{2\mu t}{\sigma^2}}\, dw \\
&= e^{\frac{2\mu m}{\sigma^2}} \int_{-\infty}^{m} \frac{4m}{\sigma^2 t\sqrt{2\pi\sigma^2 t}} e^{\frac{-(w-2m-\mu t)^2}{2\sigma^2 t}}\, dw - e^{\frac{2\mu m}{\sigma^2}} \int_{-\infty}^{m} \frac{2w}{\sigma^2 t\sqrt{2\pi\sigma^2 t}} e^{\frac{-(w-2m-\mu t)^2}{2\sigma^2 t}}\, dw.
\end{aligned}$$

Make the substitutions $z = \frac{w-2m-\mu t}{\sigma\sqrt{t}}$ and $\sigma\sqrt{t}\, dz = dw$.

$$\begin{aligned}
\hat{f}_{\overline{M}(t)}(m) &= \frac{4m}{\sigma^2 t} e^{\frac{2\mu m}{\sigma^2}} \int_{-\infty}^{-\frac{(m+\mu t)}{\sigma\sqrt{t}}} \frac{1}{\sqrt{2\pi}} e^{-\frac{z^2}{2}}\, dz - e^{\frac{2\mu m}{\sigma^2}} \int_{-\infty}^{-\frac{(m+\mu t)}{\sigma\sqrt{t}}} \frac{2\sigma\sqrt{t}z + 4m + 2\mu t}{\sigma^2 t\sqrt{2\pi}} e^{-\frac{z^2}{2}}\, dz \\
&= -e^{\frac{2\mu m}{\sigma^2}} \int_{-\infty}^{-\frac{(m+\mu t)}{\sigma\sqrt{t}}} \frac{2\sigma\sqrt{t}z + 2\mu t}{\sigma^2 t\sqrt{2\pi}} e^{-\frac{z^2}{2}}\, dz \\
&= \frac{2}{\sqrt{2\pi\sigma^2 t}} e^{\frac{2\mu m}{\sigma^2}} \int_{-\infty}^{-\frac{(m+\mu t)}{\sigma\sqrt{t}}} (-z) e^{-\frac{z^2}{2}}\, dz - \frac{2\mu}{\sigma^2} e^{\frac{2\mu m}{\sigma^2}} \int_{-\infty}^{-\frac{(m+\mu t)}{\sigma\sqrt{t}}} \frac{1}{\sqrt{2\pi}} e^{-\frac{z^2}{2}}\, dz \\
&= 2e^{\frac{2\mu m}{\sigma^2}} \left[\frac{1}{\sqrt{2\pi\sigma^2 t}} e^{-\frac{(m+\mu t)^2}{2\sigma^2 t}} - \frac{\mu}{\sigma^2}\Phi\left(-\frac{(m+\mu t)}{\sigma\sqrt{t}}\right)\right]
\end{aligned}$$

for $m > 0$. The cumulative distribution function is found by integrating the probability density function using integration by parts.

$$
\begin{aligned}
\hat{F}_{\overline{M}(t)}(m) = \mathbb{P}\left(\overline{M}(t) \le m\right) &= \int_0^m \hat{f}_{\overline{M}(t)}(x)\,dx \\
&= \int_0^m 2e^{\frac{2\mu x}{\sigma^2}} \left[\frac{1}{\sqrt{2\pi\sigma^2 t}} e^{-\frac{(x+\mu t)^2}{2\sigma^2 t}} - \frac{\mu}{\sigma^2} \Phi\left(-\frac{(x+\mu t)}{\sigma\sqrt{t}} \right) \right] dx \\
&= \int_0^m \frac{2}{\sqrt{2\pi\sigma^2 t}} e^{-\frac{(x-\mu t)^2}{2\sigma^2 t}}\,dx - \int_0^m \frac{2\mu}{\sigma^2} e^{\frac{2\mu x}{\sigma^2}} \Phi\left(-\frac{(x+\mu t)}{\sigma\sqrt{t}} \right) dx \\
&= \int_0^m \frac{2}{\sqrt{2\pi\sigma^2 t}} e^{-\frac{(x-\mu t)^2}{2\sigma^2 t}}\,dx - \left[e^{\frac{2\mu x}{\sigma^2}} \Phi\left(-\frac{(x+\mu t)}{\sigma\sqrt{t}} \right) \right]_{x=0}^{x=m} - \int_0^m \frac{1}{\sqrt{2\pi\sigma^2 t}} e^{-\frac{(x-\mu t)^2}{2\sigma^2 t}}\,dx \\
&= \int_{-\frac{\mu t}{\sigma\sqrt{t}}}^{\frac{m-\mu t}{\sigma\sqrt{t}}} \frac{1}{\sqrt{2\pi}} e^{-\frac{z^2}{2}}\,dz - e^{\frac{2\mu m}{\sigma^2}} \Phi\left(-\frac{(m+\mu t)}{\sigma\sqrt{t}} \right) + \Phi\left(-\frac{\mu t}{\sigma\sqrt{t}} \right) \\
&= \Phi\left(\frac{m-\mu t}{\sigma\sqrt{t}} \right) - e^{\frac{2\mu m}{\sigma^2}} \Phi\left(\frac{-m-\mu t}{\sigma\sqrt{t}} \right)
\end{aligned}
$$

for $m > 0$.

10.1.4. Let $F(S,t) = S^a$ and suppose this function solves the Black-Scholes partial differential equation. This implies

$$
\begin{aligned}
0 &= 0 + (r-\delta)S(aS^{a-1}) + \frac{1}{2}\sigma^2 S^2 (a(a-1)S^{a-2}) - rS^a \\
&= a(r-\delta)S^a + \frac{a(a-1)}{2}\sigma^2 S^a - rS^a \\
0 &= a(r-\delta) + \frac{a(a-1)}{2}\sigma^2 - r \\
&= \frac{\sigma^2}{2}a^2 + \left(r - \delta - \frac{\sigma^2}{2} \right) a - r \\
a &= \frac{-\left(r-\delta-\frac{\sigma^2}{2} \right) \pm \sqrt{\left(r-\delta-\frac{\sigma^2}{2} \right)^2 + 2\sigma^2 r}}{\sigma^2} \\
&= \left(\frac{1}{2} - \frac{r-\delta}{\sigma^2} \right) \pm \sqrt{\left(\frac{1}{2} - \frac{r-\delta}{\sigma^2} \right)^2 + \frac{2r}{\sigma^2}}
\end{aligned}
$$

10.1.5. Let $F(S,t) = e^t S^a$ and suppose this function solves the Black-Scholes partial differential equation.

This implies

$$\begin{aligned}
0 &= e^t S^a + (r-\delta)S(ae^t S^{a-1}) + \frac{1}{2}\sigma^2 S^2 (a(a-1)e^t S^{a-2}) - re^t S^a \\
&= S^a + (r-\delta)S(aS^{a-1}) + \frac{1}{2}\sigma^2 S^2(a(a-1)S^{a-2}) - rS^a \\
&= a(r-\delta)S^a + \frac{a(a-1)}{2}\sigma^2 S^a + (1-r)S^a \\
0 &= a(r-\delta) + \frac{a(a-1)}{2}\sigma^2 + (1-r) \\
&= \frac{\sigma^2}{2}a^2 + \left(r-\delta-\frac{\sigma^2}{2}\right)a + (1-r) \\
a &= \frac{-\left(r-\delta-\frac{\sigma^2}{2}\right) \pm \sqrt{\left(r-\delta-\frac{\sigma^2}{2}\right)^2 - 2\sigma^2(1-r)}}{\sigma^2} \\
&= \left(\frac{1}{2} - \frac{r-\delta}{\sigma^2}\right) \pm \sqrt{\left(\frac{1}{2}-\frac{r-\delta}{\sigma^2}\right)^2 + \frac{2(r-1)}{\sigma^2}}
\end{aligned}$$

10.2.3. The value of the cash-or-nothing put option will satisfy the Black-Scholes partial differential equation. The payoff of a cash-or-nothing put option with strike price K and strike time T is $F(S,T) = \mathbb{1}_{(S(T)\leq K)}$ which is the final condition for the option. If $S(t) = 0$ for some $0 \geq t \leq T$ then S remains zero thereafter. Hence the cash-or-nothing put will pay \$1 at time T and it value is the present value of \$1, namely $e^{-r(T-t)}$. Therefore the boundary condition at $S(t) = 0$ is $F(0,t) = e^{-r(T-t)}$. As $S(t) \to \infty$ the payoff of the cash-or-nothing put becomes zero and thus the boundary condition at infinity can be expressed as $\lim_{S(t)\to\infty} F(S,t) = 0$. In summary the initial, boundary-value problem for the cash-or-nothing put can be expressed as

$$\begin{aligned}
rF &= F_t + (r-\delta)SF_S + \frac{1}{2}\sigma^2 S^2 F_{SS} \text{ for } (S,t) \text{ in } \Omega, \\
F(S,T) &= \mathbb{1}_{(S(T)\leq K)} \text{ for } S > 0, \\
F(0,t) &= e^{-r(T-t)} \text{ for } 0 \leq t < T, \\
F(S,t) &= 0 \text{ as } S \to \infty.
\end{aligned}$$

10.2.5. The value of the asset-or-nothing call option will satisfy the Black-Scholes partial differential equation. The payoff of an asset-or-nothing call option with strike price K and strike time T is $F(S,T) = S(T)\,\mathbb{1}_{(S(T)>K)}$ which is the final condition for the option. If $S(t) = 0$ for some $0 \geq t \leq T$ then S remains zero thereafter. Hence the boundary condition at $S(t) = 0$ is $F(0,t) = 0$. As $S(t) \to \infty$ the payoff of the asset-or-nothing call becomes $S(T)$ and thus the value of the option is the value of $e^{-\delta(T-t)}$ shares. Thus boundary condition at infinity can be expressed as $\lim_{S(t)\to\infty} F(S,t) = e^{-\delta(T-t)}S(T)$. In summary the initial, boundary-value problem for the asset-or-nothing call can be expressed as

$$\begin{aligned}
rF &= F_t + (r-\delta)SF_S + \frac{1}{2}\sigma^2 S^2 F_{SS} \text{ for } (S,t) \text{ in } \Omega, \\
F(S,T) &= S(T)\,\mathbb{1}_{(S(T)>K)} \text{ for } S > 0, \\
F(0,t) &= 0 \text{ for } 0 \leq t < T, \\
F(S,t) &= e^{-\delta(T-t)}S(T) \text{ as } S \to \infty.
\end{aligned}$$

10.2.7. The value of the asset-or-nothing put option will satisfy the Black-Scholes partial differential equation. The payoff of an asset-or-nothing put option with strike price K and strike time T is $F(S,T) = S(T)\,\mathbb{1}_{(S(T)\leq K)}$ which is the final condition for the option. If $S(t) = 0$ for some $0 \geq t \leq T$ then S remains zero thereafter. Hence the boundary condition at $S(t) = 0$ is $F(0,t) = 0$. As $S(t) \to \infty$ the payoff of the asset-or-nothing put becomes 0 and thus the value of the option is 0. Thus boundary condition at infinity can be expressed as $\lim_{S(t)\to\infty} F(S,t) = 0$. In summary the initial, boundary-value problem for the asset-or-nothing put can be expressed as

$$\begin{aligned} rF &= F_t + (r-\delta)SF_S + \frac{1}{2}\sigma^2 S^2 F_{SS} \text{ for } (S,t) \text{ in } \Omega, \\ F(S,T) &= S(T)\,\mathbb{1}_{(S(T)\leq K)} \text{ for } S>0, \\ F(0,t) &= 0 \text{ for } 0 \leq t < T, \\ F(S,t) &= 0 \text{ as } S \to \infty. \end{aligned}$$

10.4.7. Using Eqs. (10.32), (10.33), and (10.35)

$$\begin{aligned} d_1 &= \frac{\ln\frac{55}{58} + \left(0.055 - 0.025 + \frac{(0.28)^2}{2}\right)(4/12-0)}{0.28\sqrt{4/12-0}} = -0.185844 \\ d_2 &= -0.185844 - 0.28\sqrt{4/12-0} = -0.347502 \\ P^e &= 58e^{-0.055(4/12)}\Phi(0.347502) - 55e^{-0.025(4/12)}\Phi(0.185844) = \$4.9192 \end{aligned}$$

10.4.9. The premium for the put can be calculated as follows. With $T = 6/12$, $t = 0$, $r = 0.05$, $\delta = 0.01$, $\sigma = 0.35$, $S = 69$ and $K = 70$,

$$\begin{aligned} d_1 &= \frac{\ln\frac{69}{70} + \left(0.05 - 0.01 + \frac{(0.35)^2}{2}\right)\left(\frac{6}{12}-0\right)}{0.35\sqrt{\frac{6}{12}-0}} \approx 0.146417 \\ d_2 &= 0.146417 - 0.35\sqrt{\frac{6}{12}-0} \approx -0.101071. \end{aligned}$$

Substituting these values in Eq. (10.35) yields the price of the European put option.

$$\begin{aligned} P^e &= 70e^{-0.05(6/12-0)}\Phi(0.101071) - 69e^{-0.02(6/12-0)}\Phi(-0.146417) \\ &= (68.2717)(0.540253) - (68.6559)(0.441796) \\ &= \$6.5521 \end{aligned}$$

10.4.11. Using Eq. (6.8),

$$\begin{aligned} P^e &= C^e + 120e^{-0.055(3/12-0)} - 110e^{-0.01(3/12-0)} \\ &= 2.3766 + 118.361 - 109.725 \\ &= \$11.0125. \end{aligned}$$

10.4.13. Since the stock pays no dividends,

$$\mathbb{E}\left(S(6/12)\right) = 65.87 = S(0)e^{0.10(6/12)}$$

which implies $S(0) = 62.6575$. The premium for the put can be calculated as follows. With $T = 1$, $t = 0$, $r = 0.04$, $\delta = 0.00$, $\sigma = 0.25$, $S = 62.6575$ and $K = 70$,

$$\begin{aligned} d_1 &= \frac{\ln\frac{62.6575}{70} + \left(0.04 - 0.00 + \frac{(0.25)^2}{2}\right)(1-0)}{0.25\sqrt{1-0}} \approx -0.158247 \\ d_2 &= -0.158247 - 0.25\sqrt{1-0} \approx -0.408257. \end{aligned}$$

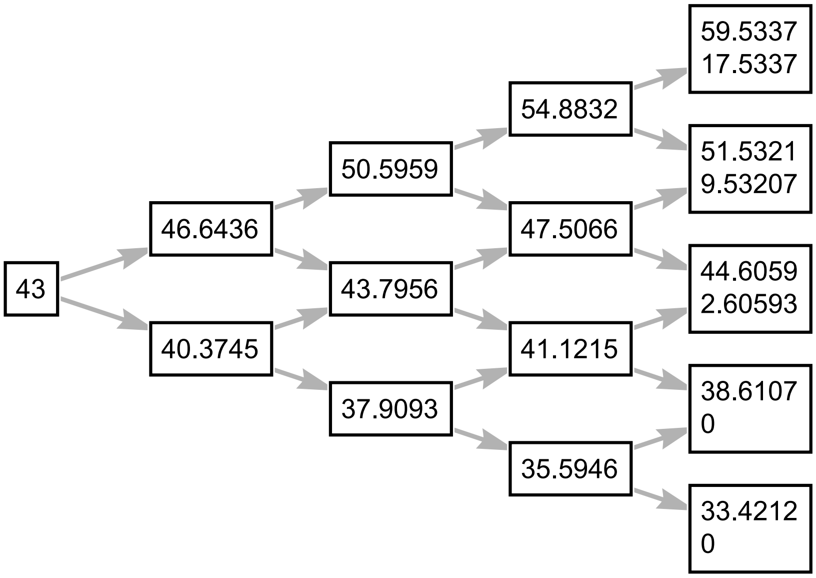
Substituting these values in Eq. (10.35) yields the price of the European put option.

$$\begin{aligned} P^e &= 70e^{-0.04(1-0)}\Phi(0.408257) - 62.6575\Phi(0.158247) \\ &= (67.2553)(0.658454) - (62.6575)(0.562869) \\ &= \$9.0165 \end{aligned}$$

10.5.5. We may calculate,

$$\begin{aligned} u &= e^{(0.11-0)(1/12)+0.25\sqrt{1/12}} = 1.08473 \\ d &= e^{(0.11-0)(1/12)-0.25\sqrt{1/12}} = 0.93894 \\ p^* &= \frac{1}{1+e^{0.25\sqrt{1/12}}} = 0.48197. \end{aligned}$$

The four-step forward tree resembles the following.



The price of the European call is approximately,

$$\begin{aligned} C^e &\approx e^{-0.11(4/12)} \sum_{j=0}^{4} \binom{4}{j} (p^*)^j (1-p^*)^{4-j} (43u^j d^{4-j} - 42)^+ \\ &= \$3.9834 \end{aligned}$$

Using the Black-Scholes formula for the European call produces,

$$\begin{aligned} d_1 &= \frac{\ln\frac{43}{42} + \left(0.11 - 0 + \frac{(0.25)^2}{2}\right)(4/12-0)}{0.25\sqrt{4/12-0}} = 0.489227 \\ d_2 &= 0.489227 - 0.25\sqrt{4/12-0} = 0.344889 \\ C^e &= 43\Phi(0.489227) - 42e^{-0.11(4/12-0)}\Phi(0.344889) = \$3.86314. \end{aligned}$$

10.5.7. We may calculate,

$$\begin{aligned} u &= e^{(0.08-0.02)(1/12)+0.40\sqrt{1/12}} = 1.12803 \\ d &= e^{(0.08-0.02)(1/12)-0.40\sqrt{1/12}} = 0.89541 \\ p^* &= \frac{1}{1+e^{0.40\sqrt{1/12}}} = 0.47117. \end{aligned}$$

The forward tree has the following values for $S(6/12)$ and call option payoffs.

$S(6/12)$	$(S(6/12)-100)^+$
206.024	106.024
163.539	63.5388
129.815	29.815
103.045	3.04545
81.7961	0
64.9287	0
51.5396	0

The price of the European call is approximately,

$$\begin{aligned} C^e &\approx e^{-0.08(6/12)} \sum_{j=0}^{6} \binom{6}{j} (p^*)^j (1-p^*)^{6-j} (100u^j d^{6-j} - 100)^+ \\ &= \$12.4398 \end{aligned}$$

Using the Black-Scholes formula for the European call produces,

$$\begin{aligned} d_1 &= \frac{\ln \frac{100}{100} + \left(0.08 - 0.02 + \frac{(0.40)^2}{2}\right)(6/12-0)}{0.40\sqrt{6/12-0}} = 0.247487 \\ d_2 &= 0.247487 - 0.40\sqrt{6/12-0} = -0.035355 \\ C^e &= 100e^{-0.02(6/12-0}\Phi\left(0.247487\right) - 100e^{-0.08(6/12-0)}\Phi\left(-0.035355\right) = \$12.4941. \end{aligned}$$

11.1.3. Using the Put-Call parity formula of Eq. (11.1),

$$\begin{aligned} P^e &= 117e^{-0.04(4/12)} - 4.05 - 110e^{-0.01(4/12)} \\ &= 1.7664. \end{aligned}$$

11.1.5. If the strike of the dollar-denominated call is \$1.05/€, the strike of the corresponding euro-denominated put is

$$\frac{1}{1.05} = 0.9524\text{€}/\$.$$

According to Eq. (11.8) the price of the put is

$$P^e_{\text{€}} = \frac{1}{1.10}\frac{1}{1.05}(0.03) = 0.025974\text{€}.$$

11.2.3. Using Eq. (11.15),

$$\begin{aligned} d_1 &= \frac{\ln \frac{27.55}{25} + \frac{(0.70)^2}{2}(6/12-0)}{0.70\sqrt{6/12-0}} = 0.443713 \\ d_2 &= 0.443713 - 0.70\sqrt{6/12-0} = -0.051262 \\ P^e &= e^{-0.035(6/12)}\left(25\Phi\left(0.051262\right) - 27.55\Phi\left(-0.443713\right)\right) = \$3.8888. \end{aligned}$$

11.2.5.

(a) The premium for the put on the underlying stock.

$$\begin{aligned} d_1 &= \frac{\ln\frac{150}{148} + \left(0.06 - 0.02 + \frac{(0.40)^2}{2}\right)(6/12)}{0.40\sqrt{6/12}} = 0.259590 \\ d_2 &= 0.259590 - 0.40\sqrt{6/12} = -0.023253 \\ P^e &= 148e^{-0.06(6/12)}\Phi(0.023253) - 150e^{-0.02(6/12)}\Phi(-0.259590) = \$14.1001 \end{aligned}$$

(b) The price of a six-month forward contract on the stock.

$$F_{0,6/12} = 150e^{(0.06-0.02)(6/12)} = \$153.03$$

(c) The price of a six-month, 148-strike put on the futures contract.

$$\begin{aligned} d_1 &= \frac{\ln\frac{153.03}{148} + \frac{(0.40)^2}{2}(6/12)}{0.40\sqrt{6/12}} = 0.259590 \\ d_2 &= 0.259590 - 0.40\sqrt{6/12} = -0.023253 \\ P^e &= e^{-0.06(6/12)}\left(148\Phi(0.023253) - 153.03\Phi(-0.259590)\right) = \$14.1001 \end{aligned}$$

(d) During the calculation of the put premium for the futures contract,

$$\begin{aligned} d_1 &= \frac{\ln\frac{153.03}{148} + \frac{(0.40)^2}{2}(6/12)}{0.40\sqrt{6/12}} \\ &= \frac{\ln\frac{150e^{(0.06-0.02)(6/12)}}{148} + \frac{(0.40)^2}{2}(6/12)}{0.40\sqrt{6/12}} \\ &= \frac{\ln\frac{150}{148} + \left(0.06 - 0.02 + \frac{(0.40)^2}{2}\right)(6/12)}{0.40\sqrt{6/12}} \end{aligned}$$

the same formula as used in the calculation of the premium for the put on the stock. Similarly the values for d_2 will be the same. The price of the put on the futures contract is

$$\begin{aligned} P^e &= e^{-0.06(6/12)}\left(148\Phi(-d_2) - 153.03\Phi(-d_1)\right) \\ &= 148e^{-0.06(6/12)}\Phi(-d_2) - e^{-0.06(6/12)}150e^{(0.06-0.02)(6/12)}\Phi(-d_1) \\ &= 148e^{-0.06(6/12)}\Phi(-d_2) - 150e^{-0.02(6/12)}\Phi(-d_1) \end{aligned}$$

which is the formula for the put on the stock. The same parameters are used in the two calculations, but the discounts they apply are used at different points in the calculations.

11.3.1. The sum of the present values of the dividends is

$$\begin{aligned} S &= \frac{D}{1+r} + \frac{D(1+g)}{(1+r)^2} + \cdots + \frac{D(1+g)^{k-1}}{(1+r)^k} \\ (1+r)S &= D\left[1 + \frac{(1+g)}{(1+r)} + \cdots + \frac{(1+g)^{k-1}}{(1+r)^{k-1}}\right] \\ &= \frac{D\left[1 - \left(\frac{1+g}{1+r}\right)^k\right]}{1 - \frac{1+g}{1+r}} \text{ (using Eq. (1.8))} \\ S &= \frac{D\left[1 - \left(\frac{1+g}{1+r}\right)^k\right]}{r-g} \end{aligned}$$

assuming $1 \neq (1+g)/(1+r)$.

11.3.7. The prepaid forward price for the stock is $F^P_{0,5/12} = 50e^{-0.03} = 48.5223$. The Put-Call parity formula produces at $t = 0$,

$$P^e = 3.9439 + 50e^{-0.025(5/12-0)} - 48.5223 = \$4.9035.$$

At $t = 2/12$,

$$P^e = 2.8722 + 50e^{-0.025(5/12-2/12)} - 48.5223 = \$4.0384.$$

11.3.9. The price of the prepaid forward for the stock is

$$F^P_{0,5/12} = 50 - 2e^{-0.025(2/12)} - 2e^{-0.025(4/12)} = 46.0249.$$

The premium for the European call can be calculated as follows.

$$\begin{aligned} d_1 &= \frac{\ln \frac{46.0249}{50} + \left(0.025 + \frac{(0.35)^2}{2}\right)(5/12-0)}{0.35\sqrt{5/12-0}} = -0.207604 \\ d_2 &= -0.207604 - 0.35\sqrt{5/12-0} = -0.433528 \\ C^e &= 46.0249\Phi\left(-0.207604\right) - 50e^{-0.025(5/12-0)}\Phi\left(-0.433528\right) = \$2.78419 \end{aligned}$$

11.3.11. Using the Put-Call parity relationship,

$$P^e = 2.78419 + 50e^{-0.025(5/12)} - 46.0249 = \$6.2412.$$

12.1.3. Using Eq. (12.2) with $S = 150$, $K = 165$, $T = 5/12$, $t = 0$, $\delta = 0$, $r = 0.025$, and $\sigma = 0.22$ we have

$$\begin{aligned} d_1 &= -0.526797 \\ \Delta_{C^e} &= 0.299167. \end{aligned}$$

12.1.5. Using Eq. (12.3) with $S = 125$, $K = 140$, $T = 2/3$, $t = 0$, $\delta = 0$, $r = 0.055$, and $\sigma = 0.15$ we have

$$\begin{aligned} d_1 &= -0.564706 \\ \Delta_{P^e} &= -0.713863. \end{aligned}$$

12.1.7. Using Eq. (12.4) with $S = 180$, $K = 175$, $T = 4/12$, $t = 0$, $\delta = 0$, $r = 0.0375$, and $\sigma = 0.30$ we have

$$\begin{aligned} d_1 &= 0.321416 \\ \Gamma^e &= 0.0121519. \end{aligned}$$

12.1.9. According to Eq. (9.3)

$$(55)^2\left(e^{\sigma^2(6/12)} - 1\right) = 140 \implies \sigma = 0.30081.$$

Using the formula for the Delta of a European call from Eq. (12.2)

$$\begin{aligned} e^{-0.01(6/12)}\Phi\left(d_1\right) &= 0.65 \\ \Phi\left(d_1\right) &= 0.653258 \\ d_1 &= 0.394132 \\ \frac{\ln \frac{50}{50} + \left(r - 0.01 + \frac{(0.30081)^2}{2}\right)(6/12)}{0.30081\sqrt{6/12}} &= 0.394132 \\ r &= 0.242659. \end{aligned}$$

The Gamma of the European call is

$$\Gamma^e = \frac{e^{-0.01(6/12)}\varphi\left(0.394132\right)}{0.30081(50)\sqrt{6/12}} = 0.0301526.$$

Note that $d_2 = 0.440533$. The price of the European call is

$$\begin{aligned} C^e &= 50e^{-0.01(6/12)}\Phi\left(0.394132\right) - 50e^{-0.242659(6/12)}\Phi\left(0.440533\right) \\ &= \$7.2923. \end{aligned}$$

12.1.11. A cash-or-nothing put has a value of $P^b = e^{-r(T-t)}\Phi\left(-d_2\right)$.

$$\begin{aligned} \frac{\partial P^b}{\partial S} &= e^{-r(T-t)}\varphi\left(-d_2\right)\frac{\partial}{\partial S}\left[-d_2\right] = \frac{-e^{-r(T-t)}}{\sigma S\sqrt{T-t}}\varphi\left(d_2\right) \\ \frac{\partial^2 P^b}{\partial S^2} &= \frac{e^{-r(T-t)}}{\sigma S^2\sqrt{T-t}}\varphi\left(d_2\right) - \frac{e^{-r(T-t)}(-d_2)}{\sigma S\sqrt{T-t}}\varphi\left(d_2\right)\frac{\partial}{\partial S}\left[-d_2\right] \\ &= \frac{e^{-r(T-t)}}{\sigma S^2\sqrt{T-t}}\varphi\left(d_2\right)\left(1 + \frac{d_2}{\sigma\sqrt{T-t}}\right) \end{aligned}$$

A cash-or-nothing call has a value of $C^b = e^{-r(T-t)}\Phi\left(d_2\right)$.

$$\begin{aligned} \frac{\partial C^b}{\partial S} &= e^{-r(T-t)}\varphi\left(d_2\right)\frac{\partial}{\partial S}\left[d_2\right] = \frac{e^{-r(T-t)}}{\sigma S\sqrt{T-t}}\varphi\left(d_2\right) \\ \frac{\partial^2 C^b}{\partial S^2} &= \frac{-e^{-r(T-t)}}{\sigma S^2\sqrt{T-t}}\varphi\left(d_2\right) + \frac{e^{-r(T-t)}(-d_2)}{\sigma S\sqrt{T-t}}\varphi\left(d_2\right)\frac{\partial}{\partial S}\left[d_2\right] \\ &= \frac{-e^{-r(T-t)}}{\sigma S^2\sqrt{T-t}}\varphi\left(d_2\right)\left(1 + \frac{d_2}{\sigma\sqrt{T-t}}\right) \end{aligned}$$

Note that the Deltas of the put and call are negatives of each. The same is true of the Gammas of the cash-or-nothing put and call.

12.1.15. Rearranging terms and differentiating Eq. (11.9) with respect to F produce,

$$\begin{aligned} P^e(F(t),K,T) &= C^e(F(t),K,T) + K^{-r(T-t)} - F(t)e^{-r(T-t)} \\ \frac{\partial P^e}{\partial F} &= e^{-r(T-t)}\Phi\left(d_1\right) - e^{-r(T-t)} \\ \Delta &= -e^{-r(T-t)}\Phi\left(-d_1\right). \end{aligned}$$

Differentiating again with respect to F yields

$$\begin{aligned} \frac{\partial^2 P^e}{\partial F^2} &= \frac{\partial}{\partial F}\left[-e^{-r(T-t)}\Phi\left(-d_1\right)\right] \\ &= e^{-r(T-t)}\varphi\left(-d_1\right)\frac{\partial d_1}{\partial F} \\ \Gamma &= \frac{e^{-r(T-t)}}{\sigma F(t)\sqrt{T-t}}\varphi\left(d_1\right). \end{aligned}$$

12.2.1. Using Eq. (12.6) with $S = 300$, $K = 310$, $T = 1/4$, $t = 0$, $\delta = 0$, $r = 0.03$, and $\sigma = 0.25$ we have

$$\begin{aligned} d_1 &= -0.139819 \\ \Theta_{C^e} &= -33.281. \end{aligned}$$

12.2.3. Using Eq. (12.6) with $S = 275$, $K = 265$, $T = 4/12$, $t = 0$, $\delta = 0$, $r = 0.02$, and $\sigma = 0.20$ we have

$$\begin{aligned} d_1 &= 0.436257 \\ \Theta_{P^e} &= -15.3073. \end{aligned}$$

12.2.5. Since the stock does not pay a dividend then $\Delta_{P^e} = 0.6371 - 1 = -0.3629$. From the Black-Scholes partial differential equation,

$$\begin{aligned} \Theta_{P^e} &= -rS\Delta_{P^e} - \frac{\sigma^2 S^2}{2}\Gamma^e + rP^e \\ -2.0132 &= -(100)(-0.3629)r - \frac{(0.30)^2(100)^2}{2}(0.0125) + 8.8846r \\ r &= 0.08. \end{aligned}$$

12.2.7. The strangle consists of a long 1-year, 160-strike European call and a short 1-year, 140-strike European call. The Theta for the 160-strike call is calculated as follows.

$$\begin{aligned} d_1 &= \frac{\ln\frac{150}{160} + \left(0.05 - 0 + \frac{(0.40)^2}{2}\right)(1-0)}{0.40\sqrt{1-0}} = 0.163654 \\ d_2 &= 0.163654 - 0.40\sqrt{1-0} = -0.236346 \\ \Theta_{160} &= -0.05(160)e^{-0.05}\Phi\left(-0.236346\right) - \frac{0.40(150)\varphi\left(0.163654\right)}{2\sqrt{1}} = -14.9031 \end{aligned}$$

Similarly the Theta for the 140-strike call is

$$\begin{aligned} d_1 &= \frac{\ln\frac{150}{140} + \left(0.05 - 0 + \frac{(0.40)^2}{2}\right)(1-0)}{0.40\sqrt{1-0}} = 0.497482 \\ d_2 &= 0.497482 - 0.40\sqrt{1-0} = 0.097482 \\ \Theta_{140} &= -0.05(140)e^{-0.05}\Phi\left(0.097482\right) - \frac{0.40(150)\varphi\left(0.497482\right)}{2\sqrt{1}} = -14.1631. \end{aligned}$$

Thus the Theta for the strangle is

$$\Theta = -14.9031 - (-14.1631) = -0.74.$$

12.3.5. Using Eq. (12.8) with $S = 123$, $K = 125$, $T = 3/12$, $t = 0$, $\delta = 0$, $r = 0.0515$, and $\sigma = 0.35$ we have

$$\begin{aligned} d_1 &= 0.0689035 \\ \mathcal{V} &= 24.4768. \end{aligned}$$

12.3.7. Using Eq. (12.9) with $S = 305$, $K = 325$, $T = 4/12$, $t = 0$, $\delta = 0$, $r = 0.0255$, and $\sigma = 0.35$ we have

$$\begin{aligned} d_2 &= -0.373282 \\ \rho_{C^e} &= 38.0758. \end{aligned}$$

12.3.9. Let $S = 47.50$, $r = 0.06$, $\sigma = 0.30$, $\delta = 0.01$, $T = 3/12$, and $t = 0$. The only quantity differing between the long put and the short put is the strike price. We will find the Vega, Rho, and Psi for the two puts separately. For the long put,

$$\begin{aligned} d_1 &= \frac{\ln\frac{47.50}{45} + \left(0.06 - 0.01 + \frac{(0.30)^2}{2}\right)(3/12 - 0)}{0.30\sqrt{3/12 - 0}} = 0.918781 \\ d_2 &= 0.918781 - 0.30\sqrt{3/12 - 0} = 0.768781. \end{aligned}$$

We may then see that

$$\begin{aligned}\mathcal{V} &= 47.50e^{-0.01(3/12-0)}\varphi\left(0.918781\right)\sqrt{3/12-0} = 6.1970\\ \rho_{P^e} &= -45(3/12-0)e^{-0.06(3/12-0)}\Phi\left(-0.768781\right) = -2.3067\\ \Psi_{P^e} &= (3/12-0)(47.50)e^{-0.01(3/12-0)}\Phi\left(-0.918781\right) = 2.1216.\end{aligned}$$

For the short put,

$$\begin{aligned}d_1 &= \frac{\ln\frac{47.50}{50} + \left(0.06 - 0.01 + \frac{(0.30)^2}{2}\right)(3/12-0)}{0.30\sqrt{3/12-0}} = 0.216378\\ d_2 &= 0.216378 - 0.30\sqrt{3/12-0} = 0.066378.\end{aligned}$$

We may then see that

$$\begin{aligned}\mathcal{V} &= 47.50e^{-0.01(3/12-0)}\varphi\left(0.216378\right)\sqrt{3/12-0} = 9.2325\\ \rho_{P^e} &= -50(3/12-0)e^{-0.06(3/12-0)}\Phi\left(-0.066378\right) = -5.4915\\ \Psi_{P^e} &= (3/12-0)(47.50)e^{-0.01(3/12-0)}\Phi\left(-0.216378\right) = 4.9081.\end{aligned}$$

Subtracting the values of the short put from the values of the long put produces the following values for the Greeks of the spread.

$$\begin{aligned}\mathcal{V} &= 6.1970 - 9.2325 = -3.0355\\ \rho &= -2.3067 - (-5.4915) = 3.1848\\ \Psi &= 2.1216 - 4.9081 = -2.7865\end{aligned}$$

12.4.1. Using the Delta/Gamma approximation we have

$$\begin{aligned}V(25+1) - V(25) &= \Delta_V(1) + \frac{1}{2}\Gamma_V(1)^2\\ -1.0061 &= \Delta_V + \frac{0.0475}{2}\\ \Delta_V &= -1.0299.\end{aligned}$$

Using the Delta/Gamma approximation again,

$$V(24) \approx V(25) + (-1.0299)(-1) + \frac{0.0475}{2}(-1)^2 = \$11.8809.$$

12.4.3. Using the Delta approximation,

$$C^{an}(28) = 14.8753 + 2.0567(0.50) = 15.9036.$$

12.4.5. Using Eq. (12.6) we have $\Theta_{C^e} = -4.73721$. Now the linear approximation can be written as

$$\begin{aligned}C_2^e &= C_1^e + (\Delta_{C^e})(54.75 - 54.00) + (\Theta)(1/12-0)\\ &= \$2.30516.\end{aligned}$$

12.4.7. Using $S = 425$, $K = 435$, $T = 6/12$, $t = 0$, $\delta = 0$, $r = 0.0525$, and $\sigma = 0.17$ we have

$$\begin{aligned}d_1 &= 0.085004\\ d_2 &= -0.035205\\ C^e &= 20.98\\ \Delta_{C^e} &= 0.533871\\ \Omega &= \frac{(425)(0.533871)}{20.98} = 10.8148.\end{aligned}$$

12.4.11. Without loss of generality assume the option is a European call option. Using the relationship between the differentials, then

$$\begin{aligned} dC^e &= \mathcal{V}\, d\sigma \\ &= (24.4768)(0.05) \\ &= 1.22384. \end{aligned}$$

12.4.13. Using the given parameter values,

$$\begin{aligned} d_1 &= \frac{\ln(1) + \left(0.065 - 0.03 + \frac{(0.45)^2}{2}\right)(3/12 - 0)}{0.45\sqrt{3/12 - 0}} = 0.151389 \\ d_2 &= 0.151389 - 0.45\sqrt{3/12 - 0} = -0.073611 \\ C^e &= Se^{-0.03(3/12)}\Phi(0.151389) - Se^{-0.065(3/12)}\Phi(-0.073611) = 0.0929S \\ \Delta_{C^e} &= e^{-0.03(3/12)}\Phi(0.151389) = 0.5560 \\ \Omega_{C^e} &= \frac{0.5560S}{0.0929S} = 5.9843 \\ \gamma &= (5.9843)(0.12 + 0.03) + (1 - 5.9843)(0.065) = 0.5737 = 57.37\%. \end{aligned}$$

13.1.1. The price of the call options is found as follows.

$$\begin{aligned} d_1 &= \frac{\ln\frac{65}{70} + \left(0.05 + 0 - \frac{(0.35)^2}{2}\right)(6/12 - 0)}{0.35\sqrt{6/12 - 0}} = -0.0746825 \\ d_2 &= d_1 - 0.35\sqrt{6/12 - 0} = -0.32217 \\ \Phi(d_1) &= 0.470234 \\ \Phi(d_2) &= 0.373662 \\ C^e &= \$5.05465 \end{aligned}$$

If the market maker sells 100 of such options and purchases 50 shares of the underlying stock, they must borrow

$$50(65) - 100(5.05465) = \$2744.53$$

at the risk-free rate.

13.1.3. The price of the call options is found as follows.

$$\begin{aligned} d_1 &= \frac{\ln\frac{50}{55} + \left(0.04 + 0.03 - \frac{(0.25)^2}{2}\right)(6/12 - 0)}{0.25\sqrt{6/12 - 0}} = -0.422483 \\ d_2 &= d_1 - 0.25\sqrt{6/12 - 0} = -0.59926 \\ \Phi(d_1) &= 0.336336 \\ \Phi(d_2) &= 0.2745 \\ C^e &= \$1.7679 \end{aligned}$$

The market maker must borrow,

$$(35)(50) - 100(1.7679) = \$1573.21.$$

13.1.5. The Delta of the call option is calculated as follows.

$$\begin{aligned} d_1 &= \frac{\ln\frac{315}{325} + \left(0.0625 + 0.035 - \frac{(0.45)^2}{2}\right)(4/12 - 0)}{0.45\sqrt{4/12 - 0}} = 0.0448952 \\ \Phi(d_1) &= 0.517905 \\ \Delta_{C^e} &= 0.511897 \end{aligned}$$

The Delta of the portfolio is

$$100(0.511897) - 50 = 1.18975.$$

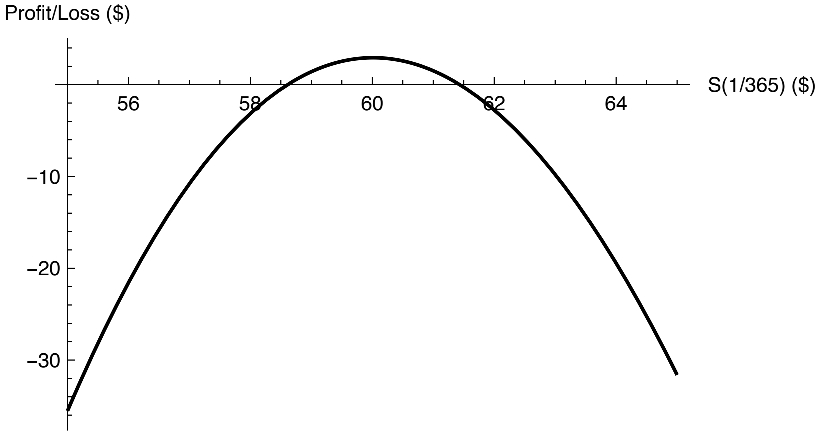
13.2.3. If the call option is at the money then $S(t) = K$. Start by considering the following limit.

$$\begin{aligned}
\lim_{t \to T^-} d_1 &= \lim_{t \to T^-} \left(\frac{\ln(S/S) + (r - \delta + \sigma^2/2)(T-t)}{\sigma\sqrt{T-t}} \right) \\
&= \lim_{t \to T^-} \left(\frac{r}{\sigma} + \frac{\sigma}{2} \right) \sqrt{T-t} \\
&= 0
\end{aligned}$$

Now since $\Delta_C = e^{-\delta(T-t)}\Phi(d_1)$ and $\Phi(d_1)$ is a continuous function then

$$\lim_{t \to T^-} \Delta_{C^e} = \lim_{d_1 \to 0} e^{-\delta(T-t)}\Phi(d_1) = \frac{1}{2}.$$

13.2.5.



13.2.7. Find the new value of the European calls.

$$\begin{aligned}
d_1 &= \frac{\ln\frac{65}{65} + \left(0.05 - 0 + \frac{(0.40)^2}{2}\right)(9/12 - 1/12)}{0.40\sqrt{9/12 - 1/12}} = 0.265361 \\
d_2 &= d_1 - 0.40\sqrt{9/12 - 1/12} = -0.061237 \\
C^e &= 65e^{-0(9/12-1/12)}\Phi(0.265361) - 65e^{-0.05(9/12-1/12)}\Phi(-0.61237) = \$9.4017
\end{aligned}$$

The net change in the value of the hedging portfolio is

$$100(9.4017 - 10.0207) + 61.0821(65 - 65) + 2968.27(e^{0.05(1/12)} - 1) = -\$49.5077.$$

13.2.9. Find the price of the put options and their Delta.

$$\begin{aligned}
d_1 &= \frac{\ln\frac{75}{70} + \left(0.04 - 0.04 + \frac{(0.35)^2}{2}\right)(6/12 - 0)}{0.35\sqrt{6/12 - 0}} = 0.402517 \\
d_2 &= d_1 - 0.35\sqrt{6/12 - 0} = 0.155030 \\
\Delta_{P^e} &= -e^{-0.04(6/12-0)}\Phi(-0.402517) = -0.336847 \\
P^e &= 70e^{-0.04(6/12-0)}\Phi(-0.155030) - 75e^{-0.04(6/12-0)}\Phi(-0.402517) = \$4.81674
\end{aligned}$$

The investor has purchased 100 put options, so they should purchase $-100\Delta_{P^e} = 33.6847$ shares. The net cash flow is

$$B = (33.6847)(75) + 100(4.81674) = \$3008.03$$

which can be borrowed at the risk-free rate.

13.2.10. Find the new value of the European puts.

$$\begin{aligned}
d_1 &= \frac{\ln \frac{72}{70} + \left(0.04 - 0.04 + \frac{(0.35)^2}{2}\right)(6/12 - 1/12)}{0.35\sqrt{6/12 - 1/12}} = 0.237654 \\
d_2 &= d_1 - 0.35\sqrt{6/12 - 1/12} = 0.011730 \\
P^e &= 70e^{-0.04(6/12-1/12)}\Phi\left(-0.011730\right) - 72e^{-0.04(6/12-1/12)}\Phi\left(-0.237654\right) = \$5.34522
\end{aligned}$$

The net change in the value of the hedging portfolio is

$$100(5.34522 - 4.81674) + 33.6847(72 - 75) - 3008.03(e^{0.04(1/12)} - 1) = -\$58.2493.$$

13.2.13. The European put options have the following characteristics.

Strike	**Premium**	Δ_{P^e}
45	1.13669	-0.220583
50	3.07642	-0.446853
55	6.17625	-0.670171

To create the Delta-neutral hedging portfolio the investor should take a position in

$$0.220583 + 0.670171 - 2(0.446853) = -0.0029512 \text{ shares.}$$

Since the overall position is negative, the investor will short 0.0029512 shares of the underlying stock. This will generate a net cash flow at time $t = 0$ of

$$B = -1.13669 - 6.17625 + 2(3.07642) + 50(0.0029512) = -\$1.01255$$

which can be borrowed at the risk-free rate.

13.2.15.

Week	$\mathbf{S(t)}$ **($)**	$\mathbf{C^e}$ **($)**	**Shares** $(\mathbf{1000\Delta_{C^e}})$	**Borrowing ($)**	**Profit ($)**
0	45.00	1.33871	408.940	17063.60	—
1	44.58	1.09932	366.642	15192.70	52.863
2	45.64	1.44956	447.836	18911.60	78.111
3	44.90	1.06097	374.621	15640.60	118.930
4	43.42	0.53486	238.290	9734.62	77.059
5	42.23	0.25565	140.594	5617.34	64.282
6	41.18	0.10797	73.032	2840.01	59.468
7	41.52	0.10299	73.130	2846.55	86.824
8	41.94	0.10190	75.921	2966.06	116.162
9	42.72	0.12987	97.654	3897.06	144.840
10	44.83	0.41097	254.169	10917.00	66.417
11	44.93	0.33393	235.189	10073.70	159.428
12	44.19	0.11545	114.050	4729.29	195.141
13	41.77	0.00075	1.621	37.19	29.744
14	39.56	0.00000	0.000	−26.88	26.884
15	40.62	0.00000	0.000	−26.90	26.90

13.2.17.

Week	**S(t)** **($)**	$\mathbf{P^e}$ **($)**	**Shares** $(\mathbf{1000\Delta_{P^e}})$	**Borrowing** **($)**	**Profit** **($)**
0	45.00	2.73256	591.060	29,330.20	—
1	44.58	2.95333	633.358	31,241.30	52.863
2	45.64	2.28377	552.164	27,562.70	78.111
3	44.90	2.67542	625.379	30,873.80	118.930
4	43.42	3.66958	761.710	36,820.10	77.059
5	42.23	4.62067	859.406	40,977.70	64.282
6	41.18	5.56334	926.968	43,795.40	59.468
7	41.52	5.25873	926.870	43,829.20	86.824
8	41.94	4.87805	924.079	43,750.10	116.162
9	42.72	4.16647	902.346	42,859.50	144.840
10	44.83	2.37805	745.831	35,880.10	66.417
11	44.93	2.24152	764.811	36,763.90	159.428
12	44.19	2.80359	885.950	42,148.80	195.141
13	41.77	5.14948	998.379	46,881.50	29.744
14	39.56	7.39934	1000.000	46,986.20	26.884
15	40.62	6.38000	1000.000	47026.90	26.879

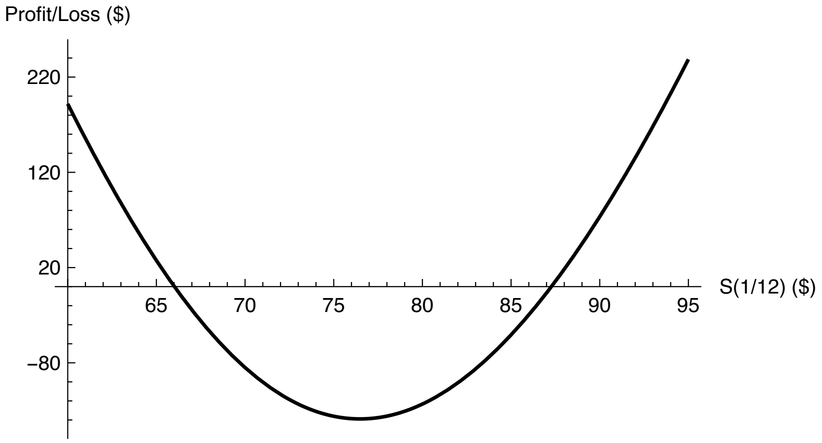
13.3.3. The price of the European put option is $P^e = \$9.4587$ and its $\Delta_{P^e} = -0.521195$. The hedge portfolio consists of 100 put options and 52.1195 shares of the underlying security. The investor may borrow

$$B = 100(9.4587) + 52.1195(75) = \$4854.84$$

at the risk-free rate to purchase the hedging portfolio. Let $P^e(S(1/12), 80, (3-1)/12)$ be the price of the option after one month. If the investor liquidates the hedging portfolio, the profit/loss may be expressed as

$$\text{P/L} = 100P^e(S(1/12), 80, (3-1)/12) + 52.1195S(1/12) - 4854.84e^{0.08(1/12)}.$$

The plot of profit/loss *versus* $S(1/12)$ resembles that shown below.



13.3.5. The exercise states that $\Delta_{C^e} = 0.655422 = \Phi(d_1)$, therefore

$$\begin{aligned} d_1 &= \Phi^{-1}(0.655422) \\ \frac{\ln\frac{60}{60} + \left(0.06 - 0 + \frac{\sigma^2}{2}\right)(4/12 - 0)}{\sigma\sqrt{4/12 - 0}} &= 0.4 \\ \sigma^2 - 0.8\sqrt{3}\sigma + 0.12 &= 0 \\ \sigma &= \frac{0.8\sqrt{3} \pm \sqrt{1.92 - 0.48}}{2} \\ \sigma = 0.092820 \text{ or } \sigma &= 1.292820. \end{aligned}$$

Thus the change in the security price leading to a self-financing portfolio is

$$\pm S(0)\sigma\sqrt{h} = \pm(60)(0.092820)\sqrt{1/52} = \pm\$0.772312.$$

13.4.1. Let Γ_4 denote the Gamma of the four-month call option and Γ_6 denote the Gamma of the six-month call option.

$$\begin{aligned} d_{1,4} &= \frac{\ln\frac{85}{88} + \left(0.055 - 0 + \frac{(0.17)^2}{2}\right)(4/12 - 0)}{0.17\sqrt{4/12 - 0}} = -0.11753 \\ \Gamma_4 &= \frac{e^{-0(4/12-0)}\varphi(-0.11753)}{85(0.17)\sqrt{4/12 - 0}} = 0.0474901 \\ d_{1,6} &= \frac{\ln\frac{85}{88} + \left(0.055 - 0 + \frac{(0.17)^2}{2}\right)(6/12 - 0)}{0.17\sqrt{6/12 - 0}} = 0.000328 \\ \Gamma_6 &= \frac{e^{-0(6/12-0)}\varphi(0.000328)}{85(0.17)\sqrt{6/12 - 0}} = 0.0390443 \end{aligned}$$

The ratio of the Gammas is

$$\frac{\Gamma_4}{\Gamma_6} = \frac{0.047901}{0.0390443} = 1.21632$$

thus a Gamma-neutral portfolio would be long 608.158 of the six-month call option.

13.4.3. Let Γ_3 denote the Gamma of the three-month call option and Γ_5 denote the Gamma of the five-month call option.

$$\begin{aligned} d_{1,3} &= \frac{\ln\frac{95}{97} + \left(0.045 - 0 + \frac{(0.23)^2}{2}\right)(3/12 - 0)}{0.23\sqrt{3/12 - 0}} = -0.0258399 \\ \Gamma_3 &= \frac{e^{-0(3/12-0)}\varphi(-0.0258399)}{95(0.23)\sqrt{3/12 - 0}} = 0.0365043 \\ d_{1,5} &= \frac{\ln\frac{95}{98} + \left(0.045 - 0 + \frac{(0.23)^2}{2}\right)(5/12 - 0)}{0.23\sqrt{5/12 - 0}} = -0.00888936 \\ \Gamma_5 &= \frac{e^{-0(5/12-0)}\varphi(-0.00888936)}{95(0.23)\sqrt{5/12 - 0}} = 0.0282844 \end{aligned}$$

The ratio of the Gammas is

$$\frac{\Gamma_3}{\Gamma_5} = \frac{0.0365043}{0.0282844} = 1.29061$$

thus a Gamma-neutral portfolio would be long 129.061 of the five-month call option.

13.4.5. Let Γ_{62} denote the Gamma of the 62-strike call option and Γ_{65} denote the Gamma of the 65-strike call option.

$$
\begin{aligned}
d_{1,62} &= \frac{\ln\frac{60}{62} + \left(0.0565 - 0.02 + \frac{(0.45)^2}{2}\right)(4/12 - 0)}{0.17\sqrt{4/12 - 0}} = 0.00732301 \\
\Gamma_{62} &= \frac{e^{-0.02(3/12-0)}\varphi\left(0.00732301\right)}{60(0.45)\sqrt{3/12 - 0}} = 0.0294031 \\
d_{1,65} &= \frac{\ln\frac{60}{65} + \left(0.0565 - 0.02 + \frac{(0.45)^2}{2}\right)(3/12 - 0)}{0.45\sqrt{3/12 - 0}} = -0.202690 \\
\Gamma_{6} &= \frac{e^{-0.02(3/12-0)}\varphi\left(-0.202690\right)}{60(0.45)\sqrt{3/12 - 0}} = 0.0288061
\end{aligned}
$$

The ratio of the Gammas is

$$\frac{\Gamma_4}{\Gamma_6} = \frac{0.0294031}{0.0288061} = 1.02073$$

thus a Gamma-neutral portfolio would be long 102.073 of the 65-strike call option.

13.4.7. The price of the 62-strike calls is \$4.72655 and the price of the 65-strike calls is \$3.62213. The time $t = 0$ cash flow to create the Delta/Gamma-neutral portfolio is

$$B = 100(4.72655) - 102.073(3.62213) - 7.41622(60) = -\$342.038,$$

which is borrowed at the risk-free rate. If the hedging portfolio is liquidated after one day, the profit/loss will depend on the stock price the next day (\$60.50), the option prices the next day ($C^e(60.50, 62, (90-1)/365) =$ \$4.94842 and $C^e(60.50, 65, (90-1)/365) = \$3.80323$), and the interest costs of the loan.

$$\text{P/L} = -100(4.94842) + 102.073(3.80323) + 7.41622(60.50) - 342.038e^{0.0565(1/365)} = -\$0.0458565$$

14.1.1. In this case $K_1 = 150$, $K_2 = 160$, $S(0) = 150$, $\sigma = 0.35$, $\delta = 0.02$, $r = 0.05$, $T = 3/12$, and $t = 0$. Using the calculations already performed in the example yields,

$$
\begin{aligned}
d_1 &= -0.238434 \\
d_2 &= -0.413434 \\
P^g &= 150e^{-0.05(3/12-0)}\Phi\left(0.238434\right) - 150e^{-0.02(3/12-0)}\Phi\left(0.413434\right) \\
&= \$9.13328.
\end{aligned}
$$

14.1.5. Starting with Eq. (6.8) and using K_2 in place of the strike price we have,

$$
\begin{aligned}
P^e(S(t); K_2, T) + S(t)e^{-\delta(T-t)} &= C^e(S(t); K_2, T) + K_2 e^{-r(T-t)} \\
P^g(S(t); K_1, K_2, T) + (K_2 - K_1)e^{-r(T-t)}\Phi\left(-d_2\right) + S(t)e^{-\delta(T-t)} &= C^g(S(t); K_1, K_2, T) - (K_2 - K_1)e^{-r(T-t)}\Phi\left(d_2\right) \\
&\quad + (K_1 + K_2 - K_1)e^{-r(T-t)} \\
P^g(S(t); K_1, K_2, T) + S(t)e^{-\delta(T-t)} &= C^g(S(t); K_1, K_2, T) + K_1 e^{-r(T-t)} \\
&\quad - (K_2 - K_1)e^{-r(T-t)}(1 - \Phi\left(d_2\right) - \Phi\left(-d_2\right)) \\
&= C^g(S(t); K_1, K_2, T) + K_1 e^{-r(T-t)}.
\end{aligned}
$$

14.2.5. Using the supplied variables and parameters,

$$
\begin{aligned}
d_1 &= \frac{\ln\frac{10}{3^2} + (0.045 - 0.02 + (\frac{1}{2} - \frac{1}{2})(0.30)^2)(3/12 - 0)}{0.30\sqrt{3/12 - 0}} = 0.74407 \\
d_2 &= \frac{\ln\frac{10}{3^2} + (0.045 - 0.02 + \frac{(0.30)^2}{2})(3/12 - 0)}{0.30\sqrt{3/12 - 0}} = 0.66907
\end{aligned}
$$

Thus the prices of the power call and put are respectively,

$$\begin{aligned}
C^p(10; 1/2, 3/12, 3) &= (10)^{1/2} e^{[(1/2-1)(0.045+(1/2)(0.30)^2/2)-(1/2)(0.02)](3/12-0)} \Phi\left(0.74407\right) - (3)e^{-0.045(3/12-0)} \Phi\left(0.66907\right) \\
&= \$0.193707 \\
P^p(10; 1/2, 3/12, 3) &= (3)e^{-0.045(3/12-0)} \Phi\left(-0.66907\right) - (10)^{1/2} e^{[(1/2-1)(0.045+(1/2)(0.30)^2/2)-(1/2)(0.02)](3/12-0)} \Phi\left(-0.74407\right) \\
&= \$0.0322672
\end{aligned}$$

14.2.9. According to Eq. (14.21) the Gamma of the sold power calls is

$$\begin{aligned}
\Gamma_{C^p} &= \frac{3}{2}(90)^{3/2-2} e^{[(3/2-1)(0.05+(3/2)(0.45)^2/2](6/12-0)} \left((3/2-1)\Phi\left(d_1\right) + \frac{\varphi\left(d_1\right)}{0.45\sqrt{6/12-0}} \right) \\
&= 0.0822199
\end{aligned}$$

The Gamma of an ordinary European call is

$$\Gamma^e = \frac{e^{-0.015(6/12-0)}\varphi\left(d_1\right)}{(0.45)(90)\sqrt{6/12-0}} = 0.013813.$$

The ratio of these Gammas is 0.0822199/0.013813 = 5.95234, thus if 100 power call options are sold, the market maker must purchase 595.234 ordinary European call options in order to create a Gamma neutral portfolio. The Delta of an ordinary European call is

$$\Delta_{C^e} = e^{-0.015(6/12-0)}\Phi\left(d_1\right) = 0.513752.$$

Since 100 power calls are sold and 595.234 ordinary calls are bought, the portfolio can be simultaneously Delta-neutral when the market maker takes a position in z shares of the underlying security, where

$$-100(14.7994) + 595.234(0.513752) + z = 0.$$

This implies the market maker should purchase $z = 1174.14$ shares in order to create the Delta/Gamma-neutral hedging portfolio.

14.3.3. This is an exchange call with underlying asset $S(t)$ and strike asset $K(t)$.

$$\begin{aligned}
\sigma^2 &= (0.40)^2 - 2(0.78)(0.40)(0.35) + (0.35)^2 = 0.0641 \\
\sigma &= 0.25318 \\
d_1 &= \frac{\ln\frac{3(17)}{50} + \left(0.013 - 0.025 + \frac{0.0641}{2}\right)(3/12-0)}{0.25318\sqrt{3/12-0}} = 0.196028 \\
d_2 &= \frac{\ln\frac{3(17)}{50} + \left(0.013 - 0.025 - \frac{0.0641}{2}\right)(3/12-0)}{0.25318\sqrt{3/12-0}} = 0.0694377
\end{aligned}$$

Using these calculated input values Eq. (14.28) yields

$$C^x((3)17, 50; \sigma, 3/12) = (3)(17)e^{-0.025(3/12-0)}\Phi\left(0.196028\right) - 50e^{-0.013(3/12-0)}\Phi\left(0.0694377\right) = \$2.98106.$$

14.3.5. Use the Put-Call parity relationship of Eq. (14.32).

$$\begin{aligned}
C^x(S(0), 4K(0); \sigma, 6/12) - P^x(S(0), 4K(0); \sigma, 6/12) &= S(0)e^{-0.01(6/12-0)} - 4K(0) \\
3 - P^x(S(0), 4K(0); \sigma, 6/12) &= 20e^{-0.01(6/12-0)} - 4(5) \\
P^x(S(0), 4K(0); \sigma, 6/12) &= \$3.09975
\end{aligned}$$

This is the price to exchange one share of security S for four shares of security K. Thus to exchange five shares of security S for twenty shares of security K the premium is (5)(3.09975) = \$15.4988.

14.3.9. The market maker must re-purchase the exchange call options and close the short position in the strike asset. After one day the value of the exchange calls can be calculated as

$$\begin{aligned}
d_1 &= \frac{\ln \frac{4(35.50)}{3(50.25)} + \left(0.02 - 0 + \frac{0.115}{2}\right)(6/12 - 1/365)}{0.339116\sqrt{6/12 - 1/365}} = -0.0888963 \\
d_2 &= \frac{\ln \frac{4(35.50)}{3(50.25)} + \left(0.02 - 0 - \frac{0.115}{2}\right)(6/12 - 1/365)}{0.339116\sqrt{6/12 - 1/365}} = -0.32803 \\
C^x &= \$10.5295.
\end{aligned}$$

Thus the one-day profit/loss is

$$\text{P/L} = 100(9.92541 + (0.353661)(50)) - 100(10.5295) - (35.3661)(50.25) = -\$69.2505.$$

14.3.11. Making use of the hint given,

$$\max\{q_1S_1(T), q_2S_2(T)\} = (q_1S_1(T) - q_2S_2(T))^+ + q_2S_2(T).$$

The time t expected value of the first term on the right-hand side of the equation is an exchange call with underlying asset $q_1S_1(t)$ and strike asset $q_2S_2(t)$. The expected value of the last term on the right-hand side is q_2 prepaid forwards for the strike asset. If $V(q_1S_1(t), q_2S_2(t))$ denotes the option which has payoff, $\max\{q_1S_1(T), q_2S_2(T)\}$, then

$$V(q_1S_1(t), q_2S_2(t)) = C^x(q_1S_1(t), q_2S_2(t); \sigma, T) + q_2F^P_{t,T}(S_2(t)).$$

14.3.13. Note that $x + y = \min\{x, y\} + \max\{x, y\}$ so

$$v(q_1S_1(t), q_2S_2(t)) + V(q_1S_1(t), q_2S_2(t)) = q_1F^P_{t,T}(S_1(t)) + q_2F^P_{t,T}(S_2(t)).$$

14.4.1. Use Eq. (14.37). For the European call option portion of this formula, $S = 100$, $K = 100$, $T = 2$, $t = 0$, $r = 0.05$, $\delta = 0.02$, and $\sigma = 0.30$.

$$\begin{aligned}
d_1 &= \frac{\ln \frac{100}{100} + \left(0.05 - 0.02 + \frac{(0.30)^2}{2}\right)(2 - 0)}{0.30\sqrt{2 - 0}} = 0.353553 \\
d_2 &= \frac{\ln \frac{100}{100} + \left(0.05 - 0.02 - \frac{(0.30)^2}{2}\right)(2 - 0)}{0.30\sqrt{2 - 0}} = -0.0707107 \\
C^e(100; 100, 2) &= 100e^{-0.02(2-0)}\Phi(0.353533) - 100e^{-0.05(2-0)}\Phi(-0.0707107) = \$18.6225
\end{aligned}$$

For the European put option portion of this formula, $S = 100$, the strike price is $100e^{-(0.05-0.02)(2-1)} = 97.0446$, expiry is at 1, $t = 0$, $r = 0.05$, $\delta = 0.02$, and $\sigma = 0.30$.

$$\begin{aligned}
d_1 &= \frac{\ln \frac{100}{97.0446} + \left(0.05 - 0.02 + \frac{(0.30)^2}{2}\right)(1 - 0)}{0.30\sqrt{1 - 0}} = 0.35 \\
d_2 &= \frac{\ln \frac{100}{97.0446} + \left(0.05 - 0.02 - \frac{(0.30)^2}{2}\right)(1 - 0)}{0.30\sqrt{1 - 0}} = 0.05 \\
P^e(100; 97.0446, 1) &= 97.0446e^{-0.05(1-0)}\Phi(-0.05) - 100e^{-0.02(1-0)}\Phi(-0.35) = \$8.71742
\end{aligned}$$

Thus the value of the chooser option at $t = 0$ is

$$V(100; 0) = 18.6225 + e^{-0.02(2-1)}8.71742 = \$27.1642.$$

14.4.3. A straddle consists of a long put and long call with the same strike price. If $S = 100$, $K = 100$, $T = 2$, $t = 0$, $r = 0.05$, $\delta = 0.02$, and $\sigma = 0.30$, then

$$\begin{aligned} C^e(100; 100, 2) &= \$18.6225 \\ P^e(100; 100, 2) &= \$13.0273. \end{aligned}$$

The total cost of the straddle is $18.6225 + 13.0273 = \$31.6499$ which is greater than the price of the chooser option at \$27.1674.

14.4.7. Using the result found in Exercise 14.4.5 and taking the partial derivative with respect to t produce,

$$\begin{aligned} \frac{\partial V}{\partial t} &= \Theta_{C^e} + rKe^{-r(T-t)}\Phi\left(-d_2^{t_1}\right) - \delta Se^{-\delta(T-t)}\Phi\left(-d_1^{t_1}\right) - Ke^{-r(T-t)}\varphi\left(d_2^{t_1}\right)\frac{\partial d_2^{t_1}}{\partial t} + Se^{-\delta(T-t)}\varphi\left(-d_1^{t_1}\right)\frac{\partial d_1^{t_1}}{\partial t} \\ &= \Theta_{C^e} + rKe^{-r(T-t)}\Phi\left(-d_2^{t_1}\right) - \delta Se^{-\delta(T-t)}\Phi\left(-d_1^{t_1}\right) - Se^{-\delta(T-t)}\varphi\left(d_1^{t_1}\right)\left(\frac{\partial d_2^{t_1}}{\partial t} - \frac{\partial d_1^{t_1}}{\partial t}\right) \\ &= \Theta_{C^e} + rKe^{-r(T-t)}\Phi\left(-d_2^{t_1}\right) - \delta Se^{-\delta(T-t)}\Phi\left(-d_1^{t_1}\right) - \frac{\sigma Se^{-\delta(T-t)}}{2\sqrt{t_1 - t}}\varphi\left(d_1^{t_1}\right) \\ &= \Theta_{C^e} + rKe^{-r(T-t)}\Phi\left(-d_2^{t_1}\right) - Se^{-\delta(T-t)}\left(\delta\Phi\left(-d_1^{t_1}\right) + \frac{\sigma\varphi\left(d_1^{t_1}\right)}{2\sqrt{t_1 - t}}\right). \end{aligned}$$

14.4.9. Using the result found in Exercise 14.4.5 and taking the partial derivative with respect to σ produce,

$$\begin{aligned} \frac{\partial V}{\partial \sigma} &= \mathcal{V}_{C^e} - Ke^{-r(T-t)}\varphi\left(d_2^{t_1}\right)\frac{\partial d_2^{t_1}}{\partial \sigma} + Se^{-\delta(T-t)}\varphi\left(d_1^{t_1}\right)\frac{\partial d_1^{t_1}}{\partial \sigma} \\ &= \mathcal{V}_{C^e} - Se^{-\delta(T-t)}\varphi\left(d_1^{t_1}\right)\left(\frac{\partial d_2^{t_1}}{\partial \sigma} - \frac{\partial d_1^{t_1}}{\partial \sigma}\right) \\ &= \mathcal{V}_{C^e} + Se^{-\delta(T-t)}\varphi\left(d_1^{t_1}\right)\sqrt{t_1 - t}. \end{aligned}$$

14.5.3.

$$P^f = 150e^{-0.03(1)}(1.05e^{-0.035(2)}\Phi(0.351415) - e^{-0.03(2)}\Phi(-0.214271)) = \$33.9163$$

14.6.1. The security price is determined by Newton's method. Suppose the $S_0 = 105$ is the initial approximation to the solution of the equation, $P^e(S, 105, 1) = 10$. The Newton's iterate formula is

$$S_n = S_{n-1} - \frac{P^e(S_{n-1}, 105, 1) - 10}{\Delta_{P^e}}$$

for $n = 1, 2, \ldots$. Evaluating the first several iterates produces the estimates in the following table.

n	S_n
0	105.000
1	109.295
2	109.574
3	109.575
4	109.575

The iterates appear to have converged to $S^* \approx \$109.575$.

14.6.3. Numerically approximating the solution S^* to the equation $P^e(S^*, 750, 9/12) = 75$ produces $S^* \approx$

\$735.97. The correlation $\rho = \sqrt{1/3}$. Hence,

$$\begin{aligned}
a_1 &= \frac{\ln \frac{775}{735.97} + \left(0.05 - 0.03 + \frac{(0.35)^2}{2}\right)\left(\frac{3}{12} - 0\right)}{0.35\sqrt{\frac{3}{12} - 0}} = 0.411351\\
a_2 &= \frac{\ln \frac{775}{735.97} + \left(0.05 - 0.03 - \frac{(0.35)^2}{2}\right)\left(\frac{3}{12} - 0\right)}{0.35\sqrt{\frac{3}{12} - 0}} = 0.236351\\
d_1 &= \frac{\ln \frac{775}{750} + \left(0.05 - 0.03 + \frac{(0.35)^2}{2}\right)(9/12 - 0)}{0.35\sqrt{1 - 0}} = 0.30922\\
d_2 &= \frac{\ln \frac{775}{750} + \left(0.05 - 0.03 - \frac{(0.35)^2}{2}\right)(9/12 - 0)}{0.35\sqrt{1 - 0}} = 0.00611108\\
\boldsymbol{\Phi}\left(-a_1, -d_1; \rho\right) &= 0.21701\\
\boldsymbol{\Phi}\left(-a_2, -d_2; \rho\right) &= 0.297211\\
\Phi\left(-a_2\right) &= 0.40658.
\end{aligned}$$

Thus the premium for this call on a put is

$$\begin{aligned}
C^p &= 750e^{-0.05(9/12-0)}(0.297211) - 775e^{-0.03(9/12-0)}(0.21701) - 75e^{-0.05(3/12-0)}(0.40658)\\
&= \$20.1479.
\end{aligned}$$

14.6.5. Numerically approximating the solution S^* to the equation $C^e(S^*, 225, 18/12) = 15$ produces $S^* \approx \$212.457$. The correlation $\rho = \sqrt{2/3}$. Hence,

$$\begin{aligned}
a_1 &= \frac{\ln \frac{230}{212.457} + \left(0.05 - 0.035 + \frac{(0.33)^2}{2}\right)(1 - 0)}{0.33\sqrt{1 - 0}} = 0.450874\\
a_2 &= \frac{\ln \frac{230}{212.457} + \left(0.05 - 0.035 - \frac{(0.33)^2}{2}\right)(1 - 0)}{0.33\sqrt{1 - 0}} = 0.120874\\
d_1 &= \frac{\ln \frac{230}{225} + \left(0.05 - 0.035 + \frac{(0.33)^2}{2}\right)(18/12 - 0)}{0.33\sqrt{18/12 - 0}} = 0.312134\\
d_2 &= \frac{\ln \frac{230}{225} + \left(0.05 - 0.035 - \frac{(0.33)^2}{2}\right)(18/12 - 0)}{0.33\sqrt{18/12 - 0}} = -0.0920318\\
\boldsymbol{\Phi}\left(-a_1, d_1; -\rho\right) &= 0.0673953\\
\boldsymbol{\Phi}\left(-a_2, d_2; -\rho\right) &= 0.0611677\\
\Phi\left(-a_2\right) &= 0.451895.
\end{aligned}$$

Thus the premium for this put on a call is

$$\begin{aligned}
P^c &= 225e^{-0.05(18/12-0)}(0.0611677) - 230e^{-0.035(18/12-0)}(0.0673953) - 15e^{-0.05(1-0)}(0.451895)\\
&= \$4.50801.
\end{aligned}$$

14.6.7. Use the Put-Call parity relationship between a call on a put and a put on a put from Eq. (14.55). The time $t = 0$ cost of the underlying put is $P^e(775, 750, 9/12) = \$72.5672$. Using Eq. (14.55),

$$P^p = 20.1479 + 75e^{-0.05(3/12-0)} - 72.5672 = \$21.649.$$

14.7.5. First calculate the dividend yield and volatility for the geometric mean.

$$\begin{aligned}\delta^* &= \frac{1}{2}\left[0.06\left(\frac{12-1}{12}\right) + \left(0.01 + \frac{(0.35)^2}{2}\right)\frac{12+1}{12} - \frac{(0.35)^2(12+1)(2(12)+1)}{6(12)^2}\right] = 0.0430541 \\ \sigma^* &= \frac{0.35}{12}\sqrt{\frac{(12+1)(2(12)+1)}{6}} = 0.214661\end{aligned}$$

Next use Eqs. (10.32) and (10.33) and Eq. (10.35) with $T = 1$ and $K = 115$.

$$\begin{aligned}d_1 &= \frac{\ln\frac{120}{115} + \left(0.06 - 0.0430541 + \frac{(0.214661)^2}{2}\right)(1-0)}{0.214661\sqrt{1-0}} = 0.384537 \\ d_2 &= \frac{\ln\frac{120}{115} + \left(0.06 - 0.0430541 - \frac{(0.214661)^2}{2}\right)(1-0)}{0.214661\sqrt{1-0}} = 0.169877 \\ P &= 115e^{-0.06(1-0)}\Phi\left(-0.169877\right) - 120e^{-0.0430541(1-0)}\Phi\left(-0.384537\right) \\ &= \$6.58337\end{aligned}$$

14.7.7.

$$\begin{aligned}\lim_{N\to\infty} \delta^* &= \lim_{N\to\infty} \delta^* = \frac{1}{2}\left[r\left(\frac{N-1}{N}\right) + \left(\delta + \frac{\sigma^2}{2}\right)\frac{N+1}{N} - \frac{\sigma^2(N+1)(2N+1)}{6N^2}\right] \\ &= \frac{1}{2}\left[r + \delta + \frac{\sigma^2}{2} - \frac{\sigma^2}{3}\right] \\ &= \frac{1}{2}\left(r + \delta + \frac{\sigma^2}{6}\right) \\ \lim_{N\to\infty} \sigma^* &= \lim_{N\to\infty} \sqrt{\frac{\sigma^2(N+1)(2N+1)}{6N^2}} \\ &= \frac{\sigma}{\sqrt{3}}\end{aligned}$$

For the situation described in Example 14.12,

$$\begin{aligned}\delta^* &= \frac{1}{2}\left(0.05 + 0.02 + \frac{(0.25)^2}{2}\right) = 0.0402083 \\ \sigma^* &= \frac{0.25}{\sqrt{3}} = 0.144338 \\ d_1 &= \frac{\ln\frac{100}{100} + \left(0.05 - 0.0402083 + \frac{(0.144338)^2}{2}\right)(1-0)}{0.144338\sqrt{1-0}} = 0.140007 \\ d_2 &= \frac{\ln\frac{100}{100} + \left(0.05 - 0.0402083 - \frac{(0.144338)^2}{2}\right)(1-0)}{0.144338\sqrt{1-0}} = -0.00433013 \\ C^{ap,G} &= 100e^{-0.0402083(1-0)}\Phi\left(0.140007\right) - 100e^{-0.05(1-0)}\Phi\left(-0.00433013\right) = \$5.9802.\end{aligned}$$

14.7.9. Using the values of δ^*, σ^{**}, d_1, and d_2 calculated in Example 14.13 generates a price for the corresponding geometric average strike call option of

$$\begin{aligned}C^{as,G} &= 100e^{-0.02(1-0)}\Phi\left(0.207512\right) - 100e^{-0.0389222(1-0)}\Phi\left(0.0722294\right) \\ &= \$6.20629.\end{aligned}$$

The prepaid forward price of the underlying stock is $F^P_{0,T}(S) = 100e^{-0.02(1)} = \$98.0199$ while the prepaid forward price of the geometric average of the stock prices is

$$F^P_{0,T}(G(T)) = 100e^{-0.0389222(1)} = 96.1826.$$

Substituting these values into the Put-Call parity relationship of Eq. (14.63) results in the following.

$$\begin{aligned} 6.20629 - 4.36898 &= 98.0199 - 96.1826 \\ 1.83731 &= 1.83731 \end{aligned}$$

Consequently the put and call values satisfy the Put-Call parity relationship for geometric average strike Asian options.

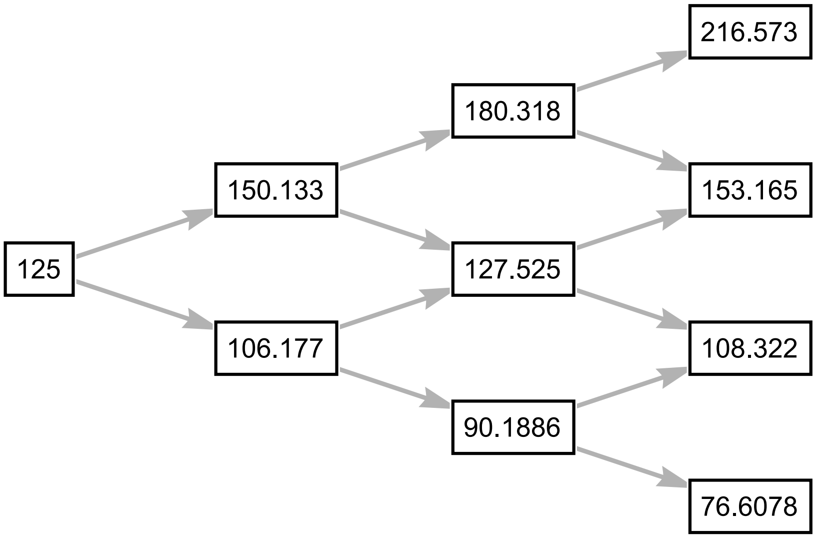
14.7.11. The factors by which the stock price increases and decreases at each time step are respectively (according to the formulas derived in Exercise 7.3.1),

$$\begin{aligned} u &= e^{(0.05-0.02)(4/12)+0.30\sqrt{4/12}} = 1.20106 \\ d &= e^{(0.05-0.02)(4/12)-0.30\sqrt{4/12}} = 0.849417. \end{aligned}$$

The risk-neutral probability of an increase in the stock price is

$$p^* = \frac{1}{1+e^{0.30\sqrt{4/12}}} = 0.456807.$$

The evolution of stock prices is shown in the following graph.



Using the graph of stock prices, the arithmetic average price $A(T)$, the call payoff, and the risk-neutral probability of the payoff are summarized in the following table.

Path	$A(T)$	$(A(T)-S(T))^+$	**Prob.**
u^3	182.341	0.000	0.095323
u^2d	161.205	8.040	0.113349
$u\,d\,u$	143.608	0.000	0.113349
$d\,u\,u$	128.956	20.634	0.113349
$u\,d^2$	128.660	0.000	0.134785
$d\,u\,d$	114.008	5.686	0.134785
d^2u	101.563	0.000	0.134785
d^3	90.991	14.383	0.160274

Therefore the present value of the risk-neutral expected value of the payoff is the value of the arithmetic average strike Asian put,

$$\begin{aligned} P^{as,A} &= e^{-0.05}\left[0 + 0.113349(8.040 + 0 + 20.634) + 0.134785(0 + 5.686 + 0) + 0.160274(14.383)\right] \\ &= \$6.43426. \end{aligned}$$

14.8.1.

(a) 0
(b) $P^e(S, K, T)$
(c) 0
(d) $C^e(S, K, T)$

14.8.13. Using the values $S(0) = 250$, $K = 260$, $H = 275$, $T = 4/12$, $r = 0.065$, $\sigma = 0.40$, and $\delta = 0.015$ then

$$\begin{array}{lll} d_3 = 0.843219 & d_5 = -0.225066 & d_7 = 0.600344 \\ d_4 = 0.612278 & d_6 = -0.456006 & d_8 = 0.369404 \end{array}$$

and $\left(\dfrac{H}{S}\right)^{\frac{2(r-\delta)}{\sigma^2}-1} = 0.96489$ and $\left(\dfrac{H}{S}\right)^{\frac{2(r-\delta)}{\sigma^2}+1} = 1.16752$. Making use of Eq. (14.94) then

$$C_{ui}^e = \$20.3586.$$