Composition of Functions and Inverse Functions

MATH 101 *College Algebra*

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Objectives

In this lesson we will learn to:

- form the **composition** of two functions,
- determine if a function is one-to-one by using the horizontal line test,
- show that two functions are **inverses** by verifying that $f(g(x)) = g(f(x)) = x$,
- find the **inverse** of a one-to-one function, and
- graph the inverses of functions, by **reflecting** the graphs of the functions across the line $y = x$.

Composite Function

Definition

For two functions f and g, the **composite function** denoted $f \circ g$ is defined as

$$(f \circ g)(x) = f(g(x)).$$

The domain of $f \circ g$ consists of those values of x in the domain of g for which $g(x)$ is in the domain of f.

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Remark: x is plugged into g to form $g(x)$ and then $g(x)$ is plugged into f to form $f(g(x))$.

Example (1 of 2)

Suppose $f(x) = \dfrac{1}{x+1}$ and $g(x) = x^2 + x - 3$ and find and simplify the following composite functions. What are their domains?

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$$\begin{aligned} f(g(x)) &= \frac{1}{g(x)+1} \\ &= \frac{1}{x^2 + x - 3 + 1} \\ &= \frac{1}{x^2 + x - 2} \\ &= \frac{1}{(x-1)(x+2)} \end{aligned}$$

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Domain of $f(g(x))$ is $(-\infty, -2) \cup (-2, 1) \cup (1, \infty)$.

Example (2 of 2)

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$$\begin{aligned} g(f(x)) &= (f(x))^2 + f(x) - 3 \\ &= \left(\frac{1}{x+1}\right)^2 + \frac{1}{x+1} - 3 \\ &= \frac{1}{(x+1)^2} + \frac{x+1}{(x+1)^2} - \frac{3(x+1)^2}{(x+1)^2} \\ &= \frac{1 + x + 1 - 3x^2 - 6x - 3}{(x+1)^2} \\ &= \frac{-3x^2 - 5x - 1}{(x+1)^2} \end{aligned}$$

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One-to-One Functions

Recall: for a function there is only one y value assigned to each x value in the domain. Graphically this can be verified using the **vertical line test**.

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A function is a **one-to-one function** (or **1–1 function**) if for each value of *y* in the range there is only one corresponding value of *x* in the domain.

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Theorem

*A function is one-to-one if no **horizontal line** intersects the graph of the function at more than one point.*

Inverse Functions

Definition

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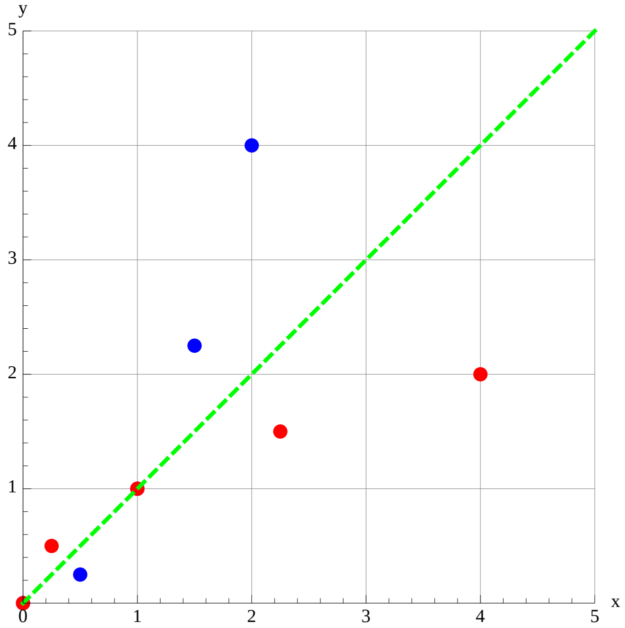
Remark: the superscript $^{-1}$ in the notation f^{-1} is part of the notation or naming of inverse functions. The $^{-1}$ is **not** an exponent.

$$f^{-1}(x) \neq \frac{1}{f(x)}$$

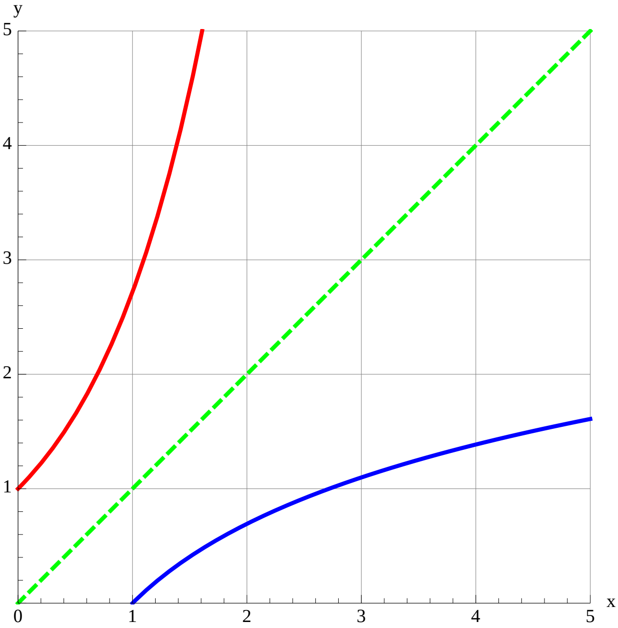
Graphs of Inverse Functions

- If f is one-to-one, then it has an inverse function f^{-1}.
- Thus if (x, y) is a point on the graph of $y = f(x)$, then (y, x) is a point on the graph of $x = f^{-1}(y)$.
- The graphs of f and f^{-1} are reflections of each other across the line $y = x$.

Example (1 of 2)



Example (2 of 2)



Properties of Inverse Functions

Theorem

If f and g are one-to-one functions and

$$\begin{aligned} f(g(x)) &= x \quad \text{for all } x \text{ in } D_g\text{, and} \\ g(f(x)) &= x \quad \text{for all } x \text{ in } D_f\text{,} \end{aligned}$$

*then f and g are **inverse functions**. We may write $g = f^{-1}$ and $f = g^{-1}$.*

Example

Show that $f(x) = -2x + 3$ and $g(x) = \frac{3-x}{2}$ are inverses of each other.

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$$
\begin{aligned}
f(g(x)) &= -2g(x) + 3 = -2\left(\frac{3-x}{2}\right) + 3 \\
&= -(3-x) + 3 = x - 3 + 3 \\
&= x \\
g(f(x)) &= \frac{3 - f(x)}{2} = \frac{3 - (-2x+3)}{2} \\
&= \frac{3 + 2x - 3}{2} = \frac{2x}{2} \\
&= x
\end{aligned}
$$

Finding Function Inverses

Steps: given a function $f(x)$,

1. let $y = f(x)$,
2. interchange y and x (that is, write $x = f(y)$),
3. in the new equation, solve for y in terms of x,
4. substitute $f^{-1}(x)$ for y.

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y &= 3x - 1 \\
x &= 3y - 1 \\
x + 1 &= 3y \\
y &= \frac{x+1}{3} \\
f^{-1}(x) &= \frac{x+1}{3}
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Example

Find the inverse of $f(x) = 3x - 1$.

$$y = 3x - 1$$
$$x = 3y - 1$$
$$x + 1 = 3y$$
$$y = \frac{x+1}{3}$$
$$f^{-1}(x) = \frac{x+1}{3}$$