Composition of Functions and Inverse Functions
MATH 101 College Algebra

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Objectives

In this lesson we will learn to:

- form the **composition** of two functions,
- determine if a function is one-to-one by using the horizontal line test,
- show that two functions are **inverses** by verifying that $f(g(x)) = g(f(x)) = x$,
- find the **inverse** of a one-to-one function, and
- graph the inverses of functions, by **reflecting** the graphs of the functions across the line $y = x$. 
Composite Function

Definition
For two functions \( f \) and \( g \), the **composite function** denoted \( f \circ g \) is defined as

\[
(f \circ g)(x) = f(g(x)).
\]

The domain of \( f \circ g \) consists of those values of \( x \) in the domain of \( g \) for which \( g(x) \) is in the domain of \( f \).
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$$(f \circ g)(x) = f(g(x)).$$

The domain of $f \circ g$ consists of those values of $x$ in the domain of $g$ for which $g(x)$ is in the domain of $f$.

Remark: $x$ is plugged into $g$ to form $g(x)$ and then $g(x)$ is plugged into $f$ to form $f(g(x))$. 
Example (1 of 2)

Suppose \(f(x) = \frac{1}{x + 1}\) and \(g(x) = x^2 + x - 3\) and find and simplify the following composite functions. What are their domains?

\[
f(g(x)) =
\]

\[
= \]

Domain of \(f(g(x))\) is \((-\infty, -2) \cup (-2, 1) \cup (1, \infty)\).
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Suppose \( f(x) = \frac{1}{x + 1} \) and \( g(x) = x^2 + x - 3 \) and find and simplify the following composite functions. What are their domains?

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\begin{align*}
  f(g(x)) &= \frac{1}{g(x) + 1} \\
  &= \frac{1}{x^2 + x - 3 + 1} \\
  &= \frac{1}{x^2 + x - 2} \\
  &= \frac{1}{(x - 1)(x + 2)}
\end{align*}
\]

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Example (2 of 2)

\[ g(f(x)) = \]

Domain of \( g(f(x)) \) is \( (-\infty, -1) \cup (-1, \infty) \).
Example (2 of 2)

\[ g(f(x)) = (f(x))^2 + f(x) - 3 \]

\[ = \left( \frac{1}{x + 1} \right)^2 + \frac{1}{x + 1} - 3 \]

\[ = \frac{1}{(x + 1)^2} + \frac{x + 1}{(x + 1)^2} - \frac{3(x + 1)^2}{(x + 1)^2} \]

\[ = \frac{1 + x + 1 - 3x^2 - 6x - 3}{(x + 1)^2} \]

\[ = \frac{-3x^2 - 5x - 1}{(x + 1)^2} \]
Example (2 of 2)

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Domain of \(g(f(x))\) is \((-\infty, -1) \cup (-1, \infty)\).
One-to-One Functions

**Recall:** for a function there is only one $y$ value assigned to each $x$ value in the domain. Graphically this can be verified using the **vertical line test**.
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**Definition**
A function is a **one-to-one function** (or **1–1 function**) if for each value of $y$ in the range there is only one corresponding value of $x$ in the domain.
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A function is a **one-to-one function** (or 1–1 function) if for each value of $y$ in the range there is only one corresponding value of $x$ in the domain.

**Theorem**
A function is one-to-one if no **horizontal line** intersects the graph of the function at more than one point.
Inverse Functions

Definition
If $f$ is a one-to-one function with ordered pairs of the form $(x, y)$, then its inverse function, denoted $f^{-1}$, is also a one-to-one function with ordered pairs of the form $(y, x)$. 

Remark: the superscript $^{-1}$ in the notation $f^{-1}$ is part of the notation or naming of inverse functions. The $^{-1}$ is not an exponent.
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\[
f^{-1}(x) \neq \frac{1}{f(x)}\]
Graphs of Inverse Functions

- If \( f \) is one-to-one, then it has an inverse function \( f^{-1} \).
- Thus if \((x, y)\) is a point on the graph of \( y = f(x) \), then \((y, x)\) is a point on the graph of \( x = f^{-1}(y) \).
- The graphs of \( f \) and \( f^{-1} \) are reflections of each other across the line \( y = x \).
Example (1 of 2)
Properties of Inverse Functions

**Theorem**
If $f$ and $g$ are one-to-one functions and

\[ f(g(x)) = x \quad \text{for all } x \text{ in } D_g, \text{ and} \]
\[ g(f(x)) = x \quad \text{for all } x \text{ in } D_f, \]

then $f$ and $g$ are **inverse functions**. We may write $g = f^{-1}$ and $f = g^{-1}$. 
Example

Show that \( f(x) = -2x + 3 \) and \( g(x) = \frac{3 - x}{2} \) are inverses of each other.
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\[
f(g(x)) = -2g(x) + 3 = -2 \left( \frac{3 - x}{2} \right) + 3
\]
\[
= -(3 - x) + 3 = x - 3 + 3
\]
\[
= x
\]

\[
g(f(x)) = \frac{3 - f(x)}{2} = \frac{3 - (-2x + 3)}{2}
\]
\[
= \frac{3 + 2x - 3}{2} = \frac{2x}{2}
\]
\[
= x
\]
Steps: given a function $f(x)$,
1. let $y = f(x),$
2. interchange $y$ and $x$ (that is, write $x = f(y))$,
3. in the new equation, solve for $y$ in terms of $x,$
4. substitute $f^{-1}(x)$ for $y.$
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x &= 3y - 1 \\
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f^{-1}(x) &= \frac{x + 1}{3}
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