Solving Quadratic Equations

MATH 101 *College Algebra*

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Objectives

In this lesson we will learn to:

- solve quadratic equations by factoring,
- solve quadratic equations using the definition of the square root,
- solve quadratic equations by completing the square, and
- find polynomials with given roots.

Review

Theorem (Zero-Factor Property)

If a product equals 0*, then at least one of the factors must be* 0*. For real numbers a and b,* ***if*** $a \cdot b = 0$ ***then*** $a = 0$ ***or*** $b = 0$ ***or both****.*

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Definition

Quadratic equations are equations of the form

$$ax^2 + bx + c = 0$$

where a, b, and c are constants and $a \neq 0$.

Solving Quadratic Equations by Factoring

Steps:

1. Add or subtract terms so that one side of the equation equals 0.
2. Factor the polynomial expression.
3. Set each factor equals to 0 and solve for the unknown.

Remark: if two of the factors are the same, then the solution is said to be a **double root** or a **root of multiplicity two**.

Example

Solve the following equation.

$$7x^2 = 11x + 6$$

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$$\begin{aligned} 7x^2 &= 11x + 6 \\ 7x^2 - 11x - 6 &= 0 \end{aligned}$$

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Solve the following equation.

$$\begin{aligned} 7x^2 &= 11x + 6 \\ 7x^2 - 11x - 6 &= 0 \\ (7x+3)(x-2) &= 0 \\ 7x + 3 &= 0 \text{or} & x - 2 &= 0 \\ x &= -\frac{3}{7} \text{or} & x &= 2 \end{aligned}$$

Solving Quadratic Equations Using Square Roots

Theorem (Square Root Property)

If $x^2 = c$, *then* $x = \pm\sqrt{c}$.

If $(x-a)^2 = c$, *then* $x - a = \pm\sqrt{c}$ *or equivalently* $x = a \pm \sqrt{c}$.

If $c < 0$ *the solutions will be non-real numbers.*

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If $c < 0$ the solutions will be non-real numbers.

Example

$$\begin{aligned}(x-3)^2 &= 7\\ x - 3 &= \pm\sqrt{7}\\ x &= 3 \pm \sqrt{7}\end{aligned}$$

Completing the Square

Recall the perfect square trinomials:

$$(x+a)^2 = x^2 + 2ax + a^2$$
$$(x-a)^2 = x^2 - 2ax + a^2$$

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Answer: if the leading coefficient is 1, take half of the coefficient of the linear term, square it, and add to the trinomial.

Example

Add the correct constant to complete the square and then factor the trinomial.

$$x^2 + 14x +$$

$$x^2 - 9x +$$

$$x^2 + \frac{1}{3}x +$$

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$$x^2 + 14x + 49 = (x + 7)^2$$

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$$x^2 + \frac{1}{3}x + \frac{1}{36} = \left(x + \frac{1}{6}\right)^2$$

Solving a Quadratic Equation by Completing the Square

Steps:

1. If necessary, multiply or divide both sides of the equation so that the leading coefficient (the coefficient of x^2) is 1.
2. If necessary, isolate the constant term on one side of the equation.
3. Find the constant that completes the square of the polynomial and add this constant to both sides of the equation. Rewrite the polynomial as the square of a binomial.
4. Use the square root property to find the solutions to the equation.

Example

Solving the following equation by completing the square.

$$x^2 - 10x + 3 = 0$$

Example

Solving the following equation by completing the square.

$$\begin{aligned} x^2 - 10x + 3 &= 0 \\ x^2 - 10x &= -3 \\ x^2 - 10x + 25 &= -3 + 25 \\ (x-5)^2 &= 22 \\ x - 5 &= \pm\sqrt{22} \\ x &= 5 \pm \sqrt{22} \end{aligned}$$

Equations With Known Roots

Recall that if $x = a$ and $x = b$ are the roots of a quadratic equation then the equation factors as

$$(x - a)(x - b) = 0$$

which implies the original equation is

$$x^2 - (a + b)x + ab = 0.$$

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Example

Suppose $x = 2 + 3i$ and $x = 2 - 3i$ are the roots of a quadratic equation, then the equation can be expressed as

$$\begin{aligned}
0 &= (x - (2 + 3i))(x - (2 - 3i)) \\
&= (x - 2 - 3i)(x - 2 + 3i) \\
&= (x - 2)^2 + 9 \\
&= x^2 - 4x + 4 + 9 \\
&= x^2 - 4x + 13.
\end{aligned}$$