Counting Techniques

MATH 130, *Elements of Statistics I*

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Fundamental Principle of Counting

Being able to count outcomes of an experiment is key to assigning probabilities to events.

Theorem (Multiplication Rule of Counting)

If a task consists of a sequence of choices in which there are p selections for the first choice, q selections for the second choice, r selections for the third choice, and so on, then the task of making these selections can be done in

$$p \cdot q \cdot r \cdots$$

different ways.

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$$26 \cdot 26 \cdot 26 \cdot 10 \cdot 10 \cdot 10 \cdot 10 = 175,760,000$$

Factorial

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Definition

If $n \geq 0$ is a whole number the quantity $n!$ read "n factorial" is

$$n! = n(n-1)(n-2)\cdots(3)(2)(1).$$

By convention $0! = 1$.

Example

Fill in the second column of the following table.

n	$n!$
1	
2	
3	
7	
10	
12	

Example

Fill in the second column of the following table.

n	$n!$
1	1
2	2
3	6
7	5,040
10	3,628,800
12	479,001,600

Permutations

Definition

A **permutation** is an ordered arrangement in which r objects are chosen from n different objects and repetition is not allowed. The symbol ${}_nP_r$ represents the number of permutations of r objects selected from n objects.

$$ {}_nP_r = \frac{n!}{(n-r)!} $$

Example

Evaluate the following permutations.

- ${}_7P_4$
- ${}_7P_3$
- ${}_{10}P_6$
- ${}_6P_6$

Example

Evaluate the following permutations.

- ${}_7P_4 = 840$
- ${}_7P_3 = 210$
- ${}_{10}P_6 = 151,200$
- ${}_6P_6 = 720$

Example

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Since finishing order matters, we should calculate

$$ {}_8P_3 = \frac{8!}{(8-3)!} = \frac{8!}{5!} = 336. $$

Combinations

Definition

A **combination** is a collection, without regard for order, of n different objects without repetition. The symbol ${}_nC_r$ represents the number of combinations of n different objects taken r at a time.

$$ {}_nC_r = \frac{n!}{r!(n-r)!} $$

Example

Evaluate the following combinations.

- ${}_{7}C_{4}$
- ${}_{7}C_{3}$
- ${}_{10}C_{6}$
- ${}_{6}C_{6}$

Example

Evaluate the following combinations.

- ${}_7C_4 = 35$
- ${}_7C_3 = 35$
- ${}_{10}C_6 = 210$
- ${}_6C_6 = 1$

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A committee of five people must be chosen from a group of fifteen people. How many different committees can be formed?

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Since order of being picked for the committee does not matter we calculate

$$_{15}C_5 = \frac{15!}{5!(15-5)!} = \frac{15!}{5!\,10!} = 3,003.$$

Permutations with Non-distinct Items

Sometimes we must arrange objects in some order but the objects are not all different.

Theorem (Permutations with Non-distinct Items)

The number of permutations of n objects of which n_1 are of one kind, n_2 are of a second kind, ..., and n_k are of a kth kind is given by

$$\frac{n!}{n_1! \cdot n_2! \cdots n_k!}.$$

Example

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We have the situation of considering the arrangements of identical items. The number of arrangements is then

$$\frac{15!}{8!\,4!\,3!} = 225,225.$$

Mississippi

How many different arrangements of the letters in the name "Mississippi" are possible?

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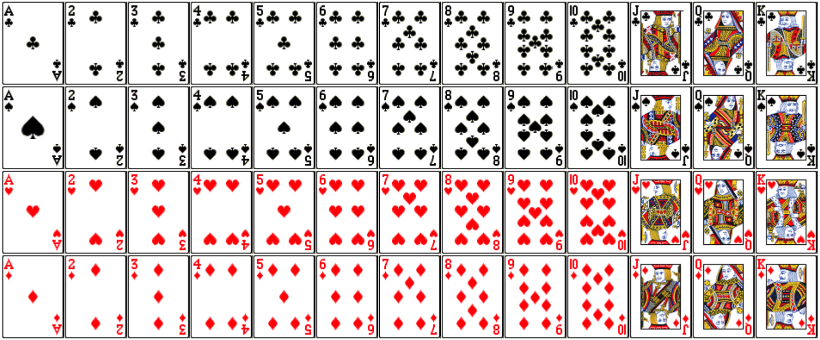
Letter	Frequency
i	4
M	1
p	2
s	4

$$\frac{11!}{4!\,1!\,2!\,4!} = 34,650$$

5-Card Poker Hands

How many 5-card poker hands can be dealt from a regular playing deck?



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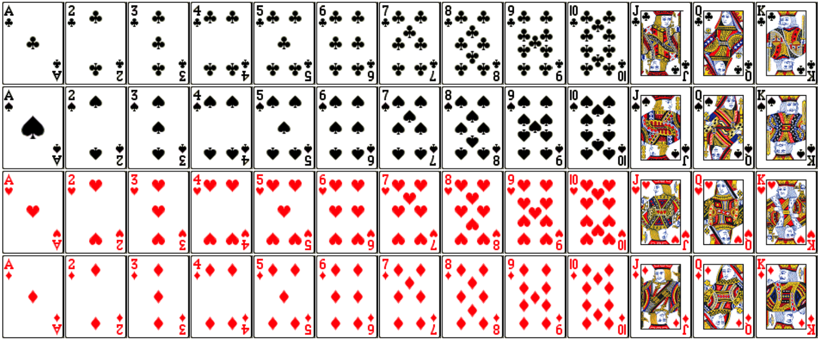
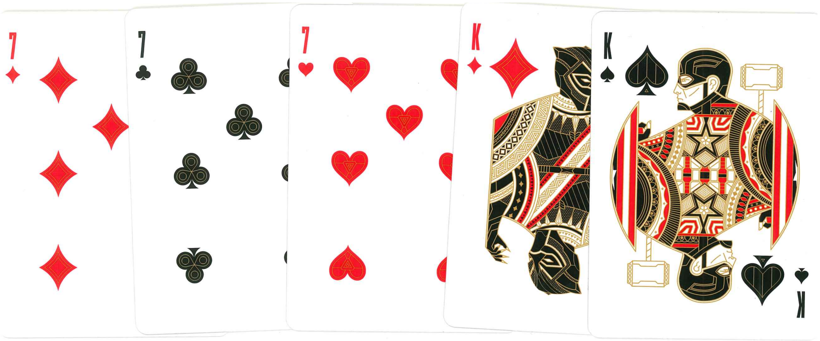


$$_{52}C_5 = \frac{52!}{5!(52-5)!} = \frac{(52)(51)(50)(49)(48)}{120} = 2,598,960$$

Full House Poker Hand



Probability of Full House

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- ${}_{13}C_2$: number of ways to choose the two denominations
- ${}_{4}C_3$: number of ways to choose 3 cards of the same denomination
- ${}_{4}C_2$: number of ways to choose 2 cards of the same denomination

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$$\text{full house hands} = 2\left({}_{13}C_2\right)\left({}_4C_3\right)\left({}_4C_2\right) = 2(78)(4)(6) = 3,744$$

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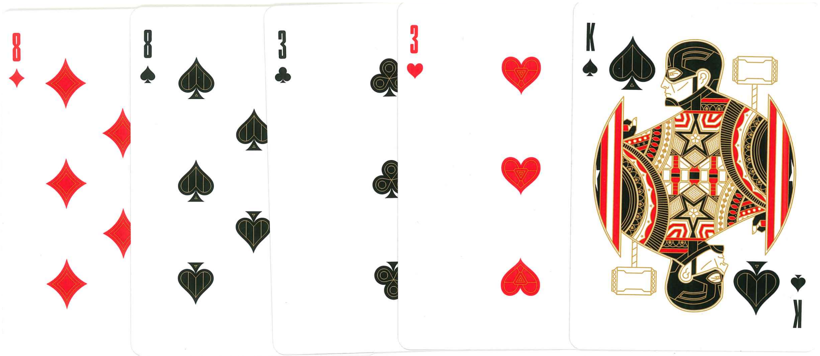
$$P(\text{full house}) = \frac{3,744}{2,598,960} \approx 0.00144$$

Two-pair Poker Hand



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$$P(\text{two pair}) = \frac{123,552}{2,598,960} \approx 0.0475$$

Three of a Kind Poker Hand

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$$\begin{aligned}\text{three kind hands} &= ({}_{13}C_1)\,({}_4C_3)\,({}_{12}C_2)\,({}_4C_1)\,({}_4C_1)\\ &= (13)(4)(66)(4)(4)\\ &= 54,912\end{aligned}$$

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$$P(\text{three kind}) = \frac{54,912}{2,598,960} \approx 0.0211$$