Arc Length

MATH 211, *Calculus II*

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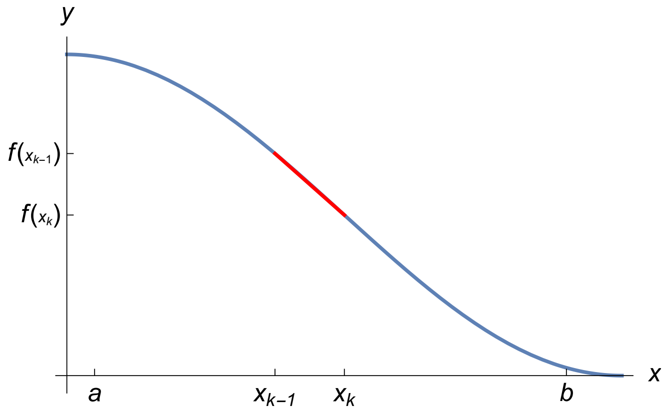
Overview

Today's discussion will focus on finding the **arc length** of a curve in the plane. This can be found via a definite integral which we will develop from a Riemann sum.

Arc Length

Suppose f is continuous on $[a, b]$ and differentiable on (a, b).



Riemann Sum Approach

Let $\Delta x = (b-a)/n$ for some $n \in \mathbb{N}$ and $x_i = a + i\Delta x$ for $i = 1, 2, \ldots, n$, then the length of the line segment from $(x_{k-1}, f(x_{k-1}))$ to $(x_k, f(x_k))$ is

$$\begin{aligned}
&\sqrt{(x_k - x_{k-1})^2 + (f(x_k) - f(x_{k-1}))^2} \\
&= \sqrt{(x_k - x_{k-1})^2 \left(1 + \frac{(f(x_k) - f(x_{k-1}))^2}{(x_k - x_{k-1})^2}\right)} \\
&= (x_k - x_{k-1})\sqrt{1 + \frac{(f(x_k) - f(x_{k-1}))^2}{(x_k - x_{k-1})^2}} \\
&= (x_k - x_{k-1})\sqrt{1 + [f'(z_k)]^2} \quad \text{(by the MVT)} \\
&= \sqrt{1 + [f'(z_k)]^2}\,\Delta x.
\end{aligned}$$

Arc Length Formula

Let s denote the arc length of the graph of $y = f(x)$ over the interval $[a, b]$, then

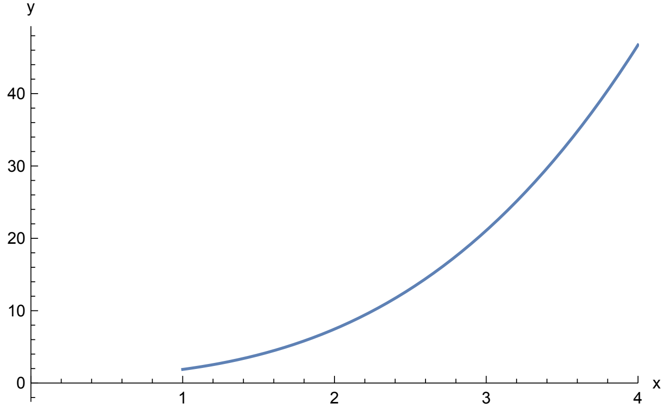
$$\begin{aligned} s &\approx \sum_{k=1}^{n} \sqrt{1 + [f'(z_k)]^2} \Delta x \\ &= \lim_{n \to \infty} \sum_{k=1}^{n} \sqrt{1 + [f'(z_k)]^2} \Delta x \\ &= \int_a^b \sqrt{1 + [f'(x)]^2}\, dx. \end{aligned}$$

The quantity $\sqrt{1 + [f'(x)]^2}\, dx$ is called **differential arc length** and sometimes denoted ds, *i.e.*

$$ds = \sqrt{1 + [f'(x)]^2}\, dx.$$

Example

Find the arc length of $f(x) = \frac{2}{3}(x^2 + 1)^{3/2}$ on the interval $[1, 4]$.

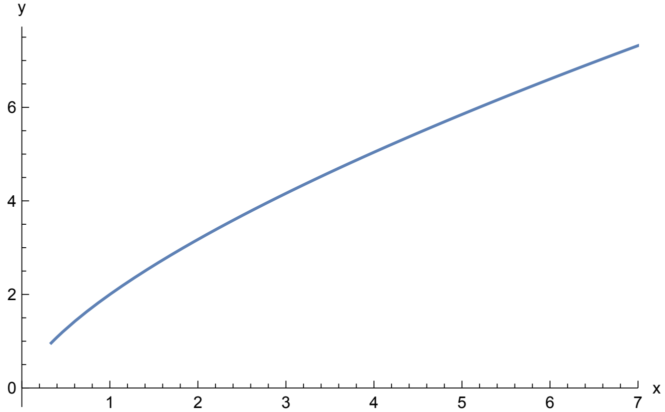


Solution

$$
\begin{aligned}
f(x) &= \frac{2}{3}(x^2+1)^{3/2} \\
f'(x) &= 2x(x^2+1)^{1/2} \\
s &= \int_1^4 \sqrt{1+\left[2x(x^2+1)^{1/2}\right]^2}\,dx \\
&= \int_1^4 \sqrt{1+4x^2(x^2+1)}\,dx \\
&= \int_1^4 \sqrt{4x^4+4x^2+1}\,dx \\
&= \int_1^4 \sqrt{(2x^2+1)^2}\,dx = \int_1^4 (2x^2+1)\,dx \\
&= \left[\frac{2}{3}x^3+x\right]_{x=1}^{x=4} \\
&= \left(\frac{128}{3}+4\right)-\left(\frac{2}{3}+1\right) = 45
\end{aligned}
$$

Example

Find the arc length of $f(x) = 2x^{2/3}$ on the interval $[\frac{1}{3}, 7]$.



Solution (1 of 2)

$$
\begin{aligned}
f(x) &= 2x^{2/3} \\
f'(x) &= \frac{4}{3}x^{-1/3} \\
s &= \int_{1/3}^{7} \sqrt{1+\left(\frac{4}{3}x^{-1/3}\right)^2}\,dx = \int_{1/3}^{7} \sqrt{1+\frac{16}{9x^{2/3}}}\,dx \\
&= \int_{1/3}^{7} \sqrt{\frac{9x^{2/3}+16}{9x^{2/3}}}\,dx = \int_{1/3}^{7} \frac{1}{3x^{1/3}}\sqrt{9x^{2/3}+16}\,dx
\end{aligned}
$$

Integrate by substitution with

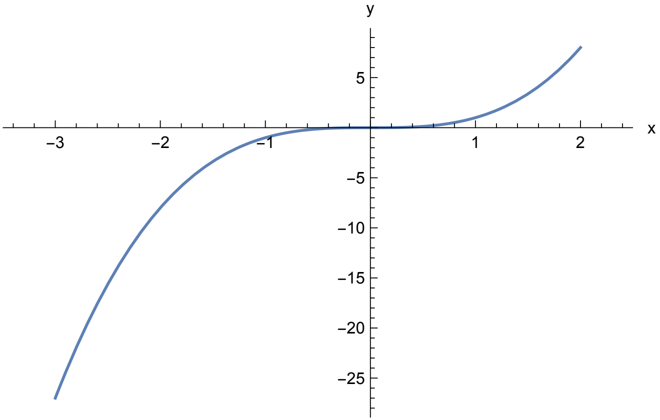
$$
\begin{aligned}
u &= 9x^{2/3}+16 \\
\frac{1}{18}\,du &= \frac{1}{3x^{1/3}}\,dx.
\end{aligned}
$$

Solution (2 of 2)

$$\begin{aligned}
s &= \int_{1/3}^{7} \frac{1}{3x^{1/3}} \sqrt{9x^{2/3} + 16}\, dx \\
&= \frac{1}{18} \int_{9(1/3)^{2/3}+16}^{9(7)^{2/3}+16} u^{1/2}\, du \\
&= \left[\frac{1}{27} u^{3/2} \right]_{u=9(1/3)^{2/3}+16}^{u=9(7)^{2/3}+16} \\
&= \frac{1}{27} \left(9(7)^{2/3} + 16 \right)^{3/2} - \frac{1}{27} \left(9(1/3)^{2/3} + 16 \right)^{3/2} \\
&\approx 9.28374
\end{aligned}$$

Numerically Estimating Arc Length

Estimate the arc length of $f(x) = x^3$ on the interval $[-3, 2]$.



Solution

$$\begin{aligned} f(x) &= x^3 \\ f'(x) &= 3x^2 \\ s &= \int_{-3}^{2} \sqrt{1 + (3x^2)^2}\, dx \\ &= \int_{-3}^{2} \sqrt{1 + 9x^4}\, dx \\ &\approx 36.2884 \end{aligned}$$

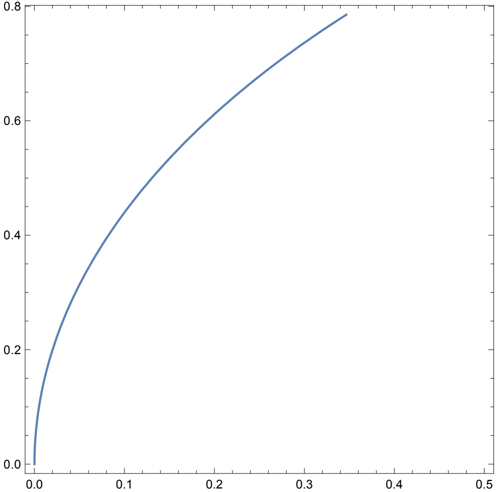
Functions of y

If a curve is described as a function of y for $c \leq y \leq d$ then the arc length of the curve is

$$s = \int_c^d \sqrt{1 + [f'(y)]^2}\, dy.$$

Example

Find the arc length of $x = \ln(\sec y)$ for $y \in [0, \pi/4]$.



Solution

$$\begin{aligned}
f(y) &= \ln(\sec y) \\
f'(y) &= \tan y \\
s &= \int_0^{\pi/4} \sqrt{1+\tan^2 y}\, dy = \int_0^{\pi/4} \sqrt{\sec^2 y}\, dy \\
&= \int_0^{\pi/4} \sec y\, dy \\
&= \left[\ln(\sec y + \tan y)\right]_{y=0}^{y=\pi/4} \\
&= \ln(\sqrt{2}+1) - \ln 1 = \ln(\sqrt{2}+1)
\end{aligned}$$

Arc Length Function

Suppose f is continuous on $[a, b]$ and differentiable on (a, b), then f is also continuous on $[a, x]$ and differentiable on (a, x), for every $a < x < b$.

Definition

The **arc length function** denoted $s(x)$ is defined to be

$$s(x) = \int_a^x \sqrt{1 + [f'(t)]^2}\, dt$$

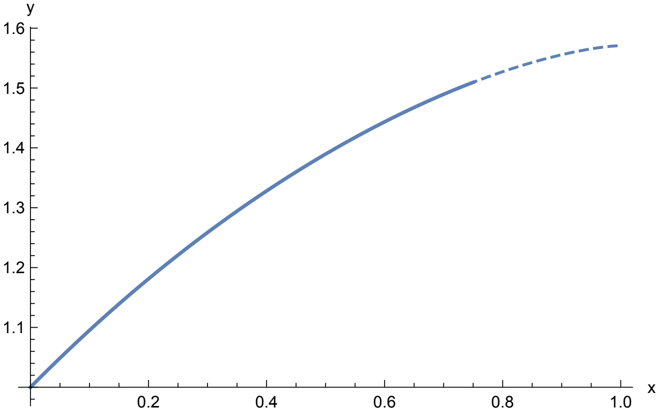
for $a < x < b$.

By the Fundamental Theorem of Calculus

$$\frac{ds}{dx} = \sqrt{1 + [f'(x)]^2}.$$

Example

Find the arc length function for the curve $y = \sin^{-1} x + \sqrt{1 - x^2}$ with starting point $(0, 1)$.

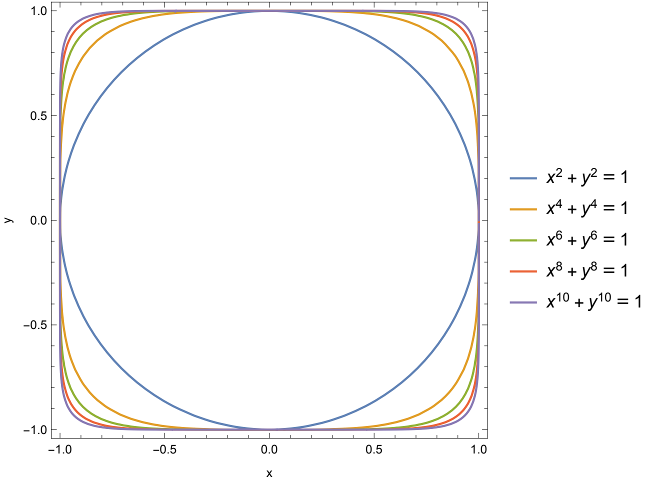


Solution

$$
\begin{aligned}
y &= \sin^{-1} x + \sqrt{1-x^2} \\
\frac{dy}{dx} &= \frac{1}{\sqrt{1-x^2}} - \frac{x}{\sqrt{1-x^2}} \\
\left(\frac{dy}{dx}\right)^2 &= \frac{1}{1-x^2} - \frac{2x}{1-x^2} + \frac{x^2}{1-x^2} \\
1 + \left(\frac{dy}{dx}\right)^2 &= \frac{2-2x}{1-x^2} = \frac{2(1-x)}{(1-x)(1+x)} = \frac{2}{1+x} \\
s(x) &= \int_0^x \sqrt{\frac{2}{1+t}}\, dt = \left[2\sqrt{2}\sqrt{1+t}\right]_{t=0}^{t=x} \\
&= 2\sqrt{2}(\sqrt{1+x} - 1)
\end{aligned}
$$

Challenging Example

The curves with equations $x^{2k} + y^{2k} = 1$ for $k = 1, 2, \ldots$ are graphed below. Set up an integral for the arc length L_{2k}. Speculate $\lim_{k \to \infty} L_{2k}$.



Solution

$$
\begin{aligned}
x^{2k} + y^{2k} &= 1 \\
y &= \left(1 - x^{2k}\right)^{1/2k} \text{ for } 0 \le x \le 1 \\
\frac{dy}{dx} &= \frac{1}{2k}\left(1 - x^{2k}\right)^{\frac{1-2k}{2k}} \\
L_{2k} &= 4\int_0^1 \sqrt{1 + \frac{1}{4k^2}\left(1 - x^{2k}\right)^{\frac{1-2k}{k}}}\, dx \\
\lim_{k\to\infty} L_{2k} &= 8
\end{aligned}
$$

Homework

- Read Section 8.1
- Exercises: WebAssign/D2L