Arc Length and Surface Area in Parametric Equations

MATH 211, *Calculus II*

J. Robert Buchanan

Department of Mathematics

Fall 2021

Background

- We have developed definite integral formulas for arc length and surface area for curves of the form $y = f(x)$ with $a \leq x \leq b$.

$$
\begin{aligned}
s &= \int_a^b \sqrt{1 + (f'(x))^2}\, dx \\
S &= 2\pi \int_a^b |f(x)|\sqrt{1 + (f'(x))^2}\, dx
\end{aligned}
$$

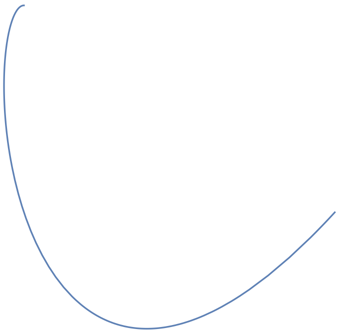
- Today we will develop formulas for calculating arc length and surface area for curves described parametrically.

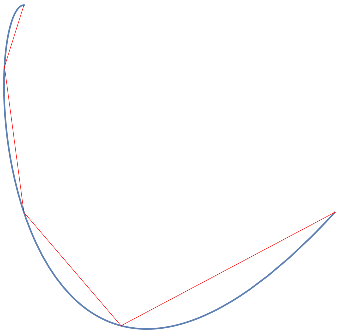
Riemann Sum Approach

Suppose a curve is described by the parametric equations:

$$\begin{aligned} x &= x(t) \\ y &= y(t) \end{aligned}$$

where $a \leq t \leq b$ and $x'(t)$ and $y'(t)$ are continuous as well.





Partition

Partition $[a, b]$ into n equal subintervals with $\Delta t = \dfrac{b-a}{n}$ and $t_k = a + k\Delta t$ for $k = 0, 1, \ldots, n$.

$$
\begin{aligned}
\Delta s_k &= \sqrt{(x(t_k) - x(t_{k-1}))^2 + (y(t_k) - y(t_{k-1}))^2} \\
&= \sqrt{(x'(v_k)\Delta t)^2 + (y'(w_k)\Delta t)^2} \quad \text{(by the MVT)} \\
&= \sqrt{(x'(v_k))^2 + (y'(w_k))^2}\Delta t \\
&\approx \sqrt{(x'(w_k))^2 + (y'(w_k))^2}\Delta t
\end{aligned}
$$

Riemann Sum

Arc length

$$\begin{aligned} s &\approx \sum_{k=1}^{n} \sqrt{(x'(w_k))^2 + (y'(w_k))^2}\Delta t \\ &= \lim_{n\to\infty} \sum_{k=1}^{n} \sqrt{(x'(w_k))^2 + (y'(w_k))^2}\Delta t \\ &= \int_a^b \sqrt{(x'(t))^2 + (y'(t))^2}\, dt \end{aligned}$$

Remark: the expression $ds = \sqrt{(x'(t))^2 + (y'(t))^2}\, dt$ is called **differential arc length.**

Result

Theorem

For the curve defined parametrically by $x = x(t)$, $y = y(t)$, $a \leq t \leq b$, *if* $x'(t)$ *and* $y'(t)$ *are continuous on* $[a, b]$ *and the curve does not intersect itself (except possibly at a finite number of points), then the arc length* s *of the curve is given by*

$$s = \int_a^b \sqrt{(x'(t))^2 + (y'(t))^2}\, dt = \int_a^b \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\, dt.$$

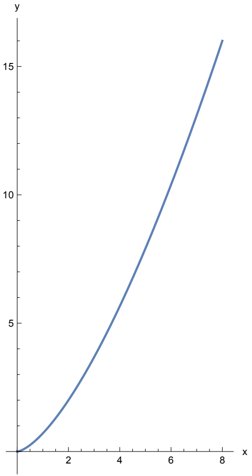
Example

Find the arc length of the curve given by

$$\begin{aligned} x &= 2t^2 \\ y &= 2t^3 \end{aligned}$$

for $0 \leq t \leq 2$.

Solution



$$s = \int_0^2 \sqrt{(4t)^2 + (6t^2)^2}\, dt = \int_0^2 2t\sqrt{4 + 9t^2}\, dt$$

Integrate by substitution, letting $u = 4 + 9t^2$ and $\frac{1}{9}\, du = 2t\, dt$.

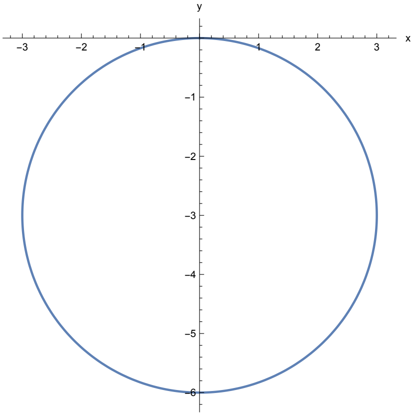
$$s = \int_4^{40} \frac{1}{9} u^{1/2}\, du = \left. \frac{2}{27} u^{3/2} \right|_4^{40} = \frac{2}{27}(40\sqrt{40} - 8) = \frac{16}{27}(10\sqrt{10} - 1)$$

Example

Find the arc length of the curve given by

$$\begin{aligned} x &= 3\sin t \\ y &= 3\cos t - 3 \end{aligned}$$

for $0 \le t \le 2\pi$.

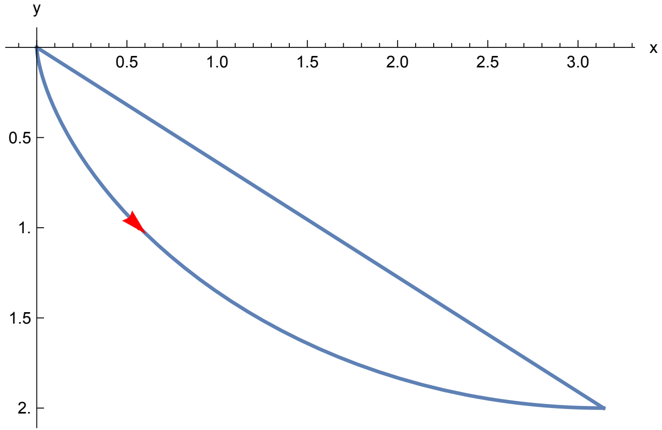


Solution

$$\begin{aligned} s &= \int_0^{2\pi} \sqrt{(3\cos t)^2 + (-3\sin t)^2}\, dt \\ &= \int_0^{2\pi} \sqrt{9\cos^2 t + 9\sin^2 t}\, dt \\ &= \int_0^{2\pi} 3\, dt \\ &= 6\pi \end{aligned}$$

Brachistochrone Problem

Problem: suppose an object is to slide down a path in the *xy*-plane from the origin to a point below the origin but not on the *y*-axis. Which path should the object follow to accomplish the trip in the least time?



Physical Background

$$\begin{aligned} \text{distance} &= \text{speed} \times \text{time} \\ d &= |v|t \end{aligned}$$

Conservation of energy:

$$\begin{aligned} \text{potential energy} &= \text{kinetic energy} \\ m\,g\,y &= \frac{1}{2}m\left(\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2\right) \end{aligned}$$

- To avoid confusion with “time” use the symbol u as the parameter of the parametric equations.
- To simplify the mathematics we will assume that $g\,y \geq 0$.

Formula for Trip Time

Suppose the path is parameterized by $x = x(u)$, $y = y(u)$ for $0 \leq u \leq 1$, then

$$\begin{aligned} m\,g\,y &= \frac{1}{2}m\left(\left(\frac{dx}{du}\right)^2 + \left(\frac{dy}{du}\right)^2\right) \\ 2g\,y &= \left(\frac{dx}{du}\right)^2 + \left(\frac{dy}{du}\right)^2 \\ 2g\,y &= (v(u))^2 \\ \sqrt{2g\,y} &= |v(u)|. \end{aligned}$$

Therefore the trip time is

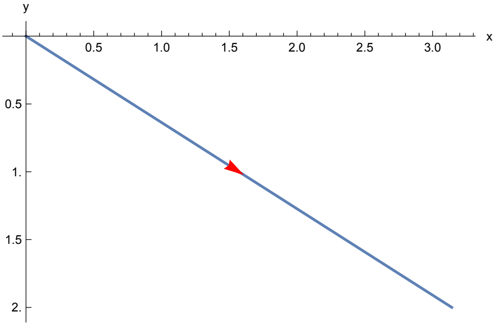
$$t = \frac{d}{|v|} = \int_0^1 \frac{\sqrt{\left(\frac{dx}{du}\right)^2 + \left(\frac{dy}{du}\right)^2}}{\sqrt{2gy}}\,du = \frac{1}{\sqrt{2g}}\int_0^1 \sqrt{\frac{\left(\frac{dx}{du}\right)^2 + \left(\frac{dy}{du}\right)^2}{y}}\,du.$$

Comparison of Paths

Suppose the object is to move from $(0,0)$ to $(\pi,2)$ along a straight line parameterized by

$$\begin{aligned} x &= \pi u \\ y &= 2u \end{aligned}$$

for $0 \leq u \leq 1$. Find the time of travel.



Calculation of Travel Time

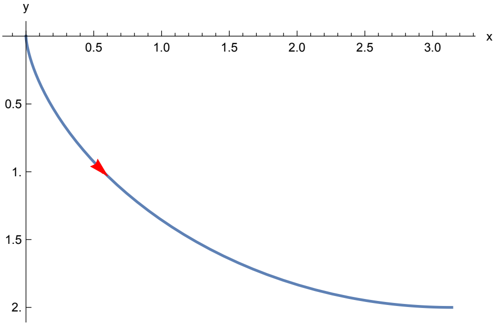
$$
\begin{aligned}
t &= \frac{1}{\sqrt{2g}} \int_0^1 \sqrt{\frac{(\pi)^2 + (2)^2}{2u}}\, du \\
&= \frac{\sqrt{\pi^2 + 4}}{2\sqrt{g}} \int_0^1 u^{-1/2}\, du \\
&= \left[\sqrt{\frac{\pi^2 + 4}{g}} u^{1/2} \right]_0^1 \\
t &= \sqrt{\frac{\pi^2 + 4}{g}}
\end{aligned}
$$

Comparison of Paths

Suppose the object is to move from $(0, 0)$ to $(\pi, 2)$ along a cycloid parameterized by

$$\begin{aligned} x &= \pi u - \sin \pi u \\ y &= 1 - \cos \pi u \end{aligned}$$

for $0 \leq u \leq 1$. Find the time of travel.



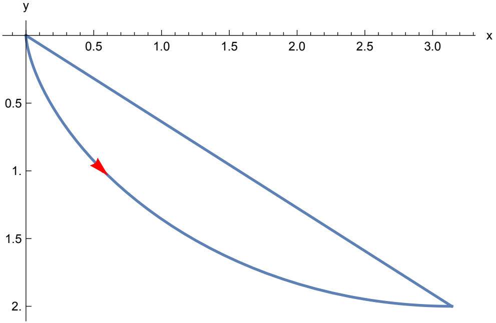
Calculation of Travel Time

$$
\begin{aligned}
t &= \frac{1}{\sqrt{2g}} \int_0^1 \sqrt{\frac{(\pi - \pi \cos \pi u)^2 + (\pi \sin \pi u)^2}{1 - \cos \pi u}}\, du \\
&= \frac{1}{\sqrt{2g}} \int_0^1 \sqrt{\frac{\pi^2 - 2\pi^2 \cos \pi u + \pi^2 \cos^2 \pi u + \pi^2 \sin^2 \pi u}{1 - \cos \pi u}}\, du \\
&= \frac{1}{\sqrt{2g}} \int_0^1 \sqrt{\frac{2\pi^2 - 2\pi^2 \cos \pi u}{1 - \cos \pi u}}\, du \\
&= \frac{\pi}{\sqrt{g}} \int_0^1 1\, du \\
t &= \sqrt{\frac{\pi^2}{g}}
\end{aligned}
$$

Result

$$\sqrt{\frac{\pi^2}{g}} < \sqrt{\frac{\pi^2 + 4}{g}}$$

Even though the cycloid is a greater *distance* (arc length), it requires less *time*.



Surface Area

Geometrically we may think of the definite integral for the surface area of a solid of revolution as

$$S = \int_a^b 2\pi(\text{radius})(\text{arc length})\, dx.$$

Thus the surface generated when the parametric curve

$$\begin{aligned} x &= x(t) \\ y &= y(t) \end{aligned}$$

for $a \le t \le b$ is revolved around the x-axis has surface area

$$S = 2\pi \int_a^b |y(t)| \sqrt{(x'(t))^2 + (y'(t))^2}\, dt.$$

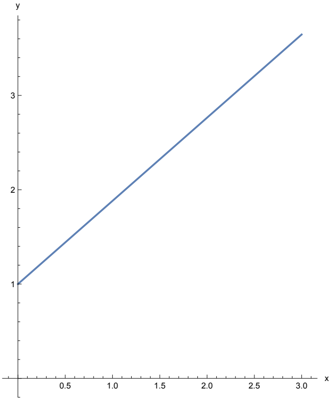
Example

Find the surface area of the solid of revolution generated when the parametric curve

$$\begin{aligned} x &= 3t \\ y &= \sqrt{7}t + 1 \end{aligned}$$

for $0 \leq t \leq 1$ is revolved around the x-axis.

Solution



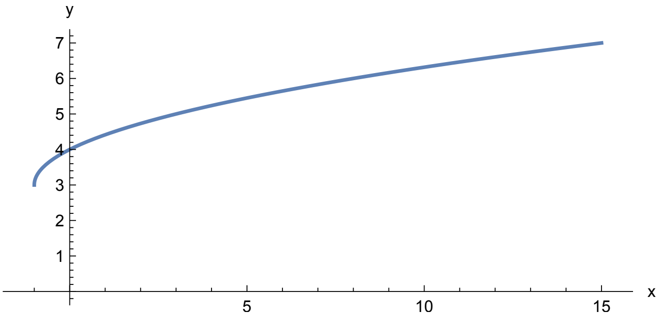
$$
\begin{aligned}
S &= 2\pi \int_0^1 (\sqrt{7}t + 1)\sqrt{(3)^2 + (\sqrt{7})^2}\, dt \\
&= 2\pi \int_0^1 4(\sqrt{7}t + 1)\, dt \\
&= 8\pi \int_0^1 (\sqrt{7}t + 1)\, dt \\
&= \left[8\pi \left(\frac{\sqrt{7}}{2} t^2 + t \right) \right]_0^1 \\
&= 8\pi \left(\frac{\sqrt{7}}{2} + 1 \right) \\
&= 4(\sqrt{7} + 2)\pi
\end{aligned}
$$

Example

Find the surface area of the solid of revolution generated when the parametric curve

$$\begin{aligned} x &= 4t^2 - 1 \\ y &= 3 - 2t \end{aligned}$$

for $-2 \leq t \leq 0$ is revolved around the x-axis.



Solution (1 of 2)

$$\begin{aligned}
S &= 2\pi \int_{-2}^{0} (3-2t)\sqrt{(8t)^2+(-2)^2}\,dt \\
&= 2\pi \int_{-2}^{0} (3-2t)\sqrt{64t^2+4}\,dt \\
&= 4\pi \int_{-2}^{0} (3-2t)\sqrt{16t^2+1}\,dt \\
&= 12\pi \int_{-2}^{0} \sqrt{16t^2+1}\,dt - 8\pi \int_{-2}^{0} t\sqrt{16t^2+1}\,dt
\end{aligned}$$

- The first integral can be handled via the trigonometric substitution, $t = \frac{1}{4}\tan\theta$ and $dt = \frac{1}{4}\sec^2\theta\,d\theta$.
- The second integral can be handled via the substitution, $u = 16t^2 + 1$ and $\frac{1}{32}\,du = t\,dt$.

Solution (2 of 2)

$$
\begin{aligned}
S &= 12\pi \int_{-2}^{0} \sqrt{16t^2+1}\, dt - 8\pi \int_{-2}^{0} t\sqrt{16t^2+1}\, dt \\
&= 3\pi \int_{-\tan^{-1} 8}^{0} \sec^3\theta \, d\theta - \frac{\pi}{4} \int_{65}^{1} u^{1/2}\, du \\
&= \left[\frac{3\pi}{2} (\sec\theta \, \tan\theta - \ln[\sec\theta + \tan\theta]) \right]_{-\tan^{-1} 8}^{0} - \left[\left(\frac{\pi}{6} u^{3/2} \right) \right]_{65}^{1} \\
&= 12\sqrt{65}\pi + \frac{3\pi}{2} \ln(8+\sqrt{65}) - \frac{\pi}{6}(65\sqrt{65}-1) \approx 590.89
\end{aligned}
$$

Homework

- Read Section 7.2
- Exercises: 109, 113, 117, 121, 125/handout