Comparison Test

MATH 211, *Calculus II*

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Introduction

Remarks:

- Determining the **convergence** or **divergence** of a **series** from its **sequence of partial sums** is difficult for most series.
- Today we will work only with **positive term** series, *i.e.*, series
$$\sum_{k=1}^{\infty} a_k \text{ where } \quad a_k \geq 0 \text{ for all } k.$$
- We need only concern ourselves with the infinite “tail” of a series since for any fixed N, the sum of the first N terms of a series must be finite.

Comparison Tests

It is helpful to compare a new infinite series to another infinite series whose convergence or divergence is already known.

Theorem (Comparison Test)

Suppose that $0 \leq a_k \leq b_k$*, for all* k*.*

1. *If* $\sum_{k=1}^{\infty} b_k$ *converges, then* $\sum_{k=1}^{\infty} a_k$ *converges too.*
2. *If* $\sum_{k=1}^{\infty} a_k$ *diverges, then* $\sum_{k=1}^{\infty} b_k$ *diverges too.*

Proof

Let B_n be the nth partial sum of the convergent series $\sum_{k=1}^{\infty} b_k$ and let B be its sum. Let A_n be the nth partial sum of the series $\sum_{k=1}^{\infty} a_k$. Since $0 \le a_k \le b_k$ then

$$0 \le A_n \le B_n \le B.$$

Thus $\{A_n\}_{n=1}^{\infty}$ is an increasing and bounded sequence and hence converges.

Questions

Questions:

- If $\sum_{k=1}^{\infty} a_k$ converges, what can you say about $\sum_{k=1}^{\infty} b_k$?
- If $\sum_{k=1}^{\infty} b_k$ diverges, what can you say about $\sum_{k=1}^{\infty} a_k$?

Examples

Use the Comparison Test to determine if the following series converge or diverge.

1. $\displaystyle\sum_{k=1}^{\infty} \frac{k}{2^k(k+1)}$ **Hint:** $\displaystyle\frac{k}{2^k(k+1)} < \frac{1}{2^k}$
2. $\displaystyle\sum_{k=1}^{\infty} \frac{k}{5k^2-4}$ **Hint:** $\displaystyle\frac{k}{5k^2-4} > \frac{1}{5k}$
3. $\displaystyle\sum_{k=1}^{\infty} \frac{\cos^{-1}(1/k)}{k^{3/2}}$ **Hint:** $\displaystyle\frac{\cos^{-1}(1/k)}{k^{3/2}} < \frac{\pi}{k^{3/2}}$

$\sum_{k=1}^{\infty} \frac{k}{2^k(k+1)}$

Since

$$0 < \frac{k}{2^k(k+1)} < \frac{1}{2^k}$$

and

$$\sum_{k=1}^{\infty} \frac{1}{2^k} = \sum_{k=1}^{\infty} \left(\frac{1}{2}\right) \left(\frac{1}{2}\right)^{k-1} = \sum_{k=0}^{\infty} \left(\frac{1}{2}\right) \left(\frac{1}{2}\right)^{k} = \frac{1/2}{1-(1/2)} = 1$$

(geometric series), then the smaller series

$$\sum_{k=1}^{\infty} \frac{k}{2^k(k+1)} \quad \text{converges.}$$

$$\sum_{k=1}^{\infty} \frac{k}{5k^2 - 4}$$

Since

$$0 < \frac{1}{5k} < \frac{k}{5k^2 - 4}$$

and

$$\int_1^{\infty} \frac{1}{5x}\,dx = \lim_{R\to\infty} \int_1^{R} \frac{1}{5x}\,dx = \lim_{R\to\infty} \frac{1}{5} \ln R = \infty$$

according to the Integral Test, then the larger series

$$\sum_{k=1}^{\infty} \frac{k}{5k^2 - 4} \quad \text{diverges.}$$

$$\sum_{k=1}^{\infty} \frac{\cos^{-1}(1/k)}{k^{3/2}}$$

Since

$$0 < \frac{\cos^{-1}(1/k)}{k^{3/2}} < \frac{\pi}{k^{3/2}}$$

and

$$\sum_{k=1}^{\infty} \frac{\pi}{k^{3/2}} = \pi \sum_{k=1}^{\infty} \frac{1}{k^{3/2}}$$

converges by the p-series test, then the smaller series

$$\sum_{k=1}^{\infty} \frac{\cos^{-1}(1/k)}{k^{3/2}} \quad \text{converges.}$$

Limit Comparison Test

Sometimes the expression a_k in the infinite series $\sum_{k=1}^{\infty} a_k$ is too complicated to compare with the b_k of another series for all k.

Theorem (Limit Comparison Test)

Suppose that $a_k > 0$, and $b_k > 0$ and there is a finite $L > 0$ such that $\lim_{k\to\infty} \frac{a_k}{b_k} = L$. Then either $\sum_{k=1}^{\infty} a_k$ and $\sum_{k=1}^{\infty} b_k$ both converge or they both diverge.

Examples

Use the Limit Comparison Test to determine whether the following infinite series converge or diverge.

1. $\displaystyle\sum_{k=2}^{\infty} \frac{1}{\sqrt{4k^2-7}}$ **Hint:** limit compare with $\dfrac{1}{k}$
2. $\displaystyle\sum_{k=1}^{\infty} \frac{k}{k^4+1}$ **Hint:** limit compare with $\dfrac{1}{k^3}$
3. $\displaystyle\sum_{k=1}^{\infty} \frac{\sqrt{k}}{k+2}$ **Hint:** limit compare with $\dfrac{1}{k^{1/2}}$

$$\sum_{k=2}^{\infty} \frac{1}{\sqrt{4k^2 - 7}}$$

$$\lim_{k\to\infty} \frac{\frac{1}{\sqrt{4k^2-7}}}{\frac{1}{k}} = \lim_{k\to\infty} \frac{k}{\sqrt{4k^2 - 7}} = \lim_{k\to\infty} \frac{1}{\sqrt{4 - \frac{7}{k^2}}} = \frac{1}{2} < \infty$$

Since $L = 1/2$ and since $\sum_{k=2}^{\infty} \frac{1}{k}$ diverges by the p-series test, then

$$\sum_{k=2}^{\infty} \frac{1}{\sqrt{4k^2 - 7}} \quad \text{diverges.}$$

$$\sum_{k=1}^{\infty} \frac{k}{k^4+1}$$

$$\lim_{k\to\infty} \frac{\frac{k}{k^4+1}}{\frac{1}{k^3}} = \lim_{k\to\infty} \frac{k^4}{k^4+1} = \lim_{k\to\infty} \frac{1}{1+\frac{1}{k^4}} = 1 < \infty$$

Since $L = 1$ and since $\sum_{k=1}^{\infty} \frac{1}{k^3}$ converges by the p-series test,

then

$$\sum_{k=1}^{\infty} \frac{k}{k^4+1} \quad \text{converges.}$$

$$\sum_{k=1}^{\infty} \frac{\sqrt{k}}{k+2}$$

$$\lim_{k\to\infty} \frac{\frac{\sqrt{k}}{k+2}}{\frac{1}{k^{1/2}}} = \lim_{k\to\infty} \frac{k}{k+2} = \lim_{k\to\infty} \frac{1}{1+\frac{2}{k}} = 1 < \infty$$

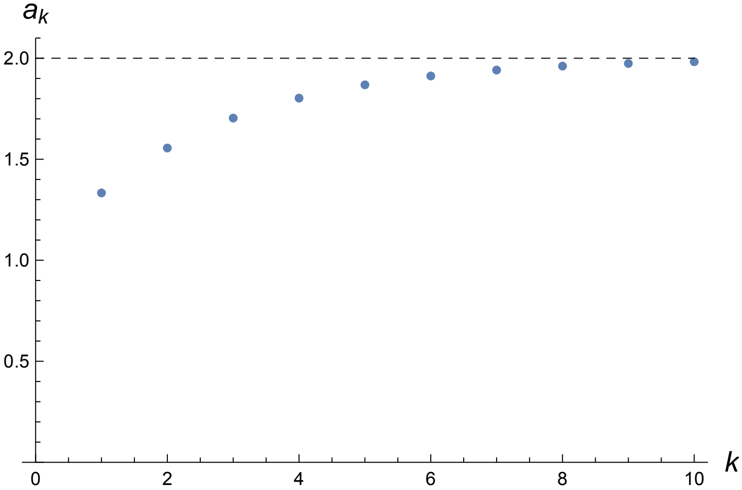
Since $L = 1$ and since $\sum_{k=1}^{\infty} \frac{1}{k^{1/2}}$ diverges by the p-series test, then

$$\sum_{k=1}^{\infty} \frac{\sqrt{k}}{k+2} \quad \text{diverges.}$$

Challenging Examples

Determine whether the following series converges or diverges.

$$\sum_{k=1}^{\infty} 2^{-k} a_k \text{ where } a_k \text{ is shown below}$$



Challenging Examples

Determine whether the following series converges or diverges.

$$\sum_{k=1}^{\infty} \sin\left(\frac{1}{k}\right)$$

Challenging Examples

Determine whether the following series converges or diverges.

$$\frac{1}{1} + \frac{1}{\ln 2} + \frac{1}{3} + \frac{1}{\ln 4} + \frac{1}{5} + \frac{1}{\ln 6} + \cdots$$

Challenging Examples

Suppose $\sum_{k=1}^{\infty} a_k$ converges and $0 < a_k < 1$ for all $k \in \mathbb{N}$.

Determine whether the following series converges or diverges.

$$\sum_{k=1}^{\infty} (a_k)^2$$

Solution

If $0 < a_k < 1$ *for all* $k \in \mathbb{N}$ *then*

$$0 < (a_k)^2 < a_k < 1 \text{ for all } k \in \mathbb{N}.$$

If $\sum_{k=1}^{\infty} a_k$ *converges, then by the Comparison Test,* $\sum_{k=1}^{\infty} (a_k)^2$ *converges.*

Homework

- Read Section 5.4
- Exercises: 195, 199, 203, . . . , 219