Infinite Series

MATH 211, *Calculus II*

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Objectives

In this lesson we will learn:

- the meaning of the sum of an infinite series,
- a formula for the sum of a geometric series, and
- to evaluate the sum of a telescoping series.

Background

Consider the repeating decimal form of $2/3$.

$$\begin{aligned}
\frac{2}{3} &= 0.666666\cdots \\
&= 0.6 + 0.06 + 0.006 + 0.0006 + \cdots \\
&= 6(0.1) + 6(0.1)^2 + 6(0.1)^3 + 6(0.1)^4 + \cdots \\
&= \sum_{k=1}^{\infty} 6(0.1)^k
\end{aligned}$$

Question: what does it mean to perform a summation of an infinite number of terms?

Infinite Series

Definition

An **infinite series** is an expression of the form

$$a_1 + a_2 + a_3 + \cdots + a_k + \cdots = \sum_{k=1}^{\infty} a_k.$$

A finite summation of the form

$$S_n = a_1 + a_2 + a_3 + \cdots + a_n = \sum_{k=1}^{n} a_k$$

is called the n^{th} **partial sum of the series.** The sequence $\{S_n\}_{n=1}^{\infty}$ is called the **sequence of partial sums** of the series.

Convergence and Divergence

Definition

If $\{S_n\}_{n=1}^{\infty}$ is the sequence of partial sums of the series $\sum_{k=1}^{\infty} a_k$ and if

$$\lim_{n \to \infty} S_n = S$$

where S is finite, then S is called the **sum of the series**, we say the series **converges** and we can write

$$S = \sum_{k=1}^{\infty} a_k.$$

If S is infinite or does not exist then we say the series **diverges.**

Previous Example (1 of 2)

Consider $\sum_{k=1}^{\infty} 6(0.1)^k$. What is the sequence of partial sums of the series?

$$\begin{aligned}
S_1 &= \sum_{k=1}^{1} 6(0.1)^k = 6(0.1)^1 = 0.6 \\
S_2 &= \sum_{k=1}^{2} 6(0.1)^k = 6(0.1)^1 + 6(0.1)^2 = 0.66 \\
S_3 &= \sum_{k=1}^{3} 6(0.1)^k = 0.666 \\
&\vdots \\
S_n &= 0.\underbrace{666\cdots 66}_{\text{n digits}}
\end{aligned}$$

Previous Example (2 of 2)

Consider $\sum_{k=1}^{\infty} 6(0.1)^k$. Does the series converge and, if so, to what number?

$$\begin{aligned}
\lim_{n\to\infty} S_n &= \lim_{n\to\infty} 0.\underbrace{666\cdots66}_{\text{n digits}} = \frac{2}{3} \\
\sum_{k=1}^{\infty} 6(0.1)^k &= \frac{2}{3}
\end{aligned}$$

The series converges to $2/3$.

Examples

Determine if the following series converge or diverge.

1. $\displaystyle\sum_{k=1}^{\infty} 2^{1-k}$
2. $\displaystyle\sum_{k=1}^{\infty} \frac{k-1}{k}$
3. $\displaystyle\sum_{k=1}^{\infty} \frac{1}{k(k+1)}$
4. $\displaystyle\sum_{k=1}^{\infty} (-1)^k$

Solution: $\sum_{k=1}^{\infty} 2^{1-k}$

$$
\begin{aligned}
S_1 &= \sum_{k=1}^{1} 2^{1-k} = 2^{1-1} = 1 \\
S_2 &= \sum_{k=1}^{2} 2^{1-k} = 1 + \frac{1}{2} = \frac{3}{2} \\
S_3 &= \sum_{k=1}^{3} 2^{1-k} = 1 + \frac{1}{2} + \frac{1}{4} = \frac{7}{4} \\
&\vdots \\
S_N &= \sum_{k=1}^{N} 2^{1-k} = 2 - \frac{1}{2^{N-1}} \\
&\vdots \\
S &= \lim_{N\to\infty} S_N = \lim_{N\to\infty} \left(2 - \frac{1}{2^{N-1}} \right) = 2
\end{aligned}
$$

The infinite series **converges** to $S = 2$.

Solution: $\sum_{k=1}^{\infty} \frac{k-1}{k}$

$$
\begin{aligned}
S_1 &= \sum_{k=1}^{1} \frac{k-1}{k} = \frac{1-1}{1} = 0 \\
S_2 &= \sum_{k=1}^{2} \frac{k-1}{k} = 0 + \frac{1}{2} = \frac{1}{2} \\
S_3 &= \sum_{k=1}^{3} \frac{k-1}{k} = 0 + \frac{1}{2} + \frac{2}{3} = \frac{7}{6} \\
&\vdots \\
S_N &= \sum_{k=1}^{N} \frac{k-1}{k} = S_{N-1} + \frac{N-1}{N}
\end{aligned}
$$

The infinite series **diverges.**

Solution: $\sum_{k=1}^{\infty} \frac{1}{k(k+1)}$

Using partial fraction decomposition $\dfrac{1}{k(k+1)} = \dfrac{1}{k} - \dfrac{1}{k+1}$. Thus

$$\begin{aligned}
S_N &= \sum_{k=1}^{N} \frac{1}{k(k+1)} \\
&= \sum_{k=1}^{N} \left(\frac{1}{k} - \frac{1}{k+1} \right) \\
&= \left(1 - \frac{1}{2} \right) + \left(\frac{1}{2} - \frac{1}{3} \right) + \cdots + \left(\frac{1}{N} - \frac{1}{N+1} \right) \\
&= 1 + \left(-\frac{1}{2} + \frac{1}{2} \right) + \left(-\frac{1}{3} + \frac{1}{3} \right) + \cdots + \left(-\frac{1}{N} + \frac{1}{N} \right) - \frac{1}{N+1} \\
&= 1 - \frac{1}{N+1} \\
S &= \lim_{N\to\infty} S_N = \lim_{N\to\infty} \left(1 - \frac{1}{N+1} \right) = 1
\end{aligned}$$

The infinite series **converges** to $S = 1$.

Solution: $\sum_{k=1}^{\infty}(-1)^k$

$$
\begin{aligned}
S_1 &= \sum_{k=1}^{1}(-1)^k = (-1)^1 = -1 \\
S_2 &= \sum_{k=1}^{2}(-1)^k = -1 + 1 = 0 \\
S_3 &= \sum_{k=1}^{3}(-1)^k = -1 + 1 - 1 = -1 \\
S_N &= \begin{cases} -1 & \text{if } N \text{ is odd,} \\ 0 & \text{if } N \text{ is even.} \end{cases}
\end{aligned}
$$

The infinite series **diverges.**

Geometric Series

Theorem

*For $a \neq 0$, the **geometric series** $\displaystyle\sum_{k=0}^{\infty} a\,r^k$ converges to $\dfrac{a}{1-r}$ if $|r| < 1$ and diverges if $|r| \geq 1$. The term r is called the **ratio**.*

Proof

$$
\begin{aligned}
S_N &= \sum_{k=0}^{N} a\, r^k = a + ar + ar^2 + ar^3 + \cdots + ar^N \\
r\, S_N &= r \sum_{k=0}^{N} a\, r^k = ar + ar^2 + ar^3 + \cdots + ar^{N+1} \\
S_N - r\, S_N &= a - ar^{N+1} \\
S_N(1 - r) &= a - ar^{N+1} \\
S_N &= \frac{a(1 - r^{N+1})}{1 - r} \\
\lim_{N\to\infty} S_N &= \lim_{N\to\infty} \frac{a(1 - r^{N+1})}{1 - r} \\
S &= \frac{a}{1 - r} \text{ if } |r| < 1.
\end{aligned}
$$

Examples (1 of 3)

Find the sum of the series

$$5 - \frac{10}{3} + \frac{20}{9} - \frac{40}{27} + \cdots$$

$$\begin{aligned}
5 - \frac{10}{3} + \frac{20}{9} - \frac{40}{27} + \cdots \\
&= 5(1 - \frac{2}{3} + \frac{4}{9} - \frac{8}{27} + \cdots) \\
&= 5\left(\left[\frac{-2}{3}\right]^0 + \left[\frac{-2}{3}\right]^1 + \left[\frac{-2}{3}\right]^2 + \left[\frac{-2}{3}\right]^3 + \cdots\right) \\
&= \sum_{k=0}^{\infty} 5\left[\frac{-2}{3}\right]^k \\
&= \frac{5}{1 - \left[\frac{-2}{3}\right]} = 3
\end{aligned}$$

Examples (2 of 3)

Is the series $\sum_{k=1}^{\infty} 2^{2k}3^{1-k}$ convergent or divergent?

$$\sum_{k=1}^{\infty} 2^{2k}3^{1-k} = \sum_{k=1}^{\infty} 4^k 3^1 3^{-k} = \sum_{k=1}^{\infty} 3\left[\frac{4}{3}\right]^k$$

Since $r = \frac{4}{3} > 1$, the infinite series diverges.

Examples (3 of 3)

Write the repeating decimal $2.3\overline{17}$ as a ratio of integers.

$$\begin{aligned}
2.3\overline{17} &= 2.3 + 0.0\overline{17} = \frac{23}{10} + \frac{17}{1000} + \frac{17}{100000} + \cdots \\
&= \frac{23}{10} + \frac{17}{1000}\left(1 + \frac{1}{100} + \frac{1}{100^2} + \cdots\right) \\
&= \frac{23}{10} + \frac{17/1000}{1 - \frac{1}{100}} \\
&= \frac{23}{10} + \frac{17}{990} = \frac{2294}{990} \\
&= \frac{1147}{495}
\end{aligned}$$

Issues of Convergence and Divergence

Theorem

If $\sum_{k=1}^{\infty} a_k$ *converges, then* $\lim_{k\to\infty} a_k = 0$.

Proof.

Let the sum of the series be L and let S_N be the nth partial sum.

$$\begin{aligned}
S_N - S_{N-1} &= \sum_{k=1}^{N} a_k - \sum_{k=1}^{N-1} a_k \\
&= (a_1 + \cdots + a_{N-1} + a_N) - (a_1 + \cdots + a_{N-1}) \\
S_N - S_{N-1} &= a_N \\
\lim_{N\to\infty} (S_N - S_{N-1}) &= \lim_{N\to\infty} a_N \\
L - L &= 0 = \lim_{N\to\infty} a_N
\end{aligned}$$

$\square$

k^{th}-Term Test for Divergence

Theorem (k^{th}-Term Test for Divergence)

If $\lim_{k\to\infty} a_k \neq 0$*, then* $\sum_{k=1}^{\infty} a_k$ *diverges.*

Example

Show that the series $\sum_{k=1}^{\infty} \frac{k^2}{5k^2+4}$ diverges.

The Harmonic Series

Remark: just because $\lim_{k\to\infty} a_k = 0$ we *cannot* conclude that $\sum_{k=1}^{\infty} a_k$ converges.

Lemma

*The **harmonic series*** $\sum_{k=1}^{\infty} \frac{1}{k}$ *diverges.*

Divergence of the Harmonic Series

$$
\begin{aligned}
S_1 &= 1 \\
S_2 &= 1 + \frac{1}{2} = \frac{3}{2} \\
S_4 &= 1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} > 2 \\
S_8 &> \frac{5}{2} \\
S_{16} &> 3 \\
&\vdots \\
S_{2^n} &> 1 + \frac{n}{2} \\
\lim_{n\to\infty} S_{2^n} &= \infty
\end{aligned}
$$

Properties of Convergent Series (1 of 2)

Theorem

If $\sum_{k=1}^{\infty} a_k$ *and* $\sum_{k=1}^{\infty} b_k$ *are convergent series with sums* A *and* B *respectively, then*

1. $\sum_{k=1}^{\infty}(a_k + b_k)$ *converges to* $A + B$*.*
2. $\sum_{k=1}^{\infty}(a_k - b_k)$ *converges to* $A - B$*.*
3. *If* c *is a constant then* $\sum_{k=1}^{\infty}(ca_k)$ *converges to* cA*.*

Properties of Convergent Series (2 of 2)

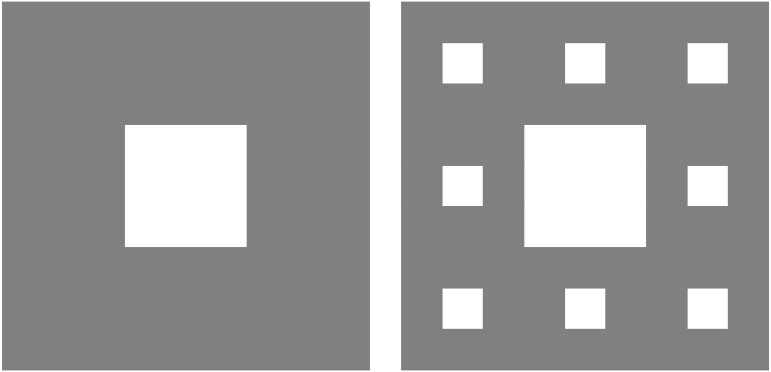
Theorem

If $\sum_{k=1}^{\infty} a_k$ *converges and* $\sum_{k=1}^{\infty} b_k$ *diverges, then*

1. $\sum_{k=1}^{\infty}(a_k + b_k)$ *diverges.*
2. *If* $c \neq 0$ *is a constant then* $\sum_{k=1}^{\infty}(c\, b_k)$ *diverges.*

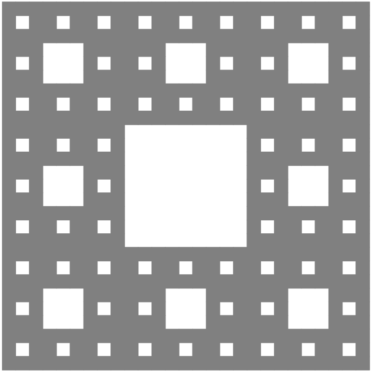
Sierpinksi Gasket (1 of 3)

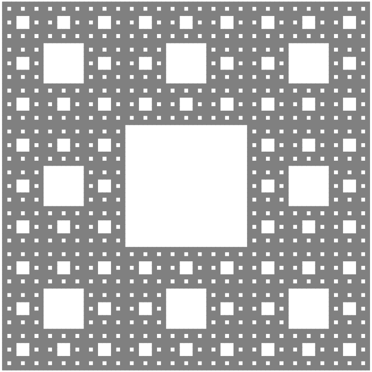
Remove the center 1/9 of a square of side 1, then remove the center ninths of the eight remaining squares, and continue. Find the total area of all the removed squares.



Sierpinksi Gasket (2 of 3)

Remove the center 1/9 of a square of side 1, then remove the center ninths of the eight remaining squares, and continue. Find the total area of all the removed squares.



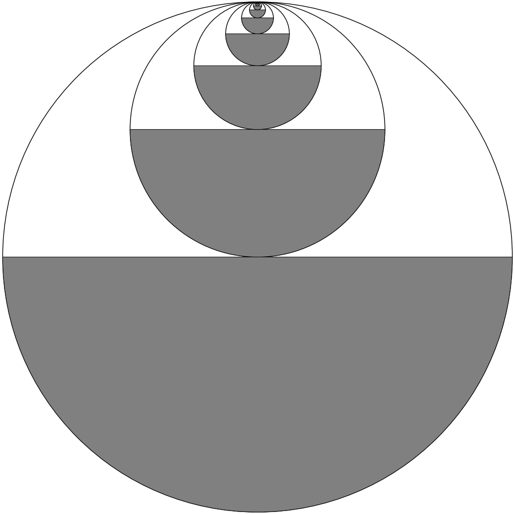


Sierpinksi Gasket (3 of 3)

Area removed:

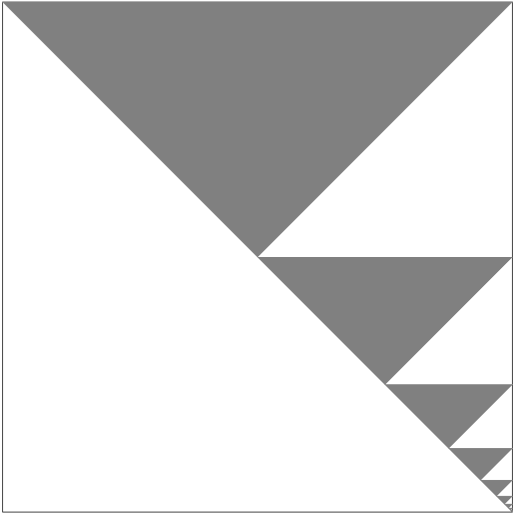
$$
\begin{aligned}
A &= \frac{1}{9} + \frac{8}{81} + \frac{64}{729} + \cdots \\
&= \frac{1}{9}\left(1 + \frac{8}{9} + \frac{64}{81} + \cdots\right) \\
&= \frac{1}{9}\left(1 + \frac{8}{9} + \left[\frac{8}{9}\right]^2 + \cdots\right) \\
&= \frac{1/9}{1 - \frac{8}{9}} \\
&= 1
\end{aligned}
$$

Half Circles



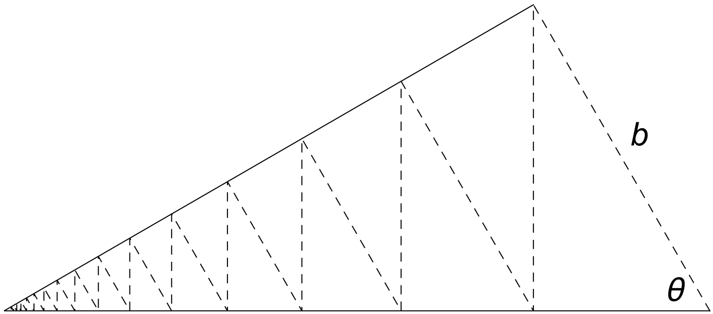
Assume the largest circle has a radius of 1 and find the sum of the shaded areas in the figure.

Triangles



Assume the square has sides of length 1 and find the sum of the shaded areas in the figure.

Right Triangles

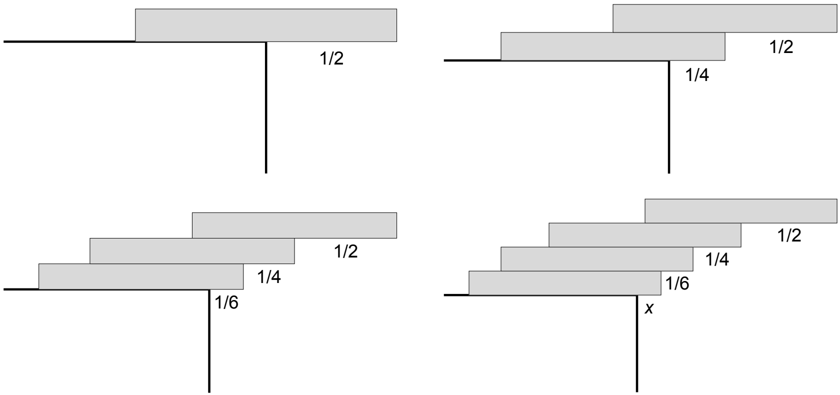


1. Express the total length of the dotted line in the triangle in terms of b and θ.
2. Compute the sum of the length of the dotted line.
3. What happens as $\theta \to \pi/2$?

Stacking Blocks

Suppose you have a large supply of rectangular blocks all of the same size and you stack them at the edge of a table with each block extending farther beyond the edge of the table than the one beneath it. Show that the top block can extend arbtrarily far beyond the edge of the table.

Stacking Blocks: Illustration



Homework

- Read Section 5.2
- Exercises: 71, 75, 79, 83, $\ldots$, 107/handout