Loan Repayment

MATH 372 *Financial Mathematics I*

J. Robert Buchanan

Department of Mathematics

Fall 2021

Objectives

In this lesson we will:

- study **loan amortization**, the issue of dividing loan payments into payment of interest and payment of principal,
- learn to use the Amortization Worksheet on the TI BA II Plus calculator,
- introduce the concept of a **sinking fund** account for repaying a loan,
- study the repayment of a loan by payments in geometric or arithmetic progression, and
- present the concept of **replacement of capital.**

Basic Principle

The total of all payments made for a loan must return the amount borrowed (the principal) and the interest owed.

A payment made against the loan can be divided into a portion which goes to pay interest and a portion which goes to repay principal.

Example

A loan of $\$50,000$ with level payments made at the end of each of the next 4 years with an effective annual interest rate of 6%.

1. Find the annual payment.
2. Find the outstanding balance on the loan immediately after the first payment.
3. Find the outstanding balance three months after the second payment.

Solutions

1. Annual payment:

$$\begin{aligned} 50,000 &= K\, a_{\overline{4}|0.06} \\ K &= \$14,429.5746 \end{aligned}$$

2. Outstanding balance after one payment: the principal has earned interest for one year.

$$50,000(1+0.06) - 14,429.5746 = \$38,570.4254$$

3. Outstanding balance after the second payment:

$$38,570.4254(1+0.06) - 14,429.5746 = \$26,455.0763.$$

Three months later the outstanding balance grows to

$$26,455.0763(1+0.06)^{0.25} = \$26,843.2740.$$

Amortized Loan

Definition

An **amortized loan** of principal amount L made at time 0 at periodic interest rate i and to be repaid by n payments in amounts $K_1, K_2, \dots, K_n$ satisfies the equation

$$L = K_1 v + K_2 v^2 + \cdots + K_n v^n = \sum_{t=1}^{n} K_t v^t.$$

Definition

The **outstanding balance** immediately after the tth payment is denoted OB_t and is the amount of principal borrowed yet to be repaid.

Amortization is the gradual reduction of the outstanding balance of a loan.

Successive Value Formula

In general,

$$\text{OB}_t(1+i) - K_{t+1} = \text{OB}_{t+1}.$$

The beginning of the payment period balance plus the interest earned during the period minus the payment at the end of the period equals the loan balance at the beginning of the next period.

This is called the **successive value formula**.

Principal and Interest Repaid

Definition

Suppose the outstanding balance on a loan immediately after the $t-1$st payment is OB_{t-1}, then the interest paid in the tth payment is denoted I_t and

$$\mathrm{I}_t = \mathrm{OB}_{t-1} i.$$

Definition

Suppose the outstanding balance on a loan immediately after the $t-1$st payment is OB_{t-1}, then the **principal repaid** in the tth payment is denoted PR_t and

$$\mathrm{PR}_t = K_t - \mathrm{I}_t = K_t - \mathrm{OB}_{t-1} i.$$

Outstanding Balances

Suppose the outstanding balance on a loan immediately after the tth payment is OB_t then

$$\begin{aligned}
\mathrm{OB}_{t+1} &= \mathrm{OB}_t(1+i) - K_{t+1} \\
&= \mathrm{OB}_t - (K_{t+1} - i\,\mathrm{OB}_t) \\
&= \mathrm{OB}_t - (K_{t+1} - \mathrm{I}_{t+1}) \\
\mathrm{OB}_{t+1} &= \mathrm{OB}_t - \mathrm{PR}_{t+1}
\end{aligned}$$

Example

A loan of \$1000 at a nominal annual interest of 10% compounded monthly is repaid in 4 monthly payments starting one month after the loan is made. The first two payments are in amount X and the last two payments are in amount $2X$. Construct the amortization table for the loan.

Solution

Treat the repayment of the loan as payments of X each month and extra payments of X in the last two months.

$$\begin{aligned} 1000 &= X a_{\overline{4}|0.10/12} + X v^2 a_{\overline{2}|0.10/12} \\ &= (3.9180)X + (1.9428)X \\ X &= 170.63 \end{aligned}$$

The amortization table will be organized in the following manner.

t	**Payment**	**Interest Due**	**Principal Repaid**	**Outstanding Balance**
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$
t	K_t	$\mathrm{I}_t = i\,\mathrm{OB}_{t-1}$	$\mathrm{PR}_t = K_t - \mathrm{I}_t$	$\mathrm{OB}_t = \mathrm{OB}_{t-1} - \mathrm{PR}_t$
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$

Amortization Table

At $t = 1$ a payment of $K_1 = 170.63$ is made. We may calculate the following quantities:

$$
\begin{aligned}
\mathrm{I}_1 &= \mathrm{OB}_0\, i = 1000\left(\frac{0.10}{12}\right) = 8.33 \\
\mathrm{PR}_1 &= K_1 - \mathrm{I}_1 = 170.63 - 8.33 = 162.30 \\
\mathrm{OB}_1 &= \mathrm{OB}_0 - \mathrm{PR}_1 = 1000 - 162.30 = 837.70
\end{aligned}
$$

t	**Payment**	**Interest Due**	**Principal Repaid**	**Outstanding Balance**
0	—	—	—	1000.00
1	170.63	8.33	162.30	837.70
2	170.63	6.98	163.65	674.05
3	341.26	5.62	335.64	338.41
4	341.26	2.82	338.44	−0.03

The other rows are calculated similarly.

Calculating Outstanding Balance

There are two common methods for calculating OB_t.

- **Retrospective method**: the outstanding balance of a loan as of time t equals the accumulated value of all the loan amounts received as of time t minus the accumulated value of all the loan repayments that have been made as of time t.
- **Prospective method**: the outstanding balance of a loan as of time t equals the present value of the loan repayments to be made after time t.

Retrospective Form of Outstanding Balance

The outstanding balances are related in the following way.

$$
\begin{aligned}
\mathrm{OB}_1 &= \mathrm{OB}_0(1+i) - K_1 \\
\mathrm{OB}_2 &= \mathrm{OB}_1(1+i) - K_2 \\
&= \mathrm{OB}_0(1+i)^2 - K_1(1+i) - K_2 \\
&\vdots \\
\mathrm{OB}_t &= \mathrm{OB}_0(1+i)^t - \sum_{j=1}^{t} K_j(1+i)^{t-j} \\
&\vdots \\
0 &= \mathrm{OB}_0(1+i)^n - \sum_{j=1}^{n} K_j(1+i)^{t-j}
\end{aligned}
$$

The accumulated amount of the original principal borrowed minus the accumulated amounts of all the payments made gives the outstanding balance.

Example

For the previous example we can use the retrospective form of the outstanding balance to find the outstanding balance just after the second payment.

$$
\begin{aligned}
\mathrm{OB}_2 &= \mathrm{OB}_0(1+i)^2 - K_1(1+i) - K_2 \\
&= 1000\left(1+\frac{0.10}{12}\right)^2 - 170.63\left(1+\frac{0.10}{12}\right) - 170.63 \\
&= 1016.74 - 172.05 - 170.63 \\
&= 674.06
\end{aligned}
$$

Prospective Form of Outstanding Balance

From the retrospective form we have

$$
\begin{aligned}
\mathrm{OB}_t &= \mathrm{OB}_0(1+i)^t - \sum_{j=1}^{t} K_j(1+i)^{t-j} \\
&= \left(\sum_{j=1}^{n} K_j v^j\right)(1+i)^t - \sum_{j=1}^{t} K_j(1+i)^{t-j} \\
&= \left(\sum_{j=1}^{n} K_j v^j\right) v^{-t} - \sum_{j=1}^{t} K_j v^{-t+j} \\
&= \sum_{j=1}^{n} K_j v^{-t+j} - \sum_{j=1}^{t} K_j v^{-t+j} = \sum_{j=t+1}^{n} K_j v^{-t+j} \\
&= K_{t+1} v + K_{t+2} v^2 + \cdots + K_n v^{n-t}.
\end{aligned}
$$

The outstanding balance is the present value (at time t) of the remaining payments.

Example

A loan of \$1500 is repaid by 6 monthly payments starting one month after the loan is made. The interest rate for the first three months is 8% compounded monthly and the interest rate for the next three months is 9% compounded monthly. The first three payments are amount X and the last three payments are amount $1.5X$. Construct the amortization table for this loan.

Solution

Treat the loan payments as the sequence $\{X, X, X, 1.5X, 1.5X, 1.5X\}$. Find the value of X.

$$
\begin{aligned}
1500 &= X\, a_{\overline{3}|0.08/12} + 1.5X\, v^3_{0.08/12}\, a_{\overline{3}|0.09/12} \\
&= (2.9604)X + (4.3458)X \\
X &= 205.30
\end{aligned}
$$

Amortization Table

t	**Payment**	**Interest Due**	**Principal Repaid**	**Outstanding Balance**
0	—	—	—	1500.00
1	205.30	10.00	195.30	1304.70
2	205.30	8.70	196.60	1108.10
3	205.30	7.39	197.91	910.19
4	307.96	6.83	301.13	609.06
5	307.96	4.57	303.39	305.67
6	307.96	2.29	305.67	0.00

Retrospective Method: Level Payment

If the payments made on the loan are level (constant), then the retrospective method can be summarized as

$$\mathrm{OB}_t = \mathrm{OB}_0(1+i)^t - K\, s_{\overline{t}|i}.$$

Example

A loan of $50,000 at a 6% annual effective rate is repaid in 10 annual level payments beginning one year from now. Use the retrospective method to find the outstanding loan balance after the sixth payment.

Prospective Method: Level Payments

If the payments made on the loan are level (constant), then the prospective method can be summarized as

$$\mathrm{OB}_t = K\, a_{\overline{n-t|}i}.$$

Example

A loan made at annual effective rate of 7.5% has 10 remaining payments of \$1051. The next payment is due in one year. What is the outstanding loan balance?

Solution

$$OB_t = K\, a_{\overline{10|}0.075} = (1051)(6.8641) = \$7,214.1491$$

Example

A loan of \$15,000 at 3.49% annual effective rate has an annual payment of $\$3,321.28$. Find the outstanding balance after the 3rd payment.

Effect of Valuation Date

For a loan of OB_0 repaid by n level payments of K when the per period effect interest rate is i, the equation of value is

$$
\begin{aligned}
OB_0 &= K\, a_{\overline{n}|i} \\
OB_0(1+i) &= K(1+i)a_{\overline{n}|i} = K + K\, a_{\overline{n-1}|i} \\
OB_0(1+i)^2 &= K(1+i)^2 a_{\overline{n}|i} = K + K(1+i) + K\, a_{\overline{n-2}|i} = K\, s_{\overline{2}|i} + K\, a_{\overline{n-2}|i} \\
&\vdots \\
OB_0(1+i)^t &= K\, s_{\overline{t}|i} + K\, a_{\overline{n-t}|i} \\
&\vdots \\
OB_0(1+i)^n &= K\, s_{\overline{n}|i}
\end{aligned}
$$

The outstanding balance after the tth payment is

$$OB_t = K\, a_{\overline{n-t}|i} = OB_0(1+i)^t - K\, s_{\overline{t}|i}.$$

Loans with Level Payments

Typically the payments on a loan remain constant and the interest rate remains constant during the life of a loan. In this case the amortization table has a regular, predictable structure.

Suppose each payment $K_t = 1$, then

- $\mathrm{OB}_0 = a_{\overline{n}|i}$
- $\mathrm{OB}_t = a_{\overline{n-t}|i}$ (prospective form)
- $\mathrm{PR}_t = v^{n-t+1}$
- $\mathrm{I}_t = 1 - v^{n-t+1}$.

Example

For a 6.5% level payment loan, the amount of principal repaid in the 3rd payment is 854.82. Find the amount of principal repaid in the 8th payment.

Solution

$$\begin{aligned}
\text{PR}_3 &= K\, v^{n-3+1} = \$854.82 \\
\text{PR}_8 &= K\, v^{n-8+1} = v^{-5}(K\, v^{n-3+1}) = (1+i)^5(K\, v^{n-3+1}) \\
\text{PR}_8 &= (1+0.065)^5 854.82 = \$1,171.18
\end{aligned}$$

Example

A homebuyer borrows \$200,000 to be repaid over a 30-year period with level monthly payments beginning one month after the loan is made. The interest rate on the loan is a nominal rate of 6% compounded monthly. Find each of the following.

- The amounts of interest and principal paid in the first year.
- The amounts of interest and principal paid in the 10th year.
- The amounts of interest and principal paid in the 30th year.

Solution (First Year)

The level payment will be

$$\begin{aligned} 200,000 &= K\, a_{\overline{360}|0.005} \\ K &= \$1,199.10. \end{aligned}$$

The outstanding balance at the end of the first year is

$$\text{OB}_{12} = K\, a_{\overline{348}|0.005} = \$197,543.98$$

which implies

$$200,000 - 197,543.98 = \$2,456.02$$

of the principal was repaid during the first year.

The amount of interest paid during the first year was

$$(12)(1199.10) - 2456.02 = \$11,933.18.$$

Solution (10th Year)

The principal repaid during the 10th year will be the difference between the outstanding balances at the end of the 9th and 10th years.

$$\text{OB}_{108} = K\, a_{\overline{252}|0.005} = \$171,580.34$$
$$\text{OB}_{120} = K\, a_{\overline{240}|0.005} = \$167,371.45$$

Thus $\$4,208.89$ was repaid in principal.

The amount of interest paid during the 10th year was

$$(12)(1199.10) - 4208.89 = \$10,180.31.$$

Solution (30th year)

The principal repaid during the 30th year (the last year of the loan) must be the outstanding balance at the end of the 29th year.

$$\text{OB}_{348} = K\, a_{\overline{12}|0.005} = \$13,932.27$$

The amount of interest paid during the 30th year was

$$(12)(1199.10) - 13932.27 = \$456.93.$$

Amortization on the Calculator

If you use the TVM worksheet to determine payments on a loan, the AMORT worksheet can help you build the amortization table.

The following linked video will explain the steps involved. https://www.youtube.com/dQXQdEbLVRY

Level Payments of Principal

Sometimes a loan is structured so that equal amounts of principal are repaid in each payment along with the interest due on the previous unpaid balance.

Example

A loan of $5,000 at an effective monthly interest rate of 1% is amortized by means of 12 monthly payments, beginning one month after the loan is made. Each payment consists of a principal repayment of 416.67 plus the interest due on the previous month's outstanding balance. Construct the amortization schedule.

Amortization Table

t	**Payment**	**Interest Due**	**Principal Repaid**	**Outstanding Balance**
0	—	—	—	5000.00
1	466.67	50.00	416.67	4583.33
2	462.50	45.83	416.67	4166.66
3	458.34	41.67	416.67	3749.99
4	454.17	37.50	416.67	3333.32
5	450.00	33.33	416.67	2916.65
6	445.84	29.17	416.67	2499.98
7	441.67	25.00	416.67	2083.31
8	437.50	20.83	416.67	1666.64
9	433.34	16.67	416.67	1249.97
10	429.17	12.50	416.67	833.30
11	425.00	8.33	416.67	416.63
12	420.84	4.17	416.67	-0.04

Variable Interest Rate Loans

A homebuyer takes out a 15-year monthly-payment variable-rate mortgage loan for $\$200,000$. The initial interest rate for the mortgage is 4.5% convertible monthly. This rate is level for the first 5 years after which it can increase or decrease.

1. Find the monthly payment for the first 5 years.
2. At the end of the first 5 years the interest rate adjusts to 8%, find the new monthly payment.

Solution

1. Monthly payment years 1–5:

$$\begin{aligned} 200,000 &= K\, a_{\overline{180}|0.045/12} \\ K &= \$1,529.9866 \end{aligned}$$

2. Monthly payment years 6-15: first find the outstanding balance after the 60th payment.

   $$\mathrm{OB}_{60} = K\, a_{\overline{120}|0.045/12} = 1529.9866(96.4893) = \$147,627.3727$$

   Treat the remaining loan as if it were for 10 years at 8%.

$$\begin{aligned} 147,627.3727 &= K\, a_{\overline{120}|0.08/12} \\ K &= \$1,791.1274 \end{aligned}$$

Loans With Graduated Payments

A loan at an effective annual rate of 6% has an initial payment of $500 and eleven additional payments. The payment amount increases by 5% each year.

1. Find the amount borrowed.
2. Find the outstanding balance after the 6th payment.

Loans With Graduated Payments

A loan at an effective annual rate of 7% has an initial payment of \$250 and seven additional payments. The payment amount increases by \$25 each year.

1. Find the amount borrowed.
2. Find the outstanding balance after the 3rd payment.

Sinking Fund

Consider a loan which calls for **periodic payments of interest only** along with **repayment of the full principal amount** at the end of the loan.

- Interest payments, $\mathrm{I}_t = L\,i$ for $t = 1, 2, \ldots, n$.
- Principal payments, $\mathrm{PR}_t = 0$ for $t = 1, 2, \ldots, n-1$ and $\mathrm{PR}_n = L$.
- the schedule of outstanding balances is

$$\mathrm{OB}_1 = \mathrm{OB}_2 = \cdots = \mathrm{OB}_{n-1} = L \text{ and } \mathrm{OB}_n = 0.$$

The borrower may accumulate L by making periodic deposits into an interest-earning savings account called a **sinking fund.**

Total periodic outlay:

$$L\,i + \frac{L}{s_{\overline{n}|j}}$$

for $t = 1, 2, \ldots, n$ where j is the interest rate earned by the sinking fund.

Example

A loan of $200,000 is to be repaid by 20 payments of interest only (with $i = 8\%$) at the end of the year and the repayment of the principal in one lump sum at the end of 20 years. The borrower will accumulate the principal in a sinking fund earning $j = 6\%$ by making 20 level deposits starting one year after the loan is made.

1. Find the borrower's total annual outlay.
2. Find the annual rate of interest i' for which the amortization of the loan at rate i' results in the same total outlay.

Solution

1. To accumulate 200,000 at 6% in 20 years requires

$$200,000 = X\, s_{\overline{20}|0.06} \implies X = 5,436.91$$

   annual deposits in the sinking fund and interest payments of

$$(200,000)(0.08) = 16,000.$$

   Thus the total annual outlay is $21,436.91$.

2. The equivalent annual rate of interest for amortization of the loan is

$$200,000 = 21,436.91\, a_{\overline{20}|i'} \implies i' = 8.6961\%.$$

Sinking Fund Schedule

Even though we think of the sinking fund method of loan payment as paying the principal back in one lump payment at the end of the loan payment, an amortization schedule can be created.

- The outstanding balance after the tth payment is the principal borrowed minus the accumulated value of the sinking fund deposits:

$$\mathrm{OB}_t = L - \frac{L}{s_{\overline{n}|j}}\, s_{\overline{t}|j} = L\left[1 - \frac{s_{\overline{t}|j}}{s_{\overline{n}|j}}\right].$$

- The principal repaid in the tth payment is the difference between OB_t and OB_{t-1}:

$$\mathrm{PR}_t = L\left[1 - \frac{s_{\overline{t}|j}}{s_{\overline{n}|j}} - 1 + \frac{s_{\overline{t-1}|j}}{s_{\overline{n}|j}}\right] = L\left[\frac{s_{\overline{t}|j} - s_{\overline{t-1}|j}}{s_{\overline{n}|j}}\right] = \frac{L(1+j)^{t-1}}{s_{\overline{n}|j}}.$$

- The net interest paid in the tth payment is

$$\mathrm{I}_t = L\,i - \frac{L\,j\,s_{\overline{t-1}|j}}{s_{\overline{n}|j}} = L\left[i - \frac{(1+j)^{t-1} - 1}{s_{\overline{n}|j}}\right].$$

Selling a Loan

- Suppose a lender is receiving payments of interest only for n periods with the principal repaid at the end of the loan.
- The sequence of payments is

$$L\,i, \quad L\,i, \quad \ldots, L\,i, \quad L(1+i).$$

- Suppose the lender sells the loan to an investor who values the loan using periodic interest rate j. In this case the investor has purchased the right to receive the interest payments and the lump sum repayment of the principal.

$$A = L\,v_j^n + (L\,i)\,a_{\overline{n}|j}$$

Capitalization of Interest

- Suppose the payment made on a loan at time t is smaller than the interest owed. In this situation the outstanding balance will increase, called **negative amortization.**
- $K_t < \mathrm{I}_t$ which implies $\mathrm{PR}_t = K_t - \mathrm{I}_t < 0$.
- In this case

$$\mathrm{OB}_t = \mathrm{OB}_{t-1} - (K_t - \mathrm{I}_t) > \mathrm{OB}_{t-1}$$

  and the outstanding balance increases by the amount of unpaid interest. The unpaid interest is **capitalized** and added to the amount owed.

Example

A loan of \$75,000 has payments at the end of each month for 10 years. The first five years the payments are X each month and for the last five years the payments are $3X$ each month. Interest is at a nominal rate of 10% compounded monthly. Find the outstanding balance at the end of the first year.

Solution (1 of 2)

Find the value of X.

$$\begin{aligned} 75,000 &= X\, a_{\overline{60}|0.10/12} + 3X\, v^{60}\, a_{\overline{60}|0.10/12} \\ &= (47.0654)X + (85.8174)X \\ &= 132.8828X \\ X &= 564.41 \end{aligned}$$

Solution (2 of 2)

- Using the retrospective method,

$$\begin{aligned} \text{OB}_{12} &= 75,000\left(1+\frac{0.10}{12}\right)^{12} - 564.41\, s_{\overline{12}|0.10/12} \\ &= 82,853.48 - 7,092.13 = 75,761.35. \end{aligned}$$

- Using the prospective method,

$$\begin{aligned} \text{OB}_{12} &= 564.41\, a_{\overline{48}|0.10/12} + (3)(564.41)v^{48}\, a_{\overline{60}|0.10/12} \\ &= 22,253.65 + 53,508.09 = 75,761.74. \end{aligned}$$

Makeham's Formula

$$
\begin{aligned}
A &= L\,v_j^n + (L\,i)\,a_{\overline{n}|j} \\
&= L\,v_j^n + (L\,i)\left(\frac{1 - v_j^n}{j}\right) \\
&= L\,v_j^n + \frac{i}{j}\left(L - L\,v_j^n\right) \\
A &= K + \frac{i}{j}(L - K)
\end{aligned}
$$

where $K = Lv_j^n$.

The last formula is called **Makeham's Formula.**

Example

A loan of \$150,000 is to be repaid in 5 annual payments of principal of \$30,000 each starting one year from now and monthly interest payments on the outstanding balance. The interest rate is $i^{(12)} = 0.06$. Three years after the loan is made (immediately after the third principal payment and 36th monthly interest payment) the lender sells the loan to an investor. Find the price paid by the investor if she values the remaining payments at a nominal annual interest rate convertible monthly of

1. 6%,
2. 10%,
3. 12%.

Solution (1 of 2)

At the time the loan is sold the outstanding balance is $L = \$60,000$ with 2 principal payments and 24 interest payments to be made. The present value of the principal payments is

$$K = 30,000 \left[v^{12} + v^{24} \right] = 30,000 v^{12} (1 + v^{12}).$$

Using Makeham's formula the amount paid for the loan by the investor is

$$A = K + \frac{i}{j}(L - K) = K + \frac{0.005}{j}(60,000 - K).$$

Solution (2 of 2)

1. When $j^{(12)} = 0.06$,

$$\begin{aligned} K &= 30,000(1.005)^{-12}(1 + (1.005)^{-12}) = 54,872.73 \\ A &= 54,872.73 + \frac{0.005}{0.005}(60,000 - 54,872.73) = 60,000 \end{aligned}$$

2. When $j^{(12)} = 0.10$,

$$\begin{aligned} K &= 30,000(1.0083)^{-12}(1 + (1.0083)^{-12}) = 51,737.95 \\ A &= 51,737.95 + \frac{0.005}{0.0083}(60,000 - 51,737.95) = 56,695.18 \end{aligned}$$

3. When $j^{(12)} = 0.12$,

$$\begin{aligned} K &= 30,000(1.01)^{-12}(1 + (1.01)^{-12}) = 50,247.67 \\ A &= 50,247.67 + \frac{0.005}{0.01}(60,000 - 50,247.67) = 50,735.29 \end{aligned}$$

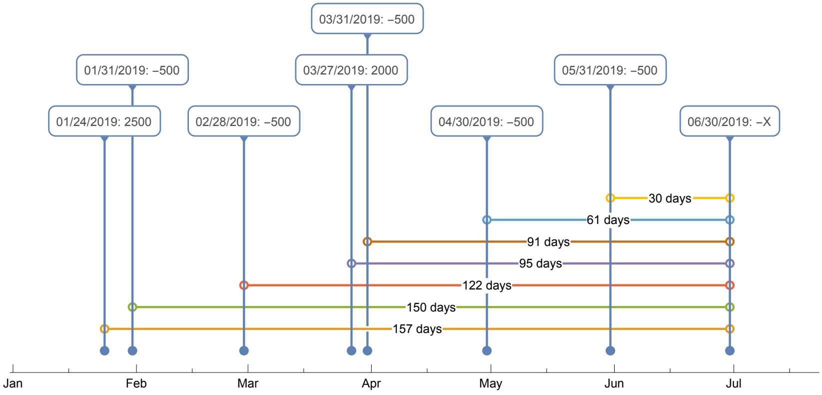
Merchant's Rule

All loan amounts advanced and all loan repayments made are accumulated with *simple interest* until the settlement date. At the settlement date the aggregate accumulated values of amounts loaned must equal the aggregate accumulated values of amounts repaid.

Example

Harry borrows $2,500 on January 24 and makes payments of $500 each on the last day of every month. On March 27 he borrows an additional $2,000 and continues payments of $500 until May 31. On June 30 he pays the remainder of the loan. What payment must be made on June 30 if the annual interest rate is 8% and the loan is based on Merchant's Rule (assume non-leap year).

Illustration



Solution

$$
\begin{aligned}
2500\left[1+0.08\left(\frac{157}{365}\right)\right]+2000\left[1+0.08\left(\frac{95}{365}\right)\right] & \\
= \quad & 500\left[5+0.08\left(\frac{150+122+91+61+30}{365}\right)\right]+X \\
4627.67 \quad = \quad & 2,549.75+X \\
X \quad = \quad & 2,077.92
\end{aligned}
$$

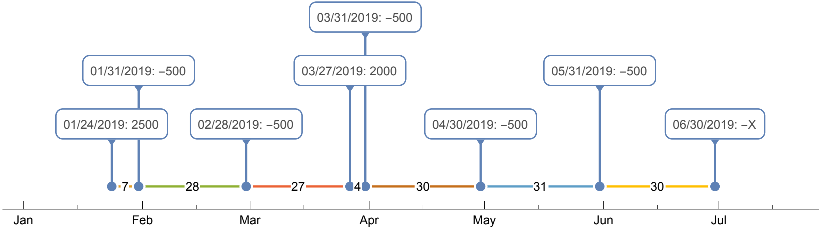
US Rule

Interest is computed each time a payment is made or an additional amount is loaned. The interest is based on simple interest from the time of the previous payment or disbursement. The balance on the loan after the current payment is the previous balance, plus interest accrued, minus the current payment (or plus the current disbursement). This is also known as the **actuarial method**.

Example

Harry borrows $2,500 on January 24 and makes payments of $500 each on the last day of every month. On March 27 he borrows an additional $2,000 and continues payments of $500 until May 31. On June 30 he pays the remainder of the loan. What payment must be made on June 30 if the annual interest rate is 8% and the loan is based on the US Rule (assume non-leap year).

Illustration



Solution (1 of 2)

The interest and outstanding balance calculations can be summarized in an amortization table.

For example, on January 31:

accrued interest: $2500(0.08)\left(\frac{7}{365}\right) = 3.84$

payment: 500

outstanding balance: $2500 + 3.84 - 500 = 2003.84$

Solution (2 of 2)

Date	Accrued Interest	Payment	Outstanding Balance
JAN 24	—	(2500)	2500.00
JAN 31	3.84	500	2003.84
FEB 28	12.30	500	1516.14
MAR 27	8.97	(2000)	3525.11
MAR 31	3.09	500	3028.20
APR 30	19.91	500	2548.11
MAY 31	17.31	500	2065.42
JUN 30	13.58	2079	0.00

Homework

- Read Chapter 3
- Exercises: on handout