Accelerating Convergence

MATH 375 *Numerical Analysis*

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Linear Convergence

Theorem

Suppose $g \in \mathcal{C}[a,b]$ and $g(x) \in [a,b]$ for all $x \in [a,b]$ and $g' \in \mathcal{C}(a,b)$ with

$$|g'(x)| \leq k < 1 \text{ for all } x \in (a,b).$$

If $g'(p) \neq 0$ then for any $p_0 \neq p$ in $[a,b]$, the sequence $p_n = g(p_{n-1})$, $n \geq 1$ converges ***linearly*** *to the unique fixed point p in $[a,b]$.*

Motivation

- We have seen that most fixed-point methods for root finding converge only linearly to a solution.
- Newton's Method is the only root finding technique we have proved converges quadratically.
- Today we describe a technique which can be used accelerate the convergence of any linearly convergent sequence.

Linear Convergence

Suppose sequence $\{p_n\}_{n=0}^{\infty}$ converges linearly to p.

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Assume the signs of $p_{n+2} - p$, $p_{n+1} - p$, and $p_n - p$ are all the same.

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Assume the signs of $p_{n+2} - p$, $p_{n+1} - p$, and $p_n - p$ are all the same.

For large n then

$$\frac{p_{n+2} - p}{p_{n+1} - p} \approx \frac{p_{n+1} - p}{p_n - p}.$$

Re-writing the Sequence (1 of 2)

$$\begin{aligned} \frac{p_{n+2} - p}{p_{n+1} - p} &\approx \frac{p_{n+1} - p}{p_n - p} \\ (p_{n+1} - p)^2 &\approx (p_{n+2} - p)(p_n - p) \end{aligned}$$

Solve this quadratic equation for p.

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Solve this quadratic equation for p.

$$\begin{aligned} p_{n+1}^2 - 2p_{n+1}p + p^2 &\approx p_{n+2}p_n - (p_{n+2} + p_n)p + p^2 \\ p_{n+1}^2 - p_{n+2}p_n + (p_{n+2} - 2p_{n+1} + p_n)p &\approx 0 \\ p &\approx \frac{p_{n+2}p_n - p_{n+1}^2}{p_{n+2} - 2p_{n+1} + p_n} \end{aligned}$$

Remark: for large n these three approximations to p can be combined to produce a better approximation to p.

Re-writing the Sequence (2 of 2)

Since $p_{n+2}p_n \approx p_{n+1}^2$ we would like to avoid the subtraction of nearly equal quantities in the numerator below.

$$p \approx \frac{p_{n+2}p_n - p_{n+1}^2}{p_{n+2} - 2p_{n+1} + p_n}$$

Add and subtract the terms p_n^2 and $2p_np_{n+1}$ in the numerator.

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Add and subtract the terms p_n^2 and $2p_np_{n+1}$ in the numerator.

$$p \approx \frac{-p_{n+1}^2 + 2p_np_{n+1} - p_n^2 + p_{n+2}p_n + p_n^2 - 2p_np_{n+1}}{p_{n+2} - 2p_{n+1} + p_n}$$

Now factor the numerator.

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Now factor the numerator.

$$\begin{aligned} p &\approx \frac{-(p_{n+1} - p_n)^2 + p_n(p_{n+2} + p_n - 2p_{n+1})}{p_{n+2} - 2p_{n+1} + p_n} \\ &= p_n - \frac{(p_{n+1} - p_n)^2}{p_{n+2} - 2p_{n+1} + p_n} \end{aligned}$$

Aitken's Δ^2 Method

Define a new sequence $\{\hat{p}_n\}_{n=0}^{\infty}$ as

$$\hat{p}_n = p_n - \frac{(p_{n+1} - p_n)^2}{p_{n+2} - 2p_{n+1} + p_n}.$$

We would like to show that

- $\lim_{n\to\infty} \hat{p}_n = p = \lim_{n\to\infty} p_n$, and
- sequence $\{\hat{p}_n\}_{n=0}^{\infty}$ converges to p faster than sequence $\{p_n\}_{n=0}^{\infty}$.

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We will come to call this iterative method as **Aitken's Δ^2 Method**.

Example (1 of 2)

Define $p_n = \sin\left(\frac{1}{n}\right)$ for $n \geq 1$. Find the limit of this sequence as $n \to \infty$ and compare its convergence to that of the sequence $\hat{p}_n$.

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n	p_n
1	0.841471
2	0.479426
3	0.327195
4	0.247404

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n	p_n
1	0.841471
2	0.479426
3	0.327195
4	0.247404

$$\hat{p}_1 = p_1 - \frac{(p_2 - p_1)^2}{p_3 - 2p_2 + p_1} \approx 0.216744$$

$$\hat{p}_2 = p_2 - \frac{(p_3 - p_2)^2}{p_4 - 2p_3 + p_2} \approx 0.159517$$

Example (2 of 2)

In the table below we have computed a few more terms of the sequences $\{p_n\}_{n=1}^{\infty}$ and $\{\hat{p}_n\}_{n=1}^{\infty}$.

n	p_n	$\hat{p}_n$
1	0.841471	0.216744
2	0.479426	0.159517
3	0.327195	0.122193
4	0.247404	0.098604
5	0.198669	0.082537
6	0.165896	0.070932
7	0.142372	0.062169
8	0.124675	0.055324
9	0.110883	0.049832
10	0.099833	0.045328
$\vdots$	$\vdots$	$\vdots$

Example

Let $p_0 = 1$ and $p_n = 3^{-p_{n-1}}$ for $n \geq 1$. Determine the first five terms of the sequence $\{\hat{p}_n\}_{n=1}^{\infty}$ using Aitken's Δ^2 method.

Solution

n	p_n	$\hat{p}_n$
1	0.333333	
2	0.693361	
3	0.466856	
4	0.598761	
5	0.517987	
6	0.566054	
7	0.536938	

Solution

n	p_n	$\hat{p}_n$
1	0.333333	0.554327
2	0.693361	
3	0.466856	
4	0.598761	
5	0.517987	
6	0.566054	
7	0.536938	

$$\hat{p}_1 = p_1 - \frac{(p_2 - p_1)^2}{p_3 - 2p_2 + p_1} \approx 0.554327$$

Solution

n	p_n	$\hat{p}_n$
1	0.333333	0.554327
2	0.693361	0.550216
3	0.466856	
4	0.598761	
5	0.517987	
6	0.566054	
7	0.536938	

$$\hat{p}_1 = p_1 - \frac{(p_2 - p_1)^2}{p_3 - 2p_2 + p_1} \approx 0.554327$$

$$\hat{p}_2 = p_2 - \frac{(p_3 - p_2)^2}{p_4 - 2p_3 + p_2} \approx 0.550216$$

Solution

n	p_n	$\hat{p}_n$
1	0.333333	0.554327
2	0.693361	0.550216
3	0.466856	0.548664
4	0.598761	0.548121
5	0.517987	0.547912
6	0.566054	
7	0.536938	

$$\hat{p}_1 = p_1 - \frac{(p_2 - p_1)^2}{p_3 - 2p_2 + p_1} \approx 0.554327$$

$$\hat{p}_2 = p_2 - \frac{(p_3 - p_2)^2}{p_4 - 2p_3 + p_2} \approx 0.550216$$

Forward Difference

Definition

Given sequence $\{p_n\}_{n=0}^{\infty}$, the **forward difference** denoted Δp_n is defined as

$$\Delta p_n = p_{n+1} - p_n, \quad \text{for } n \geq 0.$$

Higher powers of the operator Δ are defined recursively as

$$\Delta^k p_n = \Delta \left(\Delta^{k-1} p_n \right), \quad \text{for } n \geq 2.$$

Forward Difference (2 of 2)

$$
\begin{aligned}
\Delta p_n &= p_{n+1} - p_n \\
\Delta^2 p_n &= \Delta(p_{n+1} - p_n) = p_{n+2} - p_{n+1} - (p_{n+1} - p_n) \\
&= p_{n+2} - 2p_{n+1} + p_n \\
\Delta^3 p_n &= \Delta(p_{n+2} - 2p_{n+1} + p_n) \\
&= p_{n+3} - 3p_{n+2} + 3p_{n+1} - p_n \\
&\vdots
\end{aligned}
$$

Forward Difference (2 of 2)

$$
\begin{aligned}
\Delta p_n &= p_{n+1} - p_n \\
\Delta^2 p_n &= \Delta(p_{n+1} - p_n) = p_{n+2} - p_{n+1} - (p_{n+1} - p_n) \\
&= p_{n+2} - 2p_{n+1} + p_n \\
\Delta^3 p_n &= \Delta(p_{n+2} - 2p_{n+1} + p_n) \\
&= p_{n+3} - 3p_{n+2} + 3p_{n+1} - p_n \\
&\vdots
\end{aligned}
$$

We can write Aitken's Δ^2 Method as

$$\hat{p}_n = p_n - \frac{(\Delta p_n)^2}{\Delta^2 p_n}.$$

Convergence of Aitken's Δ^2 Method

Theorem

Let $\{p_n\}_{n=0}^{\infty}$ be a sequence converging linearly to p with asymptotic error constant $\lambda < 1$. The sequence $\{\hat{p}_n\}_{n=0}^{\infty}$ converges to p faster than $\{p_n\}_{n=0}^{\infty}$ in the sense that

$$\lim_{n\to\infty} \frac{\hat{p}_n - p}{p_n - p} = 0.$$

Proof (1 of 5)

- By assumption $\{p_n\}_{n=0}^{\infty}$ converges linearly to p, *i.e.*

$$0 < \lim_{n\to\infty} \frac{|p_{n+1} - p|}{|p_n - p|} = \lambda < 1.$$

- Assume the signs of $p_k - p$ are all the same for sufficiently large k.

Proof (1 of 5)

- By assumption $\{p_n\}_{n=0}^{\infty}$ converges linearly to p, *i.e.*

$$0 < \lim_{n\to\infty} \frac{|p_{n+1} - p|}{|p_n - p|} = \lambda < 1.$$

- Assume the signs of $p_k - p$ are all the same for sufficiently large k.
- Define a new sequence $\delta_n = \dfrac{p_{n+1} - p}{p_n - p} - \lambda$ for $n \geq 0$. Note that $\lim_{n\to\infty} \delta_n = 0$.

Proof (2 of 5)

Consider

$$\frac{\hat{p}_n - p}{p_n - p} = \frac{1}{p_n - p}\left(p_n - \frac{(\Delta p_n)^2}{\Delta^2 p_n} - p\right)$$

Proof (2 of 5)

Consider

$$\begin{aligned}
\frac{\hat{p}_n - p}{p_n - p} &= \frac{1}{p_n - p}\left(p_n - \frac{(\Delta p_n)^2}{\Delta^2 p_n} - p\right) \\
&= \frac{1}{p_n - p}\left(p_n - p - \frac{(\Delta p_n)^2}{\Delta^2 p_n}\right) \\
&= 1 - \frac{(\Delta p_n)^2}{(p_n - p)\Delta^2 p_n}
\end{aligned}$$

Proof (2 of 5)

Consider

$$
\begin{aligned}
\frac{\hat{p}_n - p}{p_n - p} &= \frac{1}{p_n - p}\left(p_n - \frac{(\Delta p_n)^2}{\Delta^2 p_n} - p\right) \\
&= \frac{1}{p_n - p}\left(p_n - p - \frac{(\Delta p_n)^2}{\Delta^2 p_n}\right) \\
&= 1 - \frac{(\Delta p_n)^2}{(p_n - p)\Delta^2 p_n} \\
&= 1 - \frac{(p_{n+1} - p_n)^2}{(p_n - p)(p_{n+2} - 2p_{n+1} + p_n)} \\
&= 1 - \frac{((p_{n+1} - p) - (p_n - p))^2}{(p_n - p)((p_{n+2} - p) - 2(p_{n+1} - p) + (p_n - p))}
\end{aligned}
$$

Proof (3 of 5)

Recall

$$\begin{aligned}\frac{p_{n+1}-p}{p_n-p} &= \delta_n+\lambda \\ p_{n+1}-p &= (\delta_n+\lambda)(p_n-p)\end{aligned}$$

Proof (3 of 5)

Recall

$$\begin{aligned} \frac{p_{n+1} - p}{p_n - p} &= \delta_n + \lambda \\ p_{n+1} - p &= (\delta_n + \lambda)(p_n - p) \end{aligned}$$

This implies

$$\begin{aligned} p_2 - p &= (\delta_1 + \lambda)(p_1 - p) \\ p_3 - p &= (\delta_2 + \lambda)(p_2 - p) = (\delta_1 + \lambda)(\delta_2 + \lambda)(p_1 - p) \\ &\vdots \\ p_{n+1} - p &= (p_1 - p)\prod_{i=1}^{n}(\delta_i + \lambda). \end{aligned}$$

Proof (4 of 5)

Consequently

$$
\begin{aligned}
&\frac{\hat{p}_n - p}{p_n - p} \\
&= 1 - \frac{((p_{n+1} - p) - (p_n - p))^2}{(p_n - p)((p_{n+2} - p) - 2(p_{n+1} - p) + (p_n - p))} \\
&= 1 - \frac{\left((p_1 - p) \prod_{i=1}^{n} (\delta_i + \lambda) - (p_1 - p) \prod_{i=1}^{n-1} (\delta_i + \lambda) \right)^2}{(p_1 - p) \prod_{i=1}^{n-1} (\delta_i + \lambda) \left((p_1 - p) \prod_{i=1}^{n+1} (\delta_i + \lambda) - 2(p_1 - p) \prod_{i=1}^{n} (\delta_i + \lambda) + (p_1 - p) \prod_{i=1}^{n-1} (\delta_i + \lambda) \right)} \\
&= 1 - \frac{\left(\prod_{i=1}^{n} (\delta_i + \lambda) - \prod_{i=1}^{n-1} (\delta_i + \lambda) \right)^2}{\prod_{i=1}^{n-1} (\delta_i + \lambda) \left(\prod_{i=1}^{n+1} (\delta_i + \lambda) - 2 \prod_{i=1}^{n} (\delta_i + \lambda) + \prod_{i=1}^{n-1} (\delta_i + \lambda) \right)} \\
&= 1 - \frac{\left(\prod_{i=1}^{n-1} (\delta_i + \lambda) \right)^2 (\delta_n + \lambda - 1)^2}{\left(\prod_{i=1}^{n-1} (\delta_i + \lambda) \right)^2 ((\delta_{n+1} + \lambda)(\delta_n + \lambda) - 2(\delta_n + \lambda) + 1)}
\end{aligned}
$$

Proof (5 of 5)

$$\frac{\hat{p}_n - p}{p_n - p} = 1 - \frac{\left(\prod_{i=1}^{n-1}(\delta_i + \lambda)\right)^{2} (\delta_n+\lambda-1)^2}{\left(\prod_{i=1}^{n-1}(\delta_i + \lambda)\right)^{2} ((\delta_{n+1}+\lambda)(\delta_n+\lambda)-2(\delta_n+\lambda)+1)}$$

$$= 1 - \frac{(\delta_n + \lambda - 1)^2}{(\delta_{n+1} + \lambda)(\delta_n + \lambda) - 2(\delta_n + \lambda) + 1}$$

Proof (5 of 5)

$$\begin{aligned}
\frac{\hat{p}_n - p}{p_n - p} &= 1 - \frac{\left(\prod_{i=1}^{n-1}(\delta_i + \lambda)\right)^2 (\delta_n + \lambda - 1)^2}{\left(\prod_{i=1}^{n-1}(\delta_i + \lambda)\right)^2 ((\delta_{n+1} + \lambda)(\delta_n + \lambda) - 2(\delta_n + \lambda) + 1)} \\
&= 1 - \frac{(\delta_n + \lambda - 1)^2}{(\delta_{n+1} + \lambda)(\delta_n + \lambda) - 2(\delta_n + \lambda) + 1}
\end{aligned}$$

Taking the limit as $n \to \infty$:

$$\begin{aligned}
\lim_{n\to\infty} \frac{\hat{p}_n - p}{p_n - p} &= \lim_{n\to\infty} \left[1 - \frac{(\delta_n + \lambda - 1)^2}{(\delta_{n+1} + \lambda)(\delta_n + \lambda) - 2(\delta_n + \lambda) + 1}\right] \\
&= 1 - \frac{(\lambda - 1)^2}{\lambda^2 - 2\lambda + 1} \\
&= 0
\end{aligned}$$

Example

Suppose $f(x) = \frac{1}{8}x^3 - x^2 + 2x + 1$. We saw earlier that f has a fixed point at $p = 2$ and converges linearly.

n	x_n	$\|2 - x_n\|$	$\hat{x}_n$	$\|2 - \hat{x}_n\|$	$\frac{\|2-\hat{x}_n\|}{\|2-x_n\|^2}$
0	1.750000000	0.250×10^{0}	1.995068425	0.493×10^{-2}	1.55
1	2.107421875	0.107×10^{0}	1.999022858	0.977×10^{-3}	1.30
2	1.943559146	0.564×10^{-1}	1.999737172	0.263×10^{-3}	1.36
3	2.027401560	0.274×10^{-1}	1.999937151	0.628×10^{-4}	1.32
4	1.986114081	0.139×10^{-1}	1.999983969	0.160×10^{-4}	1.34
5	2.006894420	0.689×10^{-2}	1.999996034	0.397×10^{-5}	1.33
6	1.996540948	0.346×10^{-2}	1.999999003	0.997×10^{-6}	1.33
7	2.001726530	0.173×10^{-2}	1.999999752	0.248×10^{-6}	1.33
8	1.999135991	0.864×10^{-3}	1.999999938	0.622×10^{-7}	1.33
9	2.000431818	0.432×10^{-3}	2.000000000		
10	1.999784044	0.216×10^{-3}	2.000000000		

Aitken's delta-squared sequence converges to 2 faster than the fixed point sequence.

Steffensen's Method

We can use Aitken's Δ^2 Method to accelerate the convergence of any linearly convergent sequence generated by fixed-point iteration.

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Consider the fixed point problem $g(x) = x$ and an initial approximation p_0. Calculate

$$
\begin{aligned}
p_0^{(0)} &= p_0 \\
p_1^{(0)} &= g(p_0^{(0)}) \\
p_2^{(0)} &= g(p_1^{(0)})
\end{aligned}
$$

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$$
\begin{aligned}
p_0^{(0)} &= p_0 & \qquad p_0^{(1)} &= \left\{\Delta^2\right\}(p_0^{(0)}) \\
p_1^{(0)} &= g(p_0^{(0)}) & \qquad p_1^{(1)} &= g(p_0^{(1)}) \\
p_2^{(0)} &= g(p_1^{(0)}) & \qquad p_2^{(1)} &= g(p_1^{(1)})
\end{aligned}
$$

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\begin{aligned}
p_0^{(0)} &= p_0 & \qquad p_0^{(1)} &= \{\Delta^2\}(p_0^{(0)}) & \qquad p_0^{(2)} &= \{\Delta^2\}(p_0^{(1)}) \\
p_1^{(0)} &= g(p_0^{(0)}) & \qquad p_1^{(1)} &= g(p_0^{(1)}) & \qquad p_1^{(2)} &= g(p_0^{(2)}) \\
p_2^{(0)} &= g(p_1^{(0)}) & \qquad p_2^{(1)} &= g(p_1^{(1)}) & \qquad p_2^{(2)} &= g(p_1^{(2)})
\end{aligned}
$$

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Consider the fixed point problem $g(x) = x$ and an initial approximation p_0. Calculate

$$
\begin{array}{lllllllll}
p_0^{(0)} & = & p_0 & p_0^{(1)} & = & \{\Delta^2\}(p_0^{(0)}) & p_0^{(2)} & = & \{\Delta^2\}(p_0^{(1)}) \\
p_1^{(0)} & = & g(p_0^{(0)}) & p_1^{(1)} & = & g(p_0^{(1)}) & p_1^{(2)} & = & g(p_0^{(2)}) \\
p_2^{(0)} & = & g(p_1^{(0)}) & p_2^{(1)} & = & g(p_1^{(1)}) & p_2^{(2)} & = & g(p_1^{(2)})
\end{array}
$$

Every 3rd term is calculated using the Δ^2 method, all other terms use fixed-point iteration.

Algorithm

INPUT initial approximation p_0, tolerance ϵ, maximum iterations N.

STEP 1 Set $i = 1$.

STEP 2 While $i \leq N$ do STEPS 3–7.

- **STEP 3** Set $p_1 = g(p_0)$; $p_2 = g(p_1)$.
- **STEP 4** If $|p_2 - p_1| < \epsilon$ then OUTPUT p_2; STOP.
- **STEP 5** Set $p = p_0 - \dfrac{(p_1 - p_0)^2}{p_2 - 2p_1 + p_0}$.
- **STEP 6** If $|p - p_2| < \epsilon$ then OUTPUT p; STOP.
- **STEP 7** Set $p_0 = p$; $i = i + 1$.

STEP 8 OUTPUT "Method failed after N iterations."; STOP.

Example

Use Steffensen's Method to approximate the solution to $x - 3^{-x} = 0$ for $x \in [0, 1]$ with $\epsilon = 10^{-6}$

- Let $g(x) = 3^{-x}$.
- Choose $p_0 = 0.1$.

Solution

$$
\begin{aligned}
p_0^{(0)} &= 0.100000 \\
p_1^{(0)} &= g(p_0^{(0)}) = 0.895958 \\
p_2^{(0)} &= g(p_1^{(0)}) = 0.373697 \\
p_0^{(1)} &= \{\Delta^2\}(p_0^{(0)}) = p_0^{(0)} - \frac{(p_1^{(0)} - p_0^{(0)})^2}{p_2^{(0)} - 2p_1^{(0)} + p_0^{(0)}} = 0.580610 \\
p_1^{(1)} &= g(p_0^{(1)}) = 0.528419 \\
p_2^{(1)} &= g(p_1^{(1)}) = 0.559603 \\
p_0^{(2)} &= \{\Delta^2\}(p_0^{(1)}) = p_0^{(1)} - \frac{(p_1^{(1)} - p_0^{(1)})^2}{p_2^{(1)} - 2p_1^{(1)} + p_0^{(1)}} = 0.547940 \\
p_1^{(2)} &= g(p_0^{(2)}) = 0.547730 \\
p_2^{(2)} &= g(p_1^{(2)}) = 0.547856 \\
p_0^{(3)} &= \{\Delta^2\}(p_0^{(2)}) = p_0^{(2)} - \frac{(p_1^{(2)} - p_0^{(2)})^2}{p_2^{(2)} - 2p_1^{(2)} + p_0^{(2)}} = 0.547809 \\
p_1^{(3)} &= g(p_0^{(2)}) = 0.547809 \\
p_2^{(3)} &= g(p_1^{(2)}) = 0.547809
\end{aligned}
$$

Example

Use Steffensen's Method to approximate $\sqrt[3]{25}$ to within $\epsilon = 10^{-5}$ using $p_0 = 3$.

- Let $g(x) = \frac{25}{x^2+1}$.
- Choose $p_0 = 3.0$.

Solution

$$
\begin{aligned}
p_0^{(0)} &= 3.00000 \\
p_1^{(0)} &= g(p_0^{(0)}) = 2.50000 \\
p_2^{(0)} &= g(p_1^{(0)}) = 3.44828 \\
p_0^{(1)} &= \{\Delta^2\}(p_0^{(0)}) = p_0^{(0)} - \frac{(p_1^{(0)} - p_0^{(0)})^2}{p_2^{(0)} - 2p_1^{(0)} + p_0^{(0)}} = 2.82738 \\
p_1^{(1)} &= g(p_0^{(1)}) = 2.77961 \\
p_2^{(1)} &= g(p_1^{(1)}) = 2.86493 \\
p_0^{(2)} &= \{\Delta^2\}(p_0^{(1)}) = p_0^{(1)} - \frac{(p_1^{(1)} - p_0^{(1)})^2}{p_2^{(1)} - 2p_1^{(1)} + p_0^{(1)}} = 2.81023 \\
p_1^{(2)} &= g(p_0^{(2)}) = 2.80981 \\
p_2^{(2)} &= g(p_1^{(2)}) = 2.81056 \\
p_0^{(3)} &= \{\Delta^2\}(p_0^{(2)}) = p_0^{(2)} - \frac{(p_1^{(2)} - p_0^{(2)})^2}{p_2^{(2)} - 2p_1^{(2)} + p_0^{(2)}} = 2.81008 \\
p_1^{(3)} &= g(p_0^{(2)}) = 2.81008 \\
p_2^{(3)} &= g(p_1^{(2)}) = 2.81008
\end{aligned}
$$

Final Result

Remark: Steffensen's Method appears to generate a quadratically convergent sequence for root finding without requiring computation of a derivative.

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Theorem

Suppose $g(x) = x$ *has a solution* p *with* $g'(p) \neq 1$*. If there exists* $\delta > 0$ *such that* $g \in \mathcal{C}^3[p - \delta, p + \delta]$*, the Steffensen's Method gives quadratic convergence for any* $p_0 \in [p - \delta, p + \delta]$*.*

Homework

- Read Section 2.5.
- Exercises: 1, 2, 7, 9, 11