Bisection Method

MATH 375 *Numerical Analysis*

J Robert Buchanan

Department of Mathematics

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Introduction

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- Most *interesting* equations cannot be solved algebraically.

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- We begin to study a set of **root-finding** techniques, starting with the simplest, the **Bisection Method**.

Bisection Technique

- The Bisection Method approximates the root of an equation on an interval by repeatedly halving the interval.
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Suppose $f \in \mathcal{C}[a, b]$ and $f(a)\,f(b) < 0$, then there exists $p \in (a, b)$ such that $f(p) = 0$.

Remark: The root p found is not necessarily unique.

Algorithm

Given the continuous function $f(x)$ on the interval $[a, b]$ where $f(a)\,f(b) < 0$:

INPUT endpoints a, b, tolerance ϵ, maximum iterations N.

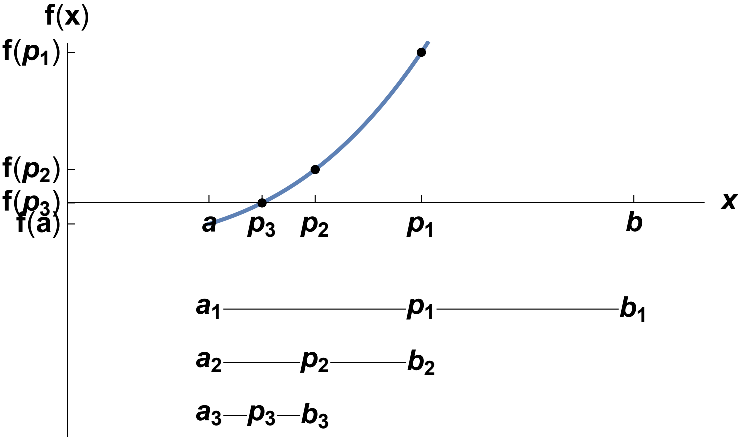
STEP 1 Set $i = 1$; $FA = f(a)$.

STEP 2 While $i \leq N$ do Steps 3–6.

- STEP 3 Set $p = a + \dfrac{b-a}{2}$; $FP = f(p)$.
- STEP 4 If $FP = 0$ or $\dfrac{b-a}{2} < \epsilon$ then OUTPUT p; STOP.
- STEP 5 Set $i = i + 1$.
- STEP 6 If $FA \cdot FP > 0$ then set $a = p$; $FA = FP$, else $b = p$.

STEP 7 OUTPUT "Method failed after N iterations."; STOP.

Illustration



Stopping Criterion

The Bisection Method generates a sequence $\{p_n\}_{n=1}^N$.

We used a stopping criterion of

- $f(p_n) = 0$ (in case we hit the root "exactly"), or
- $\dfrac{b-a}{2} < \epsilon$ (the original interval is halved enough times that the distance between p_{n-1} and p_n is smaller than a specified tolerance), or
- $i > N$ (the maximum number of iterations is reached).

Alternative Stopping Criteria

Other logic for halting the algorithm includes:

- $|p_n - p_{n-1}| < \epsilon$
- $|f(p_n)| < \epsilon$
- $\dfrac{|p_n - p_{n-1}|}{|p_n|} < \epsilon$ provided $p_n \neq 0$
- $\dfrac{|p_n - p_{n-1}|}{\min\{|a_n|, |b_n|\}} < \epsilon$

Remark: the stopping criterion chosen will depend on the equation being solved. There is no “best” criterion.

Example

Approximate the root of $f(x) = e^x - 4x$ on $[0, 1]$ with $\epsilon = 10^{-2}$ and $N = 10$.

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n	a_n	p_n	b_n	$f(p_n)$
1	0.0	0.5	1.0	-0.351279
2	0.0	0.25	0.5	0.284025
3	0.25	0.375	0.5	-0.0450086
4	0.25	0.3125	0.375	0.116838
5	0.3125	0.34375	0.375	0.035226
6	0.34375	0.359375	0.375	-0.00506614
7	0.34375	0.351563	0.359375	0.0150366

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Remark: $p \approx 0.357403$ and hence p_6 is a better approximation than p_7.

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- the Bisection Method is simple to program,
- the Bisection Method always converges to a solution, but
- the Bisection Method is slow to converge.

Rate of Convergence (1 of 2)

Theorem

If $f \in \mathcal{C}[a, b]$ *and* $f(a)\,f(b) < 0$*, the Bisection Method generates a sequence* $\{p_n\}_{n=1}^{\infty}$ *approximating a root* p *of* f *with the property that*

$$|p_n - p| \leq \frac{b - a}{2^n}, \quad \text{for } n \geq 1.$$

Rate of Convergence (2 of 2)

Proof.

- For all $n \geq 1$, $b_n - a_n \leq \dfrac{b-a}{2^{n-1}}$ and $a_n < p < b_n$.

□

Rate of Convergence (2 of 2)

Proof.

- For all $n \geq 1$, $b_n - a_n \leq \dfrac{b-a}{2^{n-1}}$ and $a_n < p < b_n$.
- For all $n \geq 1$, $p_n = \frac{a_n+b_n}{2}$ and hence

$$|p_n - p| \leq \frac{b_n - a_n}{2} = \frac{b-a}{2^n}.$$

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$$|p_n - p| \leq \frac{b_n - a_n}{2} = \frac{b-a}{2^n}.$$

- Therefore $p_n = p + O\left(\dfrac{1}{2^n}\right)$.

□

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Determine the minimum number of iterations of the Bisection Method necessary to approximate a root of $f(x) = e^x - 4x$ on $[0, 1]$ with $\epsilon = 10^{-4}$.

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$$
\begin{aligned}
|p_n - p| &\leq \frac{b-a}{2^n} \\
\frac{1-0}{2^n} &\leq 10^{-4} \\
2^n &\geq 10^4 \\
n &\geq 14
\end{aligned}
$$

Example

Determine the minimum number of iterations of the Bisection Method necessary to approximate a root of $f(x) = x^3 + x - 4$ on $[1, 4]$ with $\epsilon = 10^{-4}$. Find the approximation with this accuracy.

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\begin{aligned}
|p_n - p| &\leq \frac{b-a}{2^n} \\
\frac{4-1}{2^n} &\leq 10^{-4} \\
2^n &\geq 3 \times 10^4 \\
n &\geq 15
\end{aligned}
$$

Approximation

n	a_n	b_n	p_n	$f(p_n)$
1	1.0	4.0	2.5	14.125
2	1.0	2.5	1.75	3.10938
3	1.0	1.75	1.375	−0.0253906
4	1.375	1.75	1.5625	1.3772
5	1.375	1.5625	1.46875	0.637177
6	1.375	1.46875	1.42188	0.29652
7	1.375	1.42188	1.39844	0.13326
8	1.375	1.39844	1.38672	0.0533635
9	1.375	1.38672	1.38086	0.0138442
10	1.375	1.38086	1.37793	−0.00580869
11	1.37793	1.38086	1.37939	0.00400888
12	1.37793	1.37939	1.37866	−0.000902119
13	1.37866	1.37939	1.37903	0.00155283
14	1.37866	1.37903	1.37885	0.000325216
15	1.37866	1.37885	1.37875	−0.000288487

Homework

- Read Section 2.1.
- Exercises: 3a, 11, 13, 17, 19, 20