Chebyshev Polynomials and Economization of Power Series

MATH 375 *Numerical Analysis*

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Background

- The **Chebyshev polynomials** denoted $T_n(x)$ for $n = 0, 1, \ldots$ are a set of orthogonal polynomials on the open interval $(-1, 1)$ with respect to the weight function $w(x) = (1 - x^2)^{-1/2}$.
- Starting with $T_0(x) = 1$ we could use the **Gram-Schmidt** process to build the orthogonal set.

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- Starting with $T_0(x) = 1$ we could use the **Gram-Schmidt** process to build the orthogonal set.
- We will follow an alternative procedure to describe all the Chebyshev polynomials.
- The Chebyshev polynomials have important applications for
  - optimal placement of nodes to minimize error in Lagrange interpolation, and
  - reducing the degree of an approximating polynomial with minimal loss of accuracy.

Definition of the Chebyshev Polynomials

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For $x \in [-1, 1]$ define the nth Chebyshev polynomial as

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Claim: $T_n(x)$ is a polynomial in x of degree n for $n = 0, 1, \ldots$.

Proof (1 of 2)

- We can readily see that

$$\begin{aligned} T_0(x) &= \cos[0 \arccos x] = 1 \\ T_1(x) &= \cos[\arccos x] = x. \end{aligned}$$

- When $n \geq 1$ substitute $\theta = \arccos x$, then

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- Using the sum and difference of angles formula for the cosine we see that

$$\begin{aligned} T_{n-1}(\theta) &= \cos[(n-1)\theta] = \cos[n\theta] \cos\theta + \sin[n\theta] \sin\theta \\ T_{n+1}(\theta) &= \cos[(n+1)\theta] = \cos[n\theta] \cos\theta - \sin[n\theta] \sin\theta. \end{aligned}$$

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- If we add these expressions we can derive

$$\begin{aligned} T_{n-1}(\theta) + T_{n+1}(\theta) &= 2\cos[n\theta]\cos\theta \\ T_{n+1}(\theta) &= 2\cos[n\theta]\cos\theta - T_{n-1}(\theta), \end{aligned}$$

a **recurrence relation** for Chebyshev polynomials.

Proof (2 of 2)

$$\begin{aligned} T_{n+1}(\theta) &= 2\cos[n\,\theta]\cos\theta - T_{n-1}(\theta) \\ &= 2\cos[n\arccos x]\cos[\arccos x] - T_{n-1}(x) \\ T_{n+1}(x) &= 2x\,T_n(x) - T_{n-1}(x) \end{aligned}$$

Proof (2 of 2)

$$
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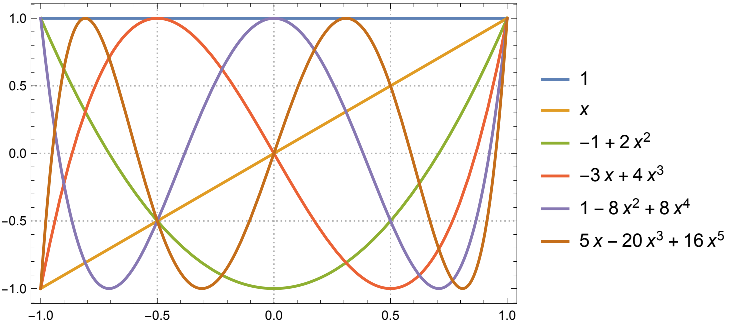
T_{n+1}(\theta) &= 2\cos[n\,\theta]\cos\theta - T_{n-1}(\theta) \\
&= 2\cos[n\arccos x]\cos[\arccos x] - T_{n-1}(x) \\
T_{n+1}(x) &= 2x\,T_n(x) - T_{n-1}(x)
\end{aligned}
$$

Since we already know that $T_0(x) = 1$ and $T_1(x) = x$ we can use the recurrence relation to develop the remaining Chebyshev polynomials.

$$
\begin{aligned}
T_2(x) &= 2x(x) - 1 = 2x^2 - 1 \\
T_3(x) &= 2x(2x^2 - 1) - x = 4x^3 - 3x \\
T_4(x) &= 8x^4 - 8x^2 + 1 \\
T_5(x) &= 16x^5 - 20x^3 + 5x \\
&\vdots
\end{aligned}
$$

Graphs of Chebyshev Polynomials



Orthogonality of Chebyshev Polynomials (1 of 4)

Suppose $n \neq m$ and consider the definite integral:

$$\int_{-1}^{1} \frac{T_n(x)\, T_m(x)}{\sqrt{1-x^2}}\, dx = \int_{-1}^{1} \frac{\cos[n \arccos x] \cos[m \arccos x]}{\sqrt{1-x^2}}\, dx.$$

Q: how can we integrate this?

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Q: how can we integrate this?
Substitute $\theta = \arccos x$ and $d\theta = -(1-x^2)^{-1/2}$.

$$\begin{aligned} \int_{-1}^{1} \frac{T_n(x)\, T_m(x)}{\sqrt{1-x^2}}\, dx &= -\int_{\pi}^{0} \cos[n\theta] \cos[m\theta]\, d\theta \\ &= \int_{0}^{\pi} \cos[n\theta] \cos[m\theta]\, d\theta \end{aligned}$$

Q: how can we integrate this?

Orthogonality of Chebyshev Polynomials (2 of 4)

Use the product-to-sum formula:

$$\cos[n\,\theta]\cos[m\,\theta] = \frac{1}{2}(\cos[(n+m)\theta] + \cos[(n-m)\theta])$$

and we have

$$\begin{aligned}\int_{-1}^{1} \frac{T_n(x)\,T_m(x)}{\sqrt{1-x^2}}\,dx &= \int_0^{\pi} \cos[n\,\theta]\cos[m\,\theta]\,d\theta \\ &= \frac{1}{2}\int_0^{\pi} (\cos[(n+m)\theta] + \cos[(n-m)\theta])\,d\theta \\ &= 0\end{aligned}$$

if $n \neq m$.

Orthogonality of Chebyshev Polynomials (3 of 4)

Suppose $n = m$ and consider the definite integral:

$$\int_{-1}^{1} \frac{T_n(x)\, T_n(x)}{\sqrt{1-x^2}}\, dx = \int_{-1}^{1} \frac{\cos^2[n \arccos x]}{\sqrt{1-x^2}}\, dx.$$

Substitute $\theta = \arccos x$ and $d\theta = -(1-x^2)^{-1/2}$.

$$\begin{aligned} \int_{-1}^{1} \frac{T_n(x)\, T_n(x)}{\sqrt{1-x^2}}\, dx &= -\int_{\pi}^{0} \cos^2[n\,\theta]\, d\theta \\ &= \int_{0}^{\pi} \cos^2[n\,\theta]\, d\theta \end{aligned}$$

Q: how can we integrate this?

Orthogonality of Chebyshev Polynomials (4 of 4)

Use the half-angle formula:

$$\cos^2[n\,\theta] = \frac{1}{2}(1 + \cos[2n\,\theta])$$

then we have

$$\begin{aligned}
\int_{-1}^{1} \frac{T_n(x)\,T_n(x)}{\sqrt{1-x^2}}\,dx &= \int_0^{\pi} \cos^2[n\,\theta]\,d\theta \\
&= \frac{1}{2}\int_0^{\pi} (1 + \cos[2n\,\theta])\,d\theta \\
&= \frac{\pi}{2}
\end{aligned}$$

if $n \geq 1$.

Roots of Chebyshev Polynomials

Theorem

The Chebyshev polynomial $T_n(x)$ for degree $n \geq 1$ has n simple roots in $[-1, 1]$ located at

$$\overline{x}_k = \cos\left(\frac{2k-1}{2n}\pi\right) \quad \text{for } k = 1, 2, \ldots, n.$$

Polynomial $T_n(x)$ assumes absolute extrema at

$$\overline{x}'_k = \cos\left(\frac{k\pi}{n}\right)$$

with $T_n(\overline{x}'_k) = (-1)^k$ for $k = 0, 1, \ldots, n$.

Proof (1 of 4)

Fix $n \geq 1$ then

$$\overline{x}_k = \cos\left(\frac{2k-1}{2n}\pi\right)$$

are distinct for $k = 1, 2, \ldots, n$.

$$T_n(\overline{x}_k) = \cos\left[n \arccos\left(\cos\left(\frac{2k-1}{2n}\pi\right)\right)\right] = \cos\left[\frac{(2k-1)\pi}{2}\right] = 0.$$

Since $T_n(x)$ is a polynomial of degree n then $\{\overline{x}_k\}_{k=1}^{n}$ is its complete set of distinct simple roots.

Proof (2 of 4)

Fix $n \geq 1$ then the derivative of $T_n(x)$ is

$$T_n'(x) = \frac{n \sin[n \arccos x]}{\sqrt{1 - x^2}}.$$

Observe that

$$T_n(\overline{x}_0') = T_n\left(\cos\frac{(0)\pi}{n}\right) = T_n(1) = \cos[n \arccos 1] = \cos[0] = 1$$

and

$$T_n(\overline{x}_n') = T_n\left(\cos\frac{n\pi}{n}\right) = T_n(-1) = \cos[n \arccos(-1)] = (-1)^n$$

Proof (3 of 4)

Fix $n \geq 1$ and let $k \in \{1, 2, \ldots, n-1\}$, then

$$\begin{aligned}
T_n'(\overline{x}_k') &= T_n'\left(\cos\left[\frac{k\pi}{n}\right]\right) \\
&= \frac{n\sin[n\arccos\left(\cos\left[\frac{k\pi}{n}\right]\right)]}{\sqrt{1-\left(\cos\left[\frac{k\pi}{n}\right]\right)^2}} \\
&= \frac{n\sin[k\pi]}{\sin\left[\frac{k\pi}{n}\right]} \\
&= 0.
\end{aligned}$$

Thus $T_n(x)$ has a critical point at each $\overline{x}_k'$.

Proof (4 of 4)

Evaluate $T_n(x)$ at each critical point.

$$\begin{aligned} T_n(\overline{x}'_k) &= T_n\left(\cos\left[\frac{k\pi}{n}\right]\right) \\ &= \cos\left[n \arccos\left(\cos\left[\frac{k\pi}{n}\right]\right)\right] \\ &= \cos[k\pi] \\ &= (-1)^k \end{aligned}$$

Remark: thus a maximum occurs at each even value of k and a minimum at each odd value of k.

Orthonormal Chebyshev Polynomials

- We can prove that the leading coefficient of $T_n(x)$ is 2^{n-1} for $n \geq 1$.
- The normalized Chebyshev polynomials are monic polynomials obtained by dividing $T_n(x)$ by 2^{n-1}.

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- We will define

$$\begin{aligned} \tilde{T}_0(x) &= 1 \\ \tilde{T}_n(x) &= \frac{1}{2^{n-1}} T_n(x) \end{aligned}$$

for $n = 1, 2, \ldots$.

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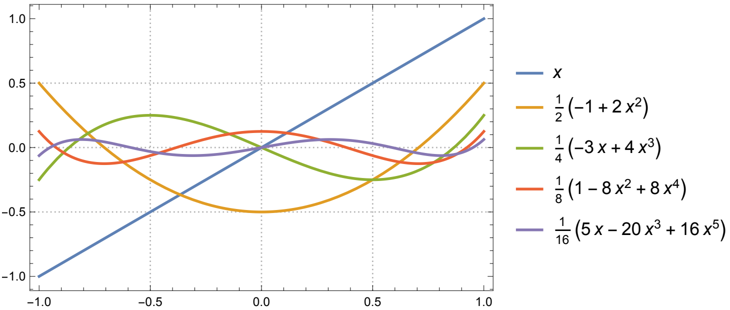
$$\begin{aligned} \tilde{T}_0(x) &= 1 \\ \tilde{T}_n(x) &= \frac{1}{2^{n-1}} T_n(x) \end{aligned}$$

  for $n = 1, 2, \ldots$.
- Denote by $\tilde{\Pi}_n$ the set of all monic polynomials of degree n or less.

Graphs



1.0
0.5
0.0
−0.5
−1.0
−1.0
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0.0
0.5
1.0
x
$\frac{1}{2}(-1 + 2x^2)$
$\frac{1}{4}(-3x + 4x^3)$
$\frac{1}{8}(1 - 8x^2 + 8x^4)$
$\frac{1}{16}(5x - 20x^3 + 16x^5)$

A Property of the $\tilde{T}_n(x)$

Theorem

The polynomials of the form $\tilde{T}_n(x)$ with $n \geq 1$ have the property that

$$\frac{1}{2^{n-1}} = \max_{-1 \leq x \leq 1} \left| \tilde{T}_n(x) \right| \leq \max_{-1 \leq x \leq 1} |P_n(x)|$$

for all $P_n(x) \in \tilde{\Pi}_n$. Furthermore, equality occurs only if $P_n \equiv \tilde{T}_n$.

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Remark: since $\max_{-1 \leq x \leq 1} |T_n(x)| = 1$ then $\max_{-1 \leq x \leq 1} \left| \tilde{T}_n(x) \right| = \dfrac{1}{2^{n-1}}$.

Proof (1 of 2)

- Suppose that $P_n(x) \in \tilde{\Pi}_n$ and

$$\max_{-1 \leq x \leq 1} |P_n(x)| \leq \frac{1}{2^{n-1}}.$$

- Define the function $Q(x) = \tilde{T}_n(x) - P_n(x)$. $Q(x)$ is a polynomial of degree at most $n-1$ (**why?**).

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- Define the function $Q(x) = \tilde{T}_n(x) - P_n(x)$. $Q(x)$ is a polynomial of degree at most $n-1$ (**why?**).
- $\tilde{T}_n(x)$ has $n+1$ extreme points $\overline{x}'_k$ and

$$Q(\overline{x}'_k) = \tilde{T}_n(\overline{x}'_k) - P_n(\overline{x}'_k) = \frac{(-1)^k}{2^{n-1}} - P_n(\overline{x}'_k).$$

Proof (2 of 2)

$$Q(\overline{x}'_k) = \tilde{T}_n(\overline{x}'_k) - P_n(\overline{x}'_k) = \frac{(-1)^k}{2^{n-1}} - P_n(\overline{x}'_k).$$

- By assumption $\left|P_n(\overline{x}'_k)\right| \leq \dfrac{1}{2^{n-1}}$ for $k = 0, 1, \ldots, n$.
- When k is even then $Q(\overline{x}'_k) \geq 0$ and when k is odd $Q(\overline{x}'_k) \leq 0$.

Proof (2 of 2)

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- By assumption $\left|P_n(\overline{x}'_k)\right| \le \dfrac{1}{2^{n-1}}$ for $k = 0, 1, \ldots, n$.
- When k is even then $Q(\overline{x}'_k) \ge 0$ and when k is odd $Q(\overline{x}'_k) \le 0$.
- Since $Q(x)$ is a polynomial, Q is continuous on $[-1, 1]$. By the **Intermediate Value Theorem** there exists $z_j \in [\overline{x}'_j, \overline{x}'_{j+1}]$ for $j = 0, 1, \ldots, n-1$ such that $Q(z_j) = 0$.

Proof (2 of 2)

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- This implies Q has at least n distinct roots, but Q is a polynomial of degree at most $n-1$. Thus $Q(x) \equiv 0$ and $P_n(x) \equiv \tilde{T}_n(x)$.

Lagrange Interpolation Error (1 of 5)

Now we may begin to apply these properties of the Chebyshev polynomials to the task of minimizing the error in a Lagrange interpolating polynomials.

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Recall an earlier theorem.

Theorem

Suppose $x_0, x_1, \ldots, x_n$ are distinct numbers in the interval $[a, b]$ and suppose $f \in \mathcal{C}^{n+1}[a, b]$. Then for each $x \in [a, b]$ there exists a number $z(x) \in (a, b)$ for which

$$f(x) = P(x) + \frac{f^{(n+1)}(z(x))}{(n+1)!}(x - x_0)(x - x_1)\cdots(x - x_n),$$

where $P(x)$ is the Lagrange Interpolating Polynomial.

Lagrange Interpolation Error (2 of 5)

If interval $[a, b] = [-1, 1]$ then

$$f(x) - P(x) = \frac{f^{(n+1)}(z(x))}{(n+1)!}(x - x_0)(x - x_1) \cdots (x - x_n),$$

where $P(x)$ is a Lagrange interpolating polynomial and $z(x) \in (-1, 1)$.

Remark: we have no control over $z(x)$ but we can attempt to make the interpolation error small by making

$$|(x - x_0)(x - x_1) \cdots (x - x_n)|$$

small on the interval $[-1, 1]$.

Lagrange Interpolation Error (3 of 5)

Q: how do we make

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Lagrange Interpolation Error (3 of 5)

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A: since the expression inside the absolute value is a monic polynomial of degree $n+1$, we should let

$$\begin{aligned}
\tilde{T}_{n+1}(x) &= (x - x_0)(x - x_1) \cdots (x - x_n) \\
&= \prod_{k=0}^{n} (x - \overline{x}_{k+1}) \\
&= \prod_{k=0}^{n} \left(x - \cos\left[\frac{2k+1}{2(n+1)} \pi \right] \right)
\end{aligned}$$

Lagrange Interpolation Error (4 of 5)

Thus the Lagrange interpolation error is bounded by

$$\begin{aligned}
|E| &\leq \max_{-1\leq x\leq 1} \left| \frac{f^{(n+1)}(z(x))}{(n+1)!}(x-x_0)(x-x_1)\cdots(x-x_n) \right| \\
&\leq \max_{-1\leq x\leq 1} \left| \frac{f^{(n+1)}(z(x))}{(n+1)!} \right| \max_{-1\leq x\leq 1} \left| \tilde{T}_{n+1}(x) \right| \\
&\leq 2^{-n} \max_{-1\leq x\leq 1} \left| \frac{f^{(n+1)}(z(x))}{(n+1)!} \right| .
\end{aligned}$$

Lagrange Interpolation Error (5 of 5)

Thus we have the following corollary.

Corollary

Suppose that $P(x)$ is the interpolating polynomial of degree at most n with nodes at the zeros of $T_{n+1}(x)$. Then

$$\max_{-1\leq x\leq 1} |f(x) - P(x)| \leq \frac{1}{2^n(n+1)!} \max_{-1\leq x\leq 1} \left|f^{(n+1)}(x)\right|$$

for every $f \in \mathcal{C}^{n+1}[-1,1]$.

Remark: if we wish to interpolate over the arbitrary interval $[a, b]$ then use the change of variables

$$t = \frac{b-a}{2}x + \frac{b+a}{2}.$$

Example (1 of 4)

Consider the function $f(x) = x \ln x$ on the interval $[1, 3]$. Compare the values given by the Lagrange interpolating polynomial found using four equally spaced nodes to the Lagrange interpolating polynomial with nodes given by the roots of $T_4(x)$.

Example (2 of 4)

With equally spaced nodes at $x_0 = 1$, $x_1 = 5/3$, $x_2 = 7/3$, and $x_3 = 3$ we have the Lagrange interpolating polynomial

$$\begin{aligned} P_3(x) &= f(1)L_0(x) + f(5/3)L_1(x) + f(7/3)L_2(x) + f(3)L_3(x) \\ &= -0.585346 + 0.0942651x + 0.536711x^2 - 0.04563x^3. \end{aligned}$$

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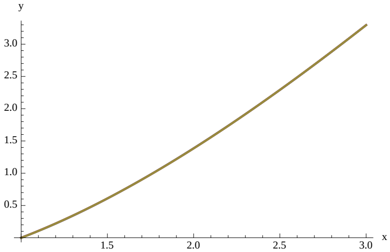
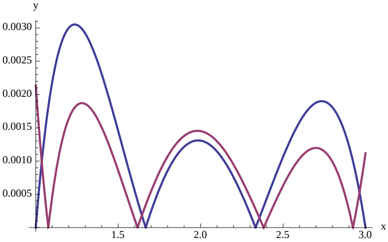
With nodes at the roots of $T_4(x)$, *i.e.*, $x_i = \cos(2i+1)\pi/8$ for $i = 0, 1, 2, 3$ we have the Lagrange interpolating polynomial

$$\begin{aligned} Q_3(x) &= f(x_0+2)L_0(x+2) + f(x_1+2)L_1(x+2) \\ &\quad + f(x_2+2)L_2(x+2) + f(x_3+2)L_3(x+2) \\ &= -0.595225 + 0.10582x + 0.532437x^2 - 0.0451646x^3. \end{aligned}$$

Example (3 of 4)



Interpolation



Absolute Error

Example (4 of 4)

Computing the error bounds we have

$$
\begin{aligned}
|E_P| &= \max_{1 \le x \le 3} \left| \frac{f^{(4)}(z(x))}{4!}(x-1)(x-5/3)(x-7/3)(x-3) \right| \\
&\le \frac{0.197531}{24} \max_{1 \le x \le 3} \left| \frac{2}{x^3} \right| \\
&= 0.0164609
\end{aligned}
$$

and

$$
\begin{aligned}
|E_Q| &= \max_{1 \le x \le 3} \left| \frac{f^{(4)}(z(x))}{4!} \prod_{k=0} \left(x - 2 - \cos\left[\frac{(2k+1)\pi}{8} \right] \right) \right| \\
&\le \frac{2}{24} \max_{1 \le x \le 3} \left| \prod_{k=0} \left(x - 2 - \cos\left[\frac{(2k+1)\pi}{8} \right] \right) \right| \\
&= \frac{0.125}{12} \\
&= 0.0104167.
\end{aligned}
$$

Reducing the Degree of Approximating Polynomials

Suppose we want to approximate a polynomial of degree n,

$$P_n(x) = a_0 + a_1 x + \cdots + a_{n-1} x^{n-1} + a_n x^n$$

by a polynomial of degree at most $n - 1$.

Objective: select $P_{n-1}(x) \in \Pi_{n-1}$ so that

$$\max_{-1 \le x \le 1} |P_n(x) - P_{n-1}(x)|$$

is as small as possible.

Selection (1 of 2)

- Recognize that

$$Q(x) = \frac{1}{a_n} \left(P_n(x) - P_{n-1}(x) \right)$$

is a monic polynomial of degree n.

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  is a monic polynomial of degree n.
- Thus the following inequality holds.

$$\frac{1}{2^{n-1}} \leq \max_{-1\leq x\leq 1} |Q(x)| = \max_{-1\leq x\leq 1} \left|\frac{1}{a_n}\left(P_n(x) - P_{n-1}(x)\right)\right|$$

Selection (1 of 2)

- Recognize that

$$Q(x) = \frac{1}{a_n}(P_n(x) - P_{n-1}(x))$$

  is a monic polynomial of degree n.
- Thus the following inequality holds.

$$\frac{1}{2^{n-1}} \leq \max_{-1 \leq x \leq 1} |Q(x)| = \max_{-1 \leq x \leq 1} \left| \frac{1}{a_n}(P_n(x) - P_{n-1}(x)) \right|$$

- Equality holds when

$$Q(x) = \frac{1}{a_n}(P_n(x) - P_{n-1}(x)) = \tilde{T}_n(x).$$

Selection (2 of 2)

- Thus we should choose

$$\begin{aligned} \frac{1}{a_n}\left(P_n(x) - P_{n-1}(x)\right) &= \tilde{T}_n(x) \\ P_{n-1}(x) &= P_n(x) - a_n \tilde{T}_n(x). \end{aligned}$$

Selection (2 of 2)

- Thus we should choose

$$\begin{aligned} \frac{1}{a_n}\left(P_n(x) - P_{n-1}(x)\right) &= \tilde{T}_n(x) \\ P_{n-1}(x) &= P_n(x) - a_n \tilde{T}_n(x). \end{aligned}$$

- The error bound is

$$\max_{-1 \le x \le 1} |P_n(x) - P_{n-1}(x)| = \max_{-1 \le x \le 1} \left| a_n \tilde{T}_n(x) \right| = \frac{|a_n|}{2^{n-1}}.$$

Example

The sixth Maclaurin polynomial for $f(x) = xe^x$ is

$$P(x) = x + x^2 + \frac{x^3}{2} + \frac{x^4}{6} + \frac{x^5}{24} + \frac{x^6}{120}.$$

A bound for the error in this approximation on $[-1, 1]$ is $E = 0.00161516$.

Use Chebyshev economization to find a polynomial of lesser degree to approximate $f(x)$ while keeping the error less than 0.01 for $-1 \leq x \leq 1$.

Solution (1 of 4)

The polynomial of degree 5 which best approximates $P(x)$ on $[-1, 1]$ is

$$
\begin{aligned}
P_5(x) &= P(x) - \frac{1}{120}\tilde{T}_6(x) \\
&= \frac{x^5}{24} + \frac{43x^4}{240} + \frac{x^3}{2} + \frac{637x^2}{640} + x + \frac{1}{3840}.
\end{aligned}
$$

Solution (1 of 4)

The polynomial of degree 5 which best approximates $P(x)$ on $[-1, 1]$ is

$$\begin{aligned} P_5(x) &= P(x) - \frac{1}{120}\tilde{T}_6(x) \\ &= \frac{x^5}{24} + \frac{43x^4}{240} + \frac{x^3}{2} + \frac{637x^2}{640} + x + \frac{1}{3840}. \end{aligned}$$

Note: $|P(x) - P_5(x)| = \left|\frac{1}{120}\tilde{T}_6(x)\right| \leq \frac{1}{120(2^5)} \approx 0.000260417$

Solution (1 of 4)

The polynomial of degree 5 which best approximates $P(x)$ on $[-1, 1]$ is

$$\begin{aligned} P_5(x) &= P(x) - \frac{1}{120}\tilde{T}_6(x) \\ &= \frac{x^5}{24} + \frac{43x^4}{240} + \frac{x^3}{2} + \frac{637x^2}{640} + x + \frac{1}{3840}. \end{aligned}$$

Note: $|P(x) - P_5(x)| = \left|\frac{1}{120}\tilde{T}_6(x)\right| \leq \frac{1}{120(2^5)} \approx 0.000260417$

Adding this to the previous error bound gives a total error bound of

$$0.00161516 + 0.000260417 = 0.00187558 < 0.01.$$

Solution (2 of 4)

The polynomial of degree 4 which best approximates $P_5(x)$ on $[-1, 1]$ is

$$\begin{aligned} P_4(x) &= P_5(x) - \frac{1}{24}\tilde{T}_5(x) \\ &= \frac{1}{3840}(688x^4 + 2120x^3 + 3822x^2 + 3790x + 1). \end{aligned}$$

Solution (2 of 4)

The polynomial of degree 4 which best approximates $P_5(x)$ on $[-1, 1]$ is

$$\begin{aligned} P_4(x) &= P_5(x) - \frac{1}{24}\tilde{T}_5(x) \\ &= \frac{1}{3840}(688x^4 + 2120x^3 + 3822x^2 + 3790x + 1). \end{aligned}$$

Note: $|P_4(x) - P_5(x)| = \left|\frac{1}{24}\tilde{T}_5(x)\right| \leq \frac{1}{24(2^4)} \approx 0.00260417$

Solution (2 of 4)

The polynomial of degree 4 which best approximates $P_5(x)$ on $[-1, 1]$ is

$$\begin{aligned} P_4(x) &= P_5(x) - \frac{1}{24}\tilde{T}_5(x) \\ &= \frac{1}{3840}(688x^4 + 2120x^3 + 3822x^2 + 3790x + 1). \end{aligned}$$

Note: $|P_4(x) - P_5(x)| = \left|\frac{1}{24}\tilde{T}_5(x)\right| \leq \frac{1}{24(2^4)} \approx 0.00260417$

Adding this to the previous error bound gives a total error bound of

$$0.00161516 + 0.000260417 + 0.00260417 = 0.00447975 < 0.01.$$

Solution (3 of 4)

The polynomial of degree 3 which best approximates $P_4(x)$ on $[-1, 1]$ is

$$\begin{aligned} P_3(x) &= P_4(x) - \frac{688}{3840}\tilde{T}_4(x) \\ &= \frac{1}{768}(424x^3 + 902x^2 + 758x - 17). \end{aligned}$$

Solution (3 of 4)

The polynomial of degree 3 which best approximates $P_4(x)$ on $[-1, 1]$ is

$$\begin{aligned} P_3(x) &= P_4(x) - \frac{688}{3840}\,\tilde{T}_4(x) \\ &= \frac{1}{768}(424x^3 + 902x^2 + 758x - 17). \end{aligned}$$

Note: $|P_3(x) - P_4(x)| = \left|\frac{688}{3840}\,\tilde{T}_4(x)\right| \leq \frac{688}{3840(2^3)} \approx 0.0223958$

Solution (3 of 4)

The polynomial of degree 3 which best approximates $P_4(x)$ on $[-1, 1]$ is

$$\begin{aligned} P_3(x) &= P_4(x) - \frac{688}{3840}\tilde{T}_4(x) \\ &= \frac{1}{768}(424x^3 + 902x^2 + 758x - 17). \end{aligned}$$

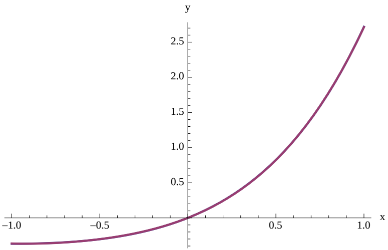
Note: $|P_3(x) - P_4(x)| = \left|\frac{688}{3840}\tilde{T}_4(x)\right| \leq \frac{688}{3840(2^3)} \approx 0.0223958$

Adding this to the previous error bound gives a total error bound of

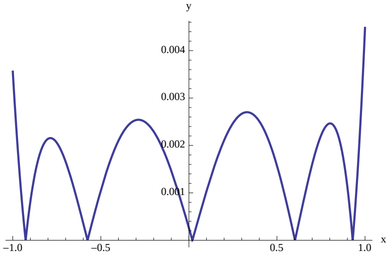
$$0.0268756 > 0.01,$$

thus we may use $P_4(x)$ to approximate $f(x)$ to within 0.01 on $[-1, 1]$.

Example (4 of 4)



Interpolation



Absolute Error

Homework

- Read Section 8.3.
- Exercises: 1ab, 3ab, 5ab, 7, 8