The **Chebyshev polynomials** denoted $T_n(x)$ for $n = 0, 1, \ldots$ are a set of orthogonal polynomials on the open interval $(-1, 1)$ with respect to the weight function $w(x) = (1 - x^2)^{-1/2}$.

Starting with $T_0(x) = 1$ we could use the **Gram-Schmidt** process to build the orthogonal set.
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We will follow an alternative procedure to describe all the Chebyshev polynomials.

The Chebyshev polynomials have important applications for

- optimal placement of nodes to minimize error in Lagrange interpolation, and
- reducing the degree of an approximating polynomial with minimal loss of accuracy.
Definition of the Chebyshev Polynomials

Definition
For $x \in [-1, 1]$ define the $n$th Chebyshev polynomial as

$$T_n(x) = \cos[n \arccos x].$$
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For $x \in [-1, 1]$ define the $n$th Chebyshev polynomial as

$$T_n(x) = \cos[n \arccos x].$$

Claim: $T_n(x)$ is a polynomial in $x$ of degree $n$ for $n = 0, 1, \ldots$. 
Proof (1 of 2)

- We can readily see that

\[
T_0(x) = \cos[0 \arccos x] = 1 \\
T_1(x) = \cos[\arccos x] = x.
\]

- When \( n \geq 1 \) substitute \( \theta = \arccos x \), then

\[
T_n(\theta) = \cos[n \theta].
\]
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\[ T_n(\theta) = \cos[n \theta]. \]

Using the sum and difference of angles formula for the cosine we see that

\[ T_{n-1}(\theta) = \cos[(n - 1)\theta] = \cos[n \theta] \cos \theta + \sin[n \theta] \sin \theta \]
\[ T_{n+1}(\theta) = \cos[(n + 1)\theta] = \cos[n \theta] \cos \theta - \sin[n \theta] \sin \theta. \]
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\[ T_{n+1}(\theta) = \cos[(n+1)\theta] = \cos[n\theta] \cos \theta - \sin[n\theta] \sin \theta. \]

If we add these expressions we can derive

\[ T_{n-1}(\theta) + T_{n+1}(\theta) = 2 \cos[n\theta] \cos \theta \]
\[ T_{n+1}(\theta) = 2 \cos[n\theta] \cos \theta - T_{n-1}(\theta), \]

a recurrence relation for Chebyshev polynomials.
Proof (2 of 2)

\[ T_{n+1}(\theta) = 2 \cos[n \theta] \cos \theta - T_{n-1}(\theta) \]
\[ = 2 \cos[n \arccos x] \cos[\arccos x] - T_{n-1}(x) \]
\[ T_{n+1}(x) = 2x T_n(x) - T_{n-1}(x) \]
Proof (2 of 2)

\[ T_{n+1}(\theta) = 2 \cos[n \theta] \cos \theta - T_{n-1}(\theta) \]
\[ = 2 \cos[n \arccos x] \cos[\arccos x] - T_{n-1}(x) \]
\[ T_{n+1}(x) = 2x T_n(x) - T_{n-1}(x) \]

Since we already know that \( T_0(x) = 1 \) and \( T_1(x) = x \) we can use the recurrence relation to develop the remaining Chebyshev polynomials.

\[ T_2(x) = 2x(x) - 1 = 2x^2 - 1 \]
\[ T_3(x) = 2x(2x^2 - 1) - x = 4x^3 - 3x \]
\[ T_4(x) = 8x^4 - 8x^2 + 1 \]
\[ T_5(x) = 16x^5 - 20x^3 + 5x \]
\[ \vdots \]
Graphs of Chebyshev Polynomials

$1$

$x$

$-1 + 2x^2$

$-3x + 4x^3$

$1 - 8x^2 + 8x^4$

$5x - 20x^3 + 16x^5$
Orthogonality of Chebyshev Polynomials (1 of 4)

Suppose $n \neq m$ and consider the definite integral:

$$\int_{-1}^{1} \frac{T_n(x) \ T_m(x)}{\sqrt{1 - x^2}} \ dx = \int_{-1}^{1} \frac{\cos[n \ \arccos x] \ \cos[m \ \arccos x]}{\sqrt{1 - x^2}} \ dx.$$

Q: how can we integrate this?
Suppose $n \neq m$ and consider the definite integral:

\[
\int_{-1}^{1} \frac{T_n(x) T_m(x)}{\sqrt{1 - x^2}} \, dx = \int_{-1}^{1} \frac{\cos[n \arccos x] \cos[m \arccos x]}{\sqrt{1 - x^2}} \, dx.
\]

Q: how can we integrate this?

Substitute $\theta = \arccos x$ and $d\theta = -(1 - x^2)^{-1/2}$.

\[
\int_{-1}^{1} \frac{T_n(x) T_m(x)}{\sqrt{1 - x^2}} \, dx = \int_{0}^{\pi} \cos[n \theta] \cos[m \theta] \, d\theta
\]

Q: how can we integrate this?
Use the product-to-sum formula:

\[ \cos[n \theta] \cos[m \theta] = \frac{1}{2}(\cos[(n + m)\theta] + \cos[(n - m)\theta]) \]

and we have

\[
\int_{-1}^{1} \frac{T_n(x) T_m(x)}{\sqrt{1 - x^2}} \, dx = \int_0^\pi \cos[n \theta] \cos[m \theta] \, d\theta
\]

\[
= \frac{1}{2} \int_0^\pi (\cos[(n + m)\theta] + \cos[(n - m)\theta]) \, d\theta
\]

\[= 0 \quad \text{if} \ n \neq m. \]
Suppose \( n = m \) and consider the definite integral:

\[
\int_{-1}^{1} \frac{T_n(x) T_n(x)}{\sqrt{1 - x^2}} \, dx = \int_{-1}^{1} \frac{\cos^2[n \arccos x]}{\sqrt{1 - x^2}} \, dx.
\]

Substitute \( \theta = \arccos x \) and \( d\theta = -(1 - x^2)^{-1/2} \).

\[
\int_{-1}^{1} \frac{T_n(x) T_n(x)}{\sqrt{1 - x^2}} \, dx = -\int_{\pi}^{0} \cos^2[n \theta] \, d\theta
\]

\[
= \int_{0}^{\pi} \cos^2[n \theta] \, d\theta
\]

Q: how can we integrate this?
Orthogonality of Chebyshev Polynomials (4 of 4)

Use the half-angle formula:

\[ \cos^2[n \theta] = \frac{1}{2} (1 + \cos[2n \theta]) \]

then we have

\[
\int_{-1}^{1} \frac{T_n(x) T_n(x)}{\sqrt{1 - x^2}} \, dx = \int_{0}^{\pi} \cos^2[n \theta] \, d\theta
\]

\[
= \frac{1}{2} \int_{0}^{\pi} (1 + \cos[2n \theta]) \, d\theta
\]

\[
= \frac{\pi}{2}
\]

if \( n \geq 1 \).
Theorem

The Chebyshev polynomial $T_n(x)$ for degree $n \geq 1$ has $n$ simple roots in $[-1, 1]$ located at

$$\bar{x}_k = \cos \left( \frac{2k - 1}{2n} \pi \right) \quad \text{for } k = 1, 2, \ldots, n.$$ 

Polynomial $T_n(x)$ assumes absolute extrema at

$$\bar{x}'_k = \cos \left( \frac{k\pi}{n} \right)$$

with $T_n(\bar{x}'_k) = (-1)^k$ for $k = 0, 1, \ldots, n.$
Fix $n \geq 1$ then

$$\bar{x}_k = \cos\left(\frac{2k - 1}{2n} \pi\right)$$

are distinct for $k = 1, 2, \ldots, n$.

$$T_n(\bar{x}_k) = \cos \left[ n \arccos \left( \cos \left( \frac{2k - 1}{2n} \pi \right) \right) \right] = \cos \left[ \frac{(2k - 1)\pi}{2} \right] = 0.$$

Since $T_n(x)$ is a polynomial of degree $n$ then $\{\bar{x}_k\}_{k=1}^n$ is its complete set of distinct simple roots.
Proof (2 of 4)

Fix $n \geq 1$ then the derivative of $T_n(x)$ is

$$T'_n(x) = \frac{n \sin[n \arccos x]}{\sqrt{1 - x^2}}.$$

Observe that

$$T_n(\bar{x}_0') = T_n \left( \cos \left( \frac{0}{n} \pi \right) \right) = T_n(1) = \cos[n \arccos 1] = \cos[0] = 1$$

and

$$T_n(\bar{x}_n') = T_n \left( \cos \left( \frac{n\pi}{n} \right) \right) = T_n(-1) = \cos[n \arccos(-1)] = (-1)^n$$
Fix \( n \geq 1 \) and let \( k \in \{1, 2, \ldots, n - 1\} \), then

\[
T_n'(x'_k) = \frac{n \sin[n \arccos(\cos(k \pi/n))]}{\sqrt{1 - (\cos(k \pi/n))^2}}
\]

\[
= \frac{n \sin[k \pi]}{\sin(k \pi/n)}
\]

\[
= 0.
\]

Thus \( T_n(x) \) has a critical point at each \( x'_k \).
Proof (4 of 4)

Evaluate $T_n(x)$ at each critical point.

\[
T_n(x'_k) = T_n \left( \cos \left( \frac{k\pi}{n} \right) \right)
\]

\[
= \cos \left[ n \arccos \left( \cos \left( \frac{k\pi}{n} \right) \right) \right]
\]

\[
= \cos[k\pi]
\]

\[
= (-1)^k
\]

**Remark**: thus a maximum occurs at each even value of $k$ and a minimum at each odd value of $k$. 
Orthonormal Chebyshev Polynomials

- We can prove that the leading coefficient of $T_n(x)$ is $2^{n-1}$ for $n \geq 1$.
- The normalized Chebyshev polynomials are monic polynomials obtained by dividing $T_n(x)$ by $2^{n-1}$.
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The normalized Chebyshev polynomials are monic polynomials obtained by dividing $T_n(x)$ by $2^{n-1}$.

We will define

$$
\tilde{T}_0(x) = 1
$$

$$
\tilde{T}_n(x) = \frac{1}{2^{n-1}} T_n(x)
$$

for $n = 1, 2, \ldots$. 
We can prove that the leading coefficient of $T_n(x)$ is $2^{n-1}$ for $n \geq 1$.

The normalized Chebyshev polynomials are monic polynomials obtained by dividing $T_n(x)$ by $2^{n-1}$.

We will define

$$\tilde{T}_0(x) = 1$$

$$\tilde{T}_n(x) = \frac{1}{2^{n-1}} T_n(x)$$

for $n = 1, 2, \ldots$.

Denote by $\tilde{\Pi}_n$ the set of all monic polynomials of degree $n$ or less.
Theorem

The polynomials of the form $\tilde{T}_n(x)$ with $n \geq 1$ have the property that

$$\frac{1}{2^{n-1}} = \max_{-1 \leq x \leq 1} |\tilde{T}_n(x)| \leq \max_{-1 \leq x \leq 1} |P_n(x)|$$

for all $P_n(x) \in \tilde{\Pi}_n$. Furthermore, equality occurs only if $P_n \equiv \tilde{T}_n$. 
A Property of the $\tilde{T}_n(x)$

**Theorem**

The polynomials of the form $\tilde{T}_n(x)$ with $n \geq 1$ have the property that

$$\frac{1}{2^{n-1}} = \max_{-1 \leq x \leq 1} |\tilde{T}_n(x)| \leq \max_{-1 \leq x \leq 1} |P_n(x)|$$

for all $P_n(x) \in \tilde{\Pi}_n$. Furthermore, equality occurs only if $P_n \equiv \tilde{T}_n$.

**Remark:** since $\max_{-1 \leq x \leq 1} |T_n(x)| = 1$ then $\max_{-1 \leq x \leq 1} |\tilde{T}_n(x)| = \frac{1}{2^{n-1}}$. 


Proof (1 of 2)

- Suppose that \( P_n(x) \in \tilde{\Pi}_n \) and
  \[
  \max_{-1 \leq x \leq 1} |P_n(x)| \leq \frac{1}{2^{n-1}}.
  \]

- Define the function \( Q(x) = \tilde{T}_n(x) - P_n(x) \). \( Q(x) \) is a polynomial of degree at most \( n - 1 \) (why?).
Proof (1 of 2)

- Suppose that $P_n(x) \in \tilde{\Pi}_n$ and

$$\max_{-1 \leq x \leq 1} |P_n(x)| \leq \frac{1}{2^{n-1}}.$$

- Define the function $Q(x) = \tilde{T}_n(x) - P_n(x)$. $Q(x)$ is a polynomial of degree at most $n - 1$ (why?).

- $\tilde{T}_n(x)$ has $n + 1$ extreme points $\bar{x}'_k$ and

$$Q(\bar{x}'_k) = \tilde{T}_n(\bar{x}'_k) - P_n(\bar{x}'_k) = \frac{(-1)^k}{2^{n-1}} - P_n(\bar{x}'_k).$$
Proof (2 of 2)

\[ Q(\bar{x}'_k) = \tilde{T}_n(\bar{x}'_k) - P_n(\bar{x}'_k) = \frac{(-1)^k}{2^{n-1}} - P_n(\bar{x}'_k). \]

- By assumption \( |P_n(\bar{x}'_k)| \leq \frac{1}{2^{n-1}} \) for \( k = 0, 1, \ldots, n \).
- When \( k \) is even then \( Q(\bar{x}'_k) \geq 0 \) and when \( k \) is odd \( Q(\bar{x}'_k) \leq 0 \).
Proof (2 of 2)

\[ Q(x'_k) = \tilde{T}_n(x'_k) - P_n(x'_k) = \frac{(-1)^k}{2^{n-1}} - P_n(x'_k). \]

- By assumption \(|P_n(x'_k)| \leq \frac{1}{2^{n-1}}\) for \(k = 0, 1, \ldots, n\).
- When \(k\) is even then \(Q(x'_k) \geq 0\) and when \(k\) is odd \(Q(x'_k) \leq 0\).
- Since \(Q(x)\) is a polynomial, \(Q\) is continuous on \([-1, 1]\). By the Intermediate Value Theorem there exists \(z_j \in [x'_j, x'_{j+1}]\) for \(j = 0, 1, \ldots, n - 1\) such that \(Q(z_j) = 0\).
\[
Q(x_k') = \tilde{T}_n(x_k') - P_n(x_k') = \frac{(-1)^k}{2^{n-1}} - P_n(x_k').
\]

- By assumption \( |P_n(x_k')| \leq \frac{1}{2^{n-1}} \) for \( k = 0, 1, \ldots, n \).
- When \( k \) is even then \( Q(x_k') \geq 0 \) and when \( k \) is odd \( Q(x_k') \leq 0 \).
- Since \( Q(x) \) is a polynomial, \( Q \) is continuous on \([-1, 1]\). By the Intermediate Value Theorem there exists \( z_j \in [x_j', x_{j+1}'] \) for \( j = 0, 1, \ldots, n - 1 \) such that \( Q(z_j) = 0 \).
- This implies \( Q \) has at least \( n \) distinct roots, but \( Q \) is a polynomial of degree at most \( n - 1 \). Thus \( Q(x) \equiv 0 \) and \( P_n(x) \equiv \tilde{T}_n(x) \).
Now we may begin to apply these properties of the Chebyshev polynomials to the task of minimizing the error in a Lagrange interpolating polynomials.
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Recall an earlier theorem.

**Theorem**

Suppose $x_0, x_1, \ldots, x_n$ are distinct numbers in the interval $[a, b]$ and suppose $f \in C^{n+1}[a, b]$. Then for each $x \in [a, b]$ there exists a number $z(x) \in (a, b)$ for which

$$f(x) = P(x) + \frac{f^{(n+1)}(z(x))}{(n + 1)!} (x - x_0)(x - x_1) \cdots (x - x_n),$$

where $P(x)$ is the Lagrange Interpolating Polynomial.
If interval \([a, b] = [-1, 1]\) then

\[
f(x) - P(x) = \frac{f^{(n+1)}(z(x))}{(n + 1)!} (x - x_0)(x - x_1) \cdots (x - x_n),
\]

where \(P(x)\) is a Lagrange interpolating polynomial and \(z(x) \in (-1, 1)\).

**Remark:** we have no control over \(z(x)\) but we can attempt to make the interpolation error small by making

\[
| (x - x_0)(x - x_1) \cdots (x - x_n) |
\]

small on the interval \([-1, 1]\).
Q: how do we make

\[ |(x - x_0)(x - x_1) \cdots (x - x_n)| \]

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Q: how do we make

\[ |(x - x_0)(x - x_1) \cdots (x - x_n)| \]

small on the interval \([-1, 1]\)?

A: since the expression inside the absolute value is a monic polynomial of degree \(n + 1\), we should let

\[
\tilde{T}_{n+1}(x) = (x - x_0)(x - x_1) \cdots (x - x_n)
\]

\[
= \prod_{k=0}^{n} (x - \overline{x}_{k+1})
\]

\[
= \prod_{k=0}^{n} \left( x - \cos \left[ \frac{2k + 1}{2(n + 1)} \pi \right] \right)
\]
Thus the Lagrange interpolation error is bounded by

\[ |E| \leq \max_{-1 \leq x \leq 1} \left| f^{(n+1)}(z(x)) \right| \frac{(x - x_0)(x - x_1) \cdots (x - x_n)}{(n + 1)!} \]

\[ \leq \max_{-1 \leq x \leq 1} \left| f^{(n+1)}(z(x)) \right| \max_{-1 \leq x \leq 1} \left| \tilde{T}_{n+1}(x) \right| \]

\[ \leq 2^{-n} \max_{-1 \leq x \leq 1} \left| f^{(n+1)}(z(x)) \right| \frac{1}{(n + 1)!} . \]
Thus we have the following corollary.

**Corollary**

*Suppose that $P(x)$ is the interpolating polynomial of degree at most $n$ with nodes at the zeros of $T_{n+1}(x)$. Then*

$$\max_{-1 \leq x \leq 1} |f(x) - P(x)| \leq \frac{1}{2^n(n + 1)!} \max_{-1 \leq x \leq 1} |f^{(n+1)}(x)|$$

*for every $f \in C^{n+1}[-1, 1]$.*

**Remark:** if we wish to interpolate over the arbitrary interval $[a, b]$ then use the change of variables

$$t = \frac{b - a}{2} x + \frac{b + a}{2}.$$
Consider the function \( f(x) = x \ln x \) on the interval \([1, 3]\). Compare the values given by the Lagrange interpolating polynomial found using four equally spaced nodes to the Lagrange interpolating polynomial with nodes given by the roots of \( T_4(x) \).
Example (2 of 4)

With equally spaced nodes at $x_0 = 1$, $x_1 = 5/3$, $x_2 = 7/3$, and $x_3 = 3$ we have the Lagrange interpolating polynomial

$$P_3(x) = f(1)L_0(x) + f(5/3)L_1(x) + f(7/3)L_2(x) + f(3)L_3(x)$$

$$= -0.585346 + 0.0942651x + 0.536711x^2 - 0.04563x^3.$$
Example (2 of 4)

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\[ P_3(x) = f(1)L_0(x) + f(5/3)L_1(x) + f(7/3)L_2(x) + f(3)L_3(x) \]
\[ = -0.585346 + 0.0942651x + 0.536711x^2 - 0.04563x^3. \]

With nodes at the roots of $T_4(x)$, i.e., $x_i = \cos(2i + 1)\pi/8$ for $i = 0, 1, 2, 3$ we have the Lagrange interpolating polynomial

\[ Q_3(x) = f(x_0 + 2)L_0(x + 2) + f(x_1 + 2)L_1(x + 2) \]
\[ + f(x_2 + 2)L_2(x + 2) + f(x_3 + 2)L_3(x + 2) \]
\[ = -0.595225 + 0.10582x + 0.532437x^2 - 0.0451646x^3. \]
Example (3 of 4)

Interpolation

Absolute Error
Example (4 of 4)

Computing the error bounds we have

\[
|E_P| = \max_{1 \leq x \leq 3} \left| \frac{f^{(4)}(z(x))}{4!} (x - 1)(x - 5/3)(x - 7/3)(x - 3) \right| \\
\leq \frac{0.197531}{24} \max_{1 \leq x \leq 3} \left| \frac{2}{x^3} \right| \\
= 0.0164609
\]

and

\[
|E_Q| = \max_{1 \leq x \leq 3} \left| \frac{f^{(4)}(z(x))}{4!} \prod_{k=0} \left( x - 2 - \cos \left[ \frac{(2k + 1)\pi}{8} \right] \right) \right| \\
\leq \frac{2}{24} \max_{1 \leq x \leq 3} \left| \prod_{k=0} \left( x - 2 - \cos \left[ \frac{(2k + 1)\pi}{8} \right] \right) \right| \\
= \frac{0.125}{12} \\
= 0.0104167.
\]
Reducing the Degree of Approximating Polynomials

Suppose we want to approximate a polynomial of degree \( n \),

\[ P_n(x) = a_0 + a_1 x + \cdots + a_{n-1} x^{n-1} + a_n x^n \]

by a polynomial of degree at most \( n - 1 \).

**Objective:** select \( P_{n-1}(x) \in \Pi_{n-1} \) so that

\[ \max_{-1 \leq x \leq 1} |P_n(x) - P_{n-1}(x)| \]

is as small as possible.
Recognize that
\[ Q(x) = \frac{1}{a_n} (P_n(x) - P_{n-1}(x)) \]
is a monic polynomial of degree \( n \).
Recognize that

\[ Q(x) = \frac{1}{a_n} (P_n(x) - P_{n-1}(x)) \]

is a monic polynomial of degree \( n \).

Thus the following inequality holds.

\[ \frac{1}{2^{n-1}} \leq \max_{-1 \leq x \leq 1} |Q(x)| = \max_{-1 \leq x \leq 1} \left| \frac{1}{a_n} (P_n(x) - P_{n-1}(x)) \right| \]
Recognize that

\[ Q(x) = \frac{1}{a_n} (P_n(x) - P_{n-1}(x)) \]

is a monic polynomial of degree \( n \).

Thus the following inequality holds.

\[
\frac{1}{2^{n-1}} \leq \max_{-1 \leq x \leq 1} |Q(x)| = \max_{-1 \leq x \leq 1} \left| \frac{1}{a_n} (P_n(x) - P_{n-1}(x)) \right|
\]

Equality holds when

\[ Q(x) = \frac{1}{a_n} (P_n(x) - P_{n-1}(x)) = \tilde{T}_n(x). \]
Thus we should choose

\[
\frac{1}{a_n} (P_n(x) - P_{n-1}(x)) = \tilde{T}_n(x)
\]

\[
P_{n-1}(x) = P_n(x) - a_n \tilde{T}_n(x).
\]
Thus we should choose

\[
\frac{1}{a_n} (P_n(x) - P_{n-1}(x)) = \tilde{T}_n(x)
\]

\[
P_{n-1}(x) = P_n(x) - a_n \tilde{T}_n(x).
\]

The error bound is

\[
\max_{-1 \leq x \leq 1} |P_n(x) - P_{n-1}(x)| = \max_{-1 \leq x \leq 1} |a_n \tilde{T}_n(x)| = \frac{|a_n|}{2^{n-1}}.
\]
The sixth Maclaurin polynomial for \( f(x) = xe^x \) is

\[
P(x) = x + x^2 + \frac{x^3}{2} + \frac{x^4}{6} + \frac{x^5}{24} + \frac{x^6}{120}.
\]

A bound for the error in this approximation on \([-1, 1]\) is \( E = 0.00161516 \).

Use Chebyshev economization to find a polynomial of lesser degree to approximate \( f(x) \) while keeping the error less than 0.01 for \(-1 \leq x \leq 1\).
The polynomial of degree 5 which best approximates $P(x)$ on $[-1, 1]$ is

$$P_5(x) = P(x) - \frac{1}{120} \tilde{T}_6(x)$$

$$= \frac{x^5}{24} + \frac{43x^4}{240} + \frac{x^3}{2} + \frac{637x^2}{640} + x + \frac{1}{3840}.$$
The polynomial of degree 5 which best approximates $P(x)$ on $[-1, 1]$ is

$$P_5(x) = P(x) - \frac{1}{120} \tilde{T}_6(x)$$

$$= \frac{x^5}{24} + \frac{43x^4}{240} + \frac{x^3}{2} + \frac{637x^2}{640} + x + \frac{1}{3840}.$$ 

**Note:** $|P(x) - P_5(x)| = \left| \frac{1}{120} \tilde{T}_6(x) \right| \leq \frac{1}{120(2^5)} \approx 0.000260417$
The polynomial of degree 5 which best approximates $P(x)$ on $[-1, 1]$ is

$$P_5(x) = P(x) - \frac{1}{120} \tilde{T}_6(x)$$

$$= \frac{x^5}{24} + \frac{43x^4}{240} + \frac{x^3}{2} + \frac{637x^2}{640} + x + \frac{1}{3840}.$$

**Note:** $|P(x) - P_5(x)| = \left| \frac{1}{120} \tilde{T}_6(x) \right| \leq \frac{1}{120(2^5)} \approx 0.000260417$

Adding this to the previous error bound gives a total error bound of

$$0.00161516 + 0.000260417 = 0.00187558 < 0.01.$$
The polynomial of degree 4 which best approximates $P_5(x)$ on $[-1, 1]$ is

$$P_4(x) = P_5(x) - \frac{1}{24} \tilde{T}_5(x)$$

$$= \frac{1}{3840}(688x^4 + 2120x^3 + 3822x^2 + 3790x + 1).$$
The polynomial of degree 4 which best approximates $P_5(x)$ on $[-1, 1]$ is

$$P_4(x) = P_5(x) - \frac{1}{24} \tilde{T}_5(x)$$

$$= \frac{1}{3840} (688x^4 + 2120x^3 + 3822x^2 + 3790x + 1).$$

**Note:** $|P_4(x) - P_5(x)| = \left| \frac{1}{24} \tilde{T}_5(x) \right| \leq \frac{1}{24(2^4)} \approx 0.00260417$
The polynomial of degree 4 which best approximates $P_5(x)$ on $[-1, 1]$ is

\[ P_4(x) = P_5(x) - \frac{1}{24} \tilde{T}_5(x) \]

\[ = \frac{1}{3840}(688x^4 + 2120x^3 + 3822x^2 + 3790x + 1). \]

**Note:** $|P_4(x) - P_5(x)| = \left| \frac{1}{24} \tilde{T}_5(x) \right| \leq \frac{1}{24(2^4)} \approx 0.00260417$

Adding this to the previous error bound gives a total error bound of

\[ 0.00161516 + 0.000260417 + 0.00260417 = 0.00447975 < 0.01. \]
Solution (3 of 4)

The polynomial of degree 3 which best approximates $P_4(x)$ on $[-1, 1]$ is

$$P_3(x) = P_4(x) - \frac{688}{3840} \tilde{T}_4(x)$$

$$= \frac{1}{768} (424x^3 + 902x^2 + 758x - 17).$$
Solution (3 of 4)

The polynomial of degree 3 which best approximates $P_4(x)$ on $[-1, 1]$ is

$$P_3(x) = P_4(x) - \frac{688}{3840} \tilde{T}_4(x)$$

$$= \frac{1}{768}(424x^3 + 902x^2 + 758x - 17).$$

**Note:** $|P_3(x) - P_4(x)| = \left| \frac{688}{3840} \tilde{T}_4(x) \right| \leq \frac{688}{3840(2^3)} \approx 0.0223958$
The polynomial of degree 3 which best approximates $P_4(x)$ on $[-1, 1]$ is

$$P_3(x) = P_4(x) - \frac{688}{3840} \tilde{T}_4(x)$$

$$= \frac{1}{768}(424x^3 + 902x^2 + 758x - 17).$$

**Note:** $|P_3(x) - P_4(x)| = \left| \frac{688}{3840} \tilde{T}_4(x) \right| \leq \frac{688}{3840(2^3)} \approx 0.0223958$

Adding this to the previous error bound gives a total error bound of $0.0268756 > 0.01$,

thus we may use $P_4(x)$ to approximate $f(x)$ to within 0.01 on $[-1, 1]$. 
Example (4 of 4)

Interpolation

Absolute Error
Homework

- Read Section 8.3.
- Exercises: 1ab, 3ab, 5ab, 7, 8