Algorithms and Convergence

MATH 375 *Numerical Analysis*

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Algorithms and Pseudocode

Definition

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A **pseudocode** is a structured language for describing algorithms.

Example

Mathematical calculation: $S = \sum_{i=1}^{N} x_i$.

Pseudocode:

Step 1 INPUT N, $x_1, x_2, \ldots, x_n$.

Step 2 Set $SUM = 0$.

Step 3 For $i = 1, 2, \ldots, N$ set $SUM = SUM + x_i$.

Step 4 OUTPUT (SUM); STOP.

Stability

Definition

An algorithm which possesses the property that small changes in its inputs produce small changes in its outputs is said to be **stable**; otherwise it is **unstable**.

Notation:

- $E_0 > 0$ denotes the error introduced at some stage in a calculation.
- E_n denotes the error n operations later.

Growth of Error

Definition

Suppose E_n represents the magnitude of error after n subsequent operations.

- If $E_n = C \cdot n \cdot E_0$ where C is a constant independent of n, then the growth rate of the error is described as **linear**.
- If $E_n = C^n E_0$ where $C > 1$ is a constant, then the growth rate of the error is described as **exponential**.

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Remark: a linear growth of error is generally acceptable in an algorithm; however, exponential growth of error should be avoided.

Example

Let c_1 and c_2 be real number constants and define the sequence $\{p_n\}_{n=0}^{\infty}$ as

$$p_n = c_1 \left(\frac{1}{3}\right)^n + c_2 3^n \quad \text{for } n = 0, 1, \ldots .$$

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1. Show that for $n = 2, 3, \ldots$ the sequence elements satisfy the equation

$$p_n = \frac{10}{3} p_{n-1} - p_{n-2}.$$

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3. Using the values found for c_1 and c_2, find a simple formula for p_n.

Solution (1 of 2)

$$
\begin{aligned}
p_n &= c_1\left(\frac{1}{3}\right)^n + c_2 3^n \\
p_{n-1} &= c_1\left(\frac{1}{3}\right)^{n-1} + c_2 3^{n-1} \\
p_{n-2} &= c_1\left(\frac{1}{3}\right)^{n-2} + c_2 3^{n-2}
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\begin{aligned}
\frac{10}{3}p_{n-1} - p_{n-2} &= \frac{10}{3}\left(c_1\left(\frac{1}{3}\right)^{n-1} + c_2 3^{n-1}\right) - c_1\left(\frac{1}{3}\right)^{n-2} - c_2 3^{n-2} \\
&= c_1\left(10\left(\frac{1}{3}\right)^n - \left(\frac{1}{3}\right)^{n-2}\right) + c_2\left(10(3)^{n-2} - (3)^{n-2}\right) \\
&= c_1\left(\left(\frac{1}{3}\right)^n + 9\left(\frac{1}{3}\right)^n - \left(\frac{1}{3}\right)^{n-2}\right) + c_2\left(9(3)^{n-2}\right) \\
&= c_1\left(\left(\frac{1}{3}\right)^n + \left(\frac{1}{3}\right)^{n-2} - \left(\frac{1}{3}\right)^{n-2}\right) + c_2 3^n \\
&= c_1\left(\frac{1}{3}\right)^n + c_2 3^n = p_n
\end{aligned}
$$

Solution (2 of 2)

Consider the simultaneous equations:

$$\begin{aligned} p_0 &= c_1 \left(\frac{1}{3}\right)^0 + c_2(3)^0 = c_1 + c_2 = 1 \\ p_1 &= c_1 \left(\frac{1}{3}\right)^1 + c_2(3)^1 = \frac{1}{3}c_1 + 3c_2 = \frac{1}{3} \end{aligned}$$

By inspection, the solutions are $c_1 = 1$ and $c_2 = 0$.

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By inspection, the solutions are $c_1 = 1$ and $c_2 = 0$.

Using the values of c_1 and c_2 then a simple formula for p_n is

$$p_n = c_1\left(\frac{1}{3}\right)^n + c_2 3^n = \left(\frac{1}{3}\right)^n.$$

Example (1 of 2)

Suppose

$$\hat{p}_n = \hat{c}_1 \left(\frac{1}{3}\right)^n + \hat{c}_2 3^n$$

and we use 5-digit rounded approximations to $\hat{p}_0 = 1$ and $\hat{p}_1 = \frac{1}{3}$. Then we must solve the system of equations:

$$\begin{aligned} \hat{c}_1 + \hat{c}_2 &= 0.10000 \times 10^1 \\ \frac{1}{3}\hat{c}_1 + 3\hat{c}_2 &= 0.33333 \times 10^0 \end{aligned}$$

Using exact arithmetic to solve for $\hat{c}_1$ and $\hat{c}_2$ we find

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These values define the recursive sequence

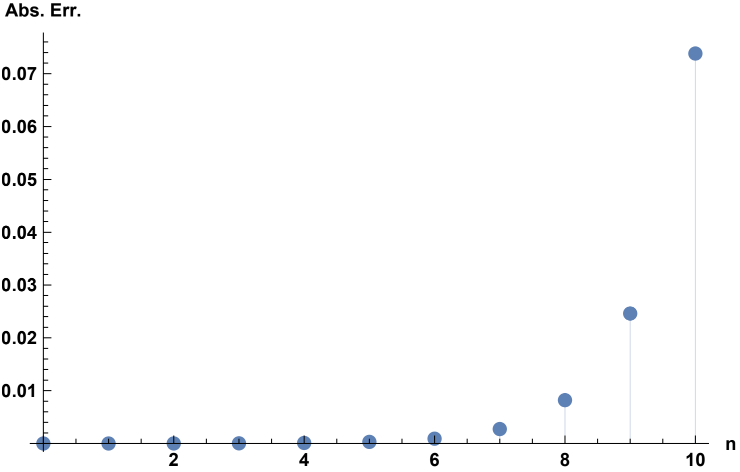
$$\hat{p}_n = (0.10000 \times 10^1) \left(\frac{1}{3}\right)^n - (0.12500 \times 10^{-5}) 3^n.$$

Example (2 of 2)

Consider the absolute error

$$|p_n - \hat{p}_n| = (0.12500 \times 10^{-5})3^n.$$

The error grows exponentially.



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p_n &= c_1 + c_2 n \\
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2p_{n-1} - p_{n-2} &= 2(c_1 + c_2(n-1)) - c_1 - c_2(n-2) \\
&= c_1(2-1) + c_2(2(n-1) - (n-2)) \\
&= c_1 + c_2 n = p_n
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Consider the simultaneous equations:

$$\begin{aligned} p_0 &= c_1 = 1 \\ p_1 &= c_1 + c_2(1) = \frac{1}{3}. \end{aligned}$$

By inspection $c_1 = 1$ and $c_2 = -2/3$.

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By inspection $c_1 = 1$ and $c_2 = -2/3$.

Using the values $c_1 = 1$ and $c_2 = -2/3$, a simple formula for p_n is

$$p_n = 1 - \frac{2}{3}n.$$

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Suppose

$$\hat{p}_n = \hat{c}_1 + \hat{c}_2 n$$

with $\hat{p}_0 = 0.10000 \times 10^1$ and $\hat{p}_1 = 0.33333 \times 10^0$.
If we use 5-digit rounding arithmetic to solve for $\hat{c}_1$ and $\hat{c}_2$ we find

$$\hat{c}_1 = 0.10000 \times 10^1 \quad \text{and} \quad \hat{c}_2 = -0.66667 \times 10^0.$$

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These values define the recursive sequence

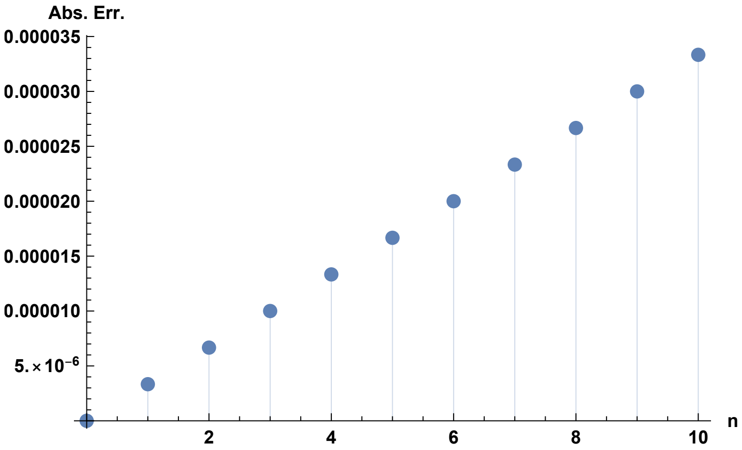
$$\hat{p}_n = 0.10000 \times 10^1 - (0.66667 \times 10^0)n.$$

Example (2 of 2)

Consider the absolute error

$$|p_n - \hat{p}_n| = \left(0.66667 - \frac{2}{3}\right) n.$$

The error grows linearly.



Rate of Convergence

Definition

Suppose $\{\beta_n\}_{n=0}^{\infty}$ is a sequence which converges to 0 and $\{\alpha_n\}_{n=0}^{\infty}$ is a sequence which converges to α. If there exists $K > 0$ such that

$$|\alpha_n - \alpha| \leq K|\beta_n|$$

for large n, then $\{\alpha_n\}_{n=0}^{\infty}$ converges to α with **rate of convergence** $O(\beta_n)$ (read as "big oh of β_n").

We will denote this as $\alpha_n = \alpha + O(\beta_n)$.

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We will denote this as $\alpha_n = \alpha + O(\beta_n)$.

Remark: We will frequently take $\beta_n = \dfrac{1}{n^p}$ where $p > 0$.

Example

Find the rate of convergence of the sequence whose terms are defined as

$$\alpha_n = \frac{2n^2 + n + 1}{n^2 + 1}.$$

Solution

- We can see that $\alpha_n \to 2$ as $n \to \infty$.

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$$\frac{2n^2+n+1}{n^2+1} - 2 = \frac{2n^2+n+1}{n^2+1} - \frac{2n^2+2}{n^2+1} = \frac{n-1}{n^2+1}.$$

Solution

- We can see that $\alpha_n \to 2$ as $n \to \infty$.
- Consider $\alpha_n - 2$,

$$\frac{2n^2+n+1}{n^2+1} - 2 = \frac{2n^2+n+1}{n^2+1} - \frac{2n^2+2}{n^2+1} = \frac{n-1}{n^2+1}.$$

- Establish a bound for $|\alpha_n - 2|$:

$$|\alpha_n - 2| = \left|\frac{n-1}{n^2+1}\right| \leq \left|\frac{n-1}{n^2}\right| \leq \left|\frac{n}{n^2}\right| = \left|\frac{1}{n}\right|$$

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- Let $\beta_n = 1/n$ (since $\beta_n \to 0$ as $n \to \infty$) and $K = 1$, then

$$|\alpha_n - 2| \leq K|\beta_n|$$

and the rate of convergence of $\{\alpha_n\}_{n=0}^{\infty}$ is $O(1/n)$.

Rate of Convergence for Functions

Definition

Suppose $\lim_{h \to 0} G(h) = 0$ and $\lim_{h \to 0} F(h) = L$. If there exists a constant $K > 0$ such that

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for sufficiently small h, then we may state $F(h) = L + O(G(h))$.

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for sufficiently small h, then we may state $F(h) = L + O(G(h))$.

Remark: We will frequently use function $G(h)$ of the form $G(h) = h^p$ where $p > 0$.

Example

Show that $\sin x + \frac{x^3}{3!} = x + O(x^5)$. (*Hint*: use Taylor's Theorem.)

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Using Taylor's Theorem

$$\begin{aligned}
\sin x &= x - \frac{x^3}{3!} + \frac{x^5}{5!}\cos z(x) \\
\sin x + \frac{x^3}{3!} - x &= \frac{x^5}{5!}\cos z(x) \\
\left|\sin x + \frac{x^3}{3!} - x\right| &= \left|\frac{x^5}{5!}\right| |\cos z(x)| \\
\left|\left(\sin x + \frac{x^3}{3!}\right) - x\right| &\leq \frac{1}{120}\left|x^5\right|.
\end{aligned}$$

Homework

- Read Section 1.3.
- Exercises: 3, 6, 7, 8, 11