Newton Polynomials and Divided Differences

MATH 375 *Numerical Analysis*

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Background

- Constructing Lagrange polynomials is relatively easy as a pencil and paper technique, but difficult to automate.
- Neville's iterated interpolation can approximate a function at a single point, but does not construct a polynomial.
- Today we learn an iterated technique for building up the Lagrange interpolating polynomials.

Polynomial Interpolation

Suppose polynomial $P_n(x)$ interpolates the data:

$$\{(x_0, f(x_0)), (x_1, f(x_1)), \ldots, (x_n, f(x_n))\}.$$

If one more data point is added, say

$$\{(x_0, f(x_0)), (x_1, f(x_1)), \ldots, (x_n, f(x_n)), (x_{n+1}, f(x_{n+1}))\},$$

we would like to use $P_n(x)$ to find $P_{n+1}(x)$.

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we would like to use $P_n(x)$ to find $P_{n+1}(x)$.

Imagine that

$$\begin{aligned} P_{n+1}(x) &= P_n(x) + q(x) \\ q(x) &= P_{n+1}(x) - P_n(x). \end{aligned}$$

Polynomial $q(x)$ interpolates the data,

$$\{(x_0, 0), (x_1, 0), \ldots, (x_n, 0), (x_{n+1}, f(x_{n+1}) - P_n(x_{n+1}))\},$$

Lagrange Form of $q(x)$

Polynomial $q(x)$ can be expressed as a single Lagrange basis polynomial.

$$q(x) = (f(x_{n+1}) - P_n(x_{n+1})) \prod_{k=0}^{n} \frac{x - x_k}{x_{n+1} - x_k}$$

Lagrange Interpolating Polynomial

Suppose $f(x)$ is a function and $P_n(x)$ is the Lagrange interpolating polynomial of degree at most n which agrees with $f(x)$ at the distinct points $\{x_0, x_1, \ldots, x_n\}$.

Lagrange Interpolating Polynomial

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We can think of $P_n(x)$ as

$$\begin{aligned} P_n(x) &= a_0 + a_1(x - x_0) + a_2(x - x_0)(x - x_1) \\ &\quad + \cdots + a_n(x - x_0)\cdots(x - x_{n-1}) \\ &= a_0 + \sum_{i=1}^{n} a_i \prod_{j=0}^{i-1}(x - x_j) \end{aligned}$$

for an appropriate choice of constants a_0, a_1, ..., a_n.

Question: how can we find these constants?

Evaluation of $P_n(x)$

$$P_n(x) = a_0 + a_1(x - x_0) + a_2(x - x_0)(x - x_1) + \cdots + a_n(x - x_0)\cdots(x - x_{n-1})$$

- If $x = x_0$ then $P_n(x_0) = f(x_0) = a_0$.

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- If $x = x_0$ then $P_n(x_0) = f(x_0) = a_0$.
- If $x = x_1$ then $P_n(x_1) = f(x_1)$ and

$$P_n(x_1) = a_0 + a_1(x_1 - x_0)$$
$$f(x_1) = f(x_0) + a_1(x_1 - x_0)$$
$$a_1 = \frac{f(x_1) - f(x_0)}{x_1 - x_0}$$

- and so on.

Find a_2

$$P_n(x_2) = a_0 + a_1(x_2 - x_0) + a_2(x_2 - x_0)(x_2 - x_1)$$

$$f(x_2) = f(x_0) + \frac{f(x_1) - f(x_0)}{x_1 - x_0}(x_2 - x_0) + a_2(x_2 - x_0)(x_2 - x_1)$$

$$a_2(x_2 - x_0)(x_2 - x_1) = f(x_2) - f(x_0) - \frac{f(x_1) - f(x_0)}{x_1 - x_0}(x_2 - x_0)$$

$$a_2 = \frac{f(x_2) - f(x_0) - \frac{f(x_1)-f(x_0)}{x_1-x_0}(x_2 - x_0)}{(x_2 - x_0)(x_2 - x_1)} = \frac{\frac{f(x_2)-f(x_1)}{x_2-x_1} - \frac{f(x_1)-f(x_0)}{x_1-x_0}}{x_2 - x_0}$$

Divided Difference Notation (1 of 2)

- Denote the **zeroth divided difference** of f with respect to x_i by

$$f[x_i] = f(x_i).$$

- Denote the **first divided difference** of f with respect to x_i and x_{i+1} by

$$f[x_i, x_{i+1}] = \frac{f[x_{i+1}] - f[x_i]}{x_{i+1} - x_i}.$$

- Denote the **second divided difference** of f with respect to x_i, x_{i+1}, and x_{i+2} by

$$f[x_i, x_{i+1}, x_{i+2}] = \frac{f[x_{i+1}, x_{i+2}] - f[x_i, x_{i+1}]}{x_{i+2} - x_i}.$$

Divided Difference Notation (2 of 2)

Proceeding recursively,

- Denote the k**th divided difference** of f with respect to x_i, x_{i+1}, x_{i+2}, ..., x_{i+k} by

$$f[x_i, x_{i+1}, \ldots, x_{i+k-1}, x_{i+k}] = \frac{f[x_{i+1}, x_{i+2}, \ldots, x_{i+k}] - f[x_i, x_{i+1}, \ldots, x_{i+k-1}]}{x_{i+k} - x_i}.$$

- Finally, denote the n**th divided difference** of f with respect to x_0, x_1, x_2, ..., x_n by

$$f[x_0, x_1, \ldots, x_n] = \frac{f[x_1, x_2, \ldots, x_n] - f[x_0, x_1, \ldots, x_{n-1}]}{x_n - x_0}.$$

Summary and Connections

Recall that

$$P_n(x) = a_0 + \sum_{k=1}^{n} a_k \prod_{j=0}^{k-1} (x - x_j).$$

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$$P_n(x) = a_0 + \sum_{k=1}^{n} a_k \prod_{j=0}^{k-1} (x - x_j).$$

Using the divided difference notation we see that

$$\begin{aligned} a_0 &= f[x_0] \\ a_1 &= f[x_0, x_1] \\ a_2 &= f[x_0, x_1, x_2] \\ &\vdots \\ a_n &= f[x_0, x_1, x_2, \ldots, x_n], \text{ and thus} \end{aligned}$$

$$P_n(x) = f[x_0] + \sum_{k=1}^{n} f[x_0, \ldots, x_k] \prod_{j=0}^{k-1} (x - x_j).$$

This is called **Newton's interpolatory divided difference formula**.

Table Format

x	$f(x)$	First	Second	Third
x_0	$f[x_0]$			
		$f[x_0, x_1] = \dfrac{f[x_1] - f[x_0]}{x_1 - x_0}$		
x_1	$f[x_1]$		$f[x_0, x_1, x_2] = \dfrac{f[x_1, x_2] - f[x_0, x_1]}{x_2 - x_0}$	
		$f[x_1, x_2] = \dfrac{f[x_2] - f[x_1]}{x_2 - x_1}$		$f[x_0, x_1, x_2, x_3] = \dfrac{f[x_1, x_2, x_3] - f[x_0, x_1, x_2]}{x_3 - x_0}$
x_2	$f[x_2]$		$f[x_1, x_2, x_3] = \dfrac{f[x_2, x_3] - f[x_1, x_2]}{x_3 - x_1}$	
		$f[x_2, x_3] = \dfrac{f[x_3] - f[x_2]}{x_3 - x_2}$		
x_3	$f[x_3]$			

Divided Difference Algorithm

INPUT nodes $\{(x_0, f(x_0)), \ldots, (x_n, f(x_n))\}$

STEP 1 For $i = 0, 1, \ldots, n$ set $F_{i,0} = f(x_i)$.

STEP 2 For $i = 1, 2, \ldots, n$

For $j = 1, 2, \ldots, i$ set

$$F_{i,j} = \frac{F_{i,j-1} - F_{i-1,j-1}}{x_i - x_{i-j}}$$

STEP 3 OUTPUT $F_{0,0}, F_{1,1}, \ldots, F_{n,n}$. STOP.

Remark: the output values are the top entries in the columns of the preceding table.

Example (1 of 2)

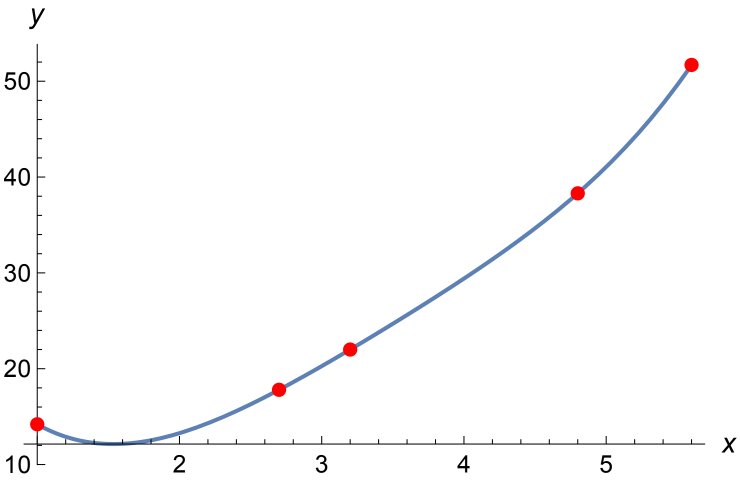
Complete the divided difference table and construct the interpolating polynomial.

x_i	$f(x_i)$	**First**	**Second**	**Third**	**Fourth**
3.2	22.0				
		8.4			
2.7	17.8		2.85561		
		2.11765		-0.52748	
1.0	14.2		2.01165		0.255838
		6.34211		0.0865307	
4.8	38.3		2.26259		
		16.75			
5.6	51.7				

Example (2 of 2)

$$\begin{aligned}
P_4(x) &= 22.0 + 8.4(x - 3.2) + 2.85561(x - 3.2)(x - 2.7) \\
&\quad - 0.52748(x - 3.2)(x - 2.7)(x - 1.0) \\
&\quad + 0.255838(x - 3.2)(x - 2.7)(x - 1.0)(x - 4.8) \\
&= 34.96 - 36.1836x + 18.6885x^2 - 3.52078x^3 + 0.255838x^4
\end{aligned}$$

Graph



Implications of the Mean Value Theorem

$$f[x_i, x_j] = \frac{f(x_j) - f(x_i)}{x_j - x_i} = f'(z)$$

for some z between x_i and x_j according to the MVT.

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This can be generalized.

Theorem

Suppose $f \in \mathcal{C}^n[a, b]$ and $x_0, x_1, \ldots, x_n$ are distinct numbers in $[a, b]$. There exists $z \in (a, b)$ such that

$$f[x_0, x_1, \ldots, x_n] = \frac{f^{(n)}(z)}{n!}.$$

Proof

- Define $g(x) = f(x) - P_n(x)$.
- Since $f(x_i) = P_n(x_i)$ for $i = 0, 1, \ldots, n$, then function g has $n + 1$ distinct roots in $[a, b]$.
- According to the Generalized Rolle's Theorem, $g^{(n)}(z) = 0$ for some $z \in (a, b)$.

$$\begin{aligned}
0 &= g^{(n)}(z) \\
&= f^{(n)}(z) - P_n^{(n)}(z) \\
P_n^{(n)}(z) &= f^{(n)}(z) \\
n!\, f[x_0, x_1, \ldots, x_n] &= f^{(n)}(z) \\
f[x_0, x_1, \ldots, x_n] &= \frac{f^{(n)}(z)}{n!}
\end{aligned}$$

Remarks

- The coordinates of the nodes $x_0, x_1, \ldots, x_n$ need not be in ascending order.
- The spacing between the nodes $\Delta x_i = x_{i+1} - x_i$ need not be uniform.

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- The coordinates of the nodes $x_0, x_1, \ldots, x_n$ need not be in ascending order.
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However, if the nodes are in ascending order and the spacing between nodes is uniform, we can modify Newton's divided difference formula.

Forward Differences (1 of 4)

Suppose $x_{i+1} - x_i = h > 0$ for $i = 0, 1, \ldots, n-1$, then

- For any x there exists s such that $x = x_0 + s\,h$.
- In particular $x_i = x_0 + ih$ for $i = 0, 1, \ldots, n$.
- For $i = 0, 1, \ldots, n$ the difference

$$x - x_i = (x_0 + s\,h) - x_i = (x_0 + s\,h) - (x_0 + ih) = (s - i)h.$$

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$$P_n(x) = f[x_0] + \sum_{k=1}^{n} f[x_0, \ldots, x_k] \prod_{j=0}^{k-1} (x - x_j)$$

$$\begin{aligned} P_n(x_0 + sh) &= f[x_0] + \sum_{k=1}^{n} f[x_0, \ldots, x_k] \prod_{j=0}^{k-1} (x_0 + s\,h - x_0 - j\,h) \\ &= f[x_0] + \sum_{k=1}^{n} f[x_0, \ldots, x_k] \prod_{j=0}^{k-1} ((s - j)h) \\ &= f[x_0] + \sum_{k=1}^{n} h^k f[x_0, \ldots, x_k] \prod_{j=0}^{k-1} (s - j) \end{aligned}$$

Forward Differences (2 of 4)

Using the binomial coefficient notation

$$\binom{s}{k} = \frac{s!}{(s-k)!\,k!} = \frac{s(s-1)\cdots(s-k+1)}{k!} = \frac{\prod_{j=0}^{k-1}(s-j)}{k!}$$

$$s(s-1)\cdots(s-k+1) = k!\binom{s}{k},$$

we can write

$$P_n(x) = f[x_0] + \sum_{k=1}^{n} h^k f[x_0,\ldots,x_k] \prod_{j=0}^{k-1}(s-j) = f[x_0] + \sum_{k=1}^{n} h^k k!\binom{s}{k} f[x_0,\ldots,x_k].$$

Forward Differences (3 of 4)

Recalling Aitken's Δ^2 notation we may write

$$
\begin{aligned}
f[x_0, x_1] &= \frac{f(x_1) - f(x_0)}{x_1 - x_0} = \frac{\Delta f(x_0)}{h} \\
f[x_0, x_1, x_2] &= \frac{f[x_1, x_2] - f[x_0, x_1]}{x_2 - x_0} = \frac{1}{2h}\left(\frac{\Delta f(x_1)}{h} - \frac{\Delta f(x_0)}{h}\right) = \frac{\Delta^2 f(x_0)}{2h^2} \\
&\vdots \\
f[x_0, x_1, \ldots, x_k] &= \frac{f[x_1, x_2, \ldots, x_k] - f[x_0, x_1, \ldots, x_{k-1}]}{x_k - x_0} \\
&= \frac{1}{k\,h}\left(\frac{\Delta^{k-1} f(x_1)}{(k-1)!h^{k-1}} - \frac{\Delta^{k-1} f(x_0)}{(k-1)!h^{k-1}}\right) = \frac{\Delta^k f(x_0)}{k!\,h^k}.
\end{aligned}
$$

Forward Differences (4 of 4)

Finally, we may write the **Newton Forward-Difference Formula**:

$$\begin{aligned} P_n(x) &= f[x_0] + \sum_{k=1}^{n} h^k k! \binom{s}{k} f[x_0, \ldots, x_k] \\ &= f[x_0] + \sum_{k=1}^{n} h^k k! \binom{s}{k} \frac{1}{k!\, h^k} \Delta^k f(x_0) \\ &= f[x_0] + \sum_{k=1}^{n} \binom{s}{k} \Delta^k f(x_0) \end{aligned}$$

Comments

- Forward-differences on the nodes

$$x_0 < x_1 < \cdots < x_{n-1} < x_n$$

are useful when x is nearer to x_0 than to x_n since generally $f(x_0)$ will be closer to $f(x)$ than will $f(x_n)$.

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- If we need to approximate f at x near x_n then we should reorder the nodes as

$$x_n > x_{n-1} > \cdots > x_1 > x_0.$$

The interpolating polynomial becomes

$$P_n(x) = f[x_n] + \sum_{i=1}^{n} f[x_n, \ldots, x_{n-i}](x - x_n) \cdots (x - x_{n-i+1}).$$

Backward Differences (1 of 4)

Definition

Given the sequence $\{p_n\}_{n=0}^{\infty}$ we define the **backward difference** ∇p_n as

$$\nabla p_n = p_n - p_{n-1}, \quad \text{for } n \geq 1.$$

For $k \geq 2$ we define the kth order backward difference as

$$\nabla^k p_n = \nabla(\nabla^{k-1} p_n).$$

Backward Differences (2 of 4)

Using the backward difference notation we may write

$$\begin{aligned} f[x_n, x_{n-1}] &= \frac{f(x_{n-1}) - f(x_n)}{x_{n-1} - x_n} = \frac{f(x_n) - f(x_{n-1})}{x_n - x_{n-1}} = \frac{1}{h}\nabla f(x_n) \\ f[x_n, x_{n-1}, x_{n-2}] &= \frac{1}{2h^2}\nabla^2 f(x_n) \\ &\vdots \\ f[x_n, x_{n-1}, \ldots, x_{n-k}] &= \frac{1}{k!\,h^k}\nabla^k f(x_n). \end{aligned}$$

Backward Differences (3 of 4)

Writing $x = x_n + sh$ where $s < 0$ and $x - x_i = (s + n - i)h$ then the interpolating polynomial can be written as

$$
\begin{aligned}
P_n(x) &= f[x_n] + \sum_{i=1}^{n} f[x_n, \ldots, x_{n-i}](x - x_n) \cdots (x - x_{n-i+1}) \\
&= f[x_n] + \sum_{i=1}^{n} h^i s(s+1) \cdots (s+n-i) f[x_n, \ldots, x_{n-i}] \\
&= f[x_n] + \sum_{i=1}^{n} \frac{s(s+1) \cdots (s+n-i)}{i!} \nabla^i f[x_n].
\end{aligned}
$$

Backward Differences (4 of 4)

Since $s < 0$ we must modify the binomial coefficient notation.

$$\binom{-s}{k} = \frac{-s(-s-1)\cdots(-s-k+1)}{k!} = (-1)^k \frac{s(s+1)\cdots(s+k-1)}{k!}$$

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Then we may write the interpolating polynomial as

$$\begin{aligned} P_n(x) &= f[x_n] + \sum_{i=1}^{n} \frac{s(s+1)\cdots(s+n-i)}{i!} \nabla^i f[x_n] \\ &= f[x_n] + \sum_{i=1}^{n} (-1)^i \binom{-s}{i} \nabla^i f[x_n] \end{aligned}$$

This is known as the **Newton backward-difference formula**.

Example (1 of 4)

Suppose we create forward and backward difference interpolating polynomials for $f(x) = \cos x$ using nodes $x_i = 0.2(i+1)$ for $i = 0, 1, 2, 3$.

x	$\cos x$	**First**	**Second**	**Third**
0.2	0.980067			
		−0.295028		
0.4	0.921061		−0.458997	
		−0.478627		0.0795056
0.6	0.825336		−0.411294	
		−0.643145		
0.8	0.696707			

Example (2 of 4)

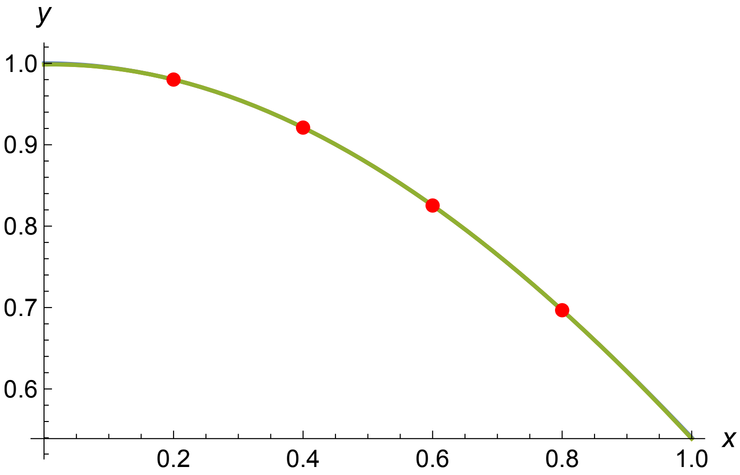
Forward divided difference:

$$P_3(x) = 0.998536 + 0.015353x - 0.554404x^2 + 0.0795056x^3$$

Backward divided difference:

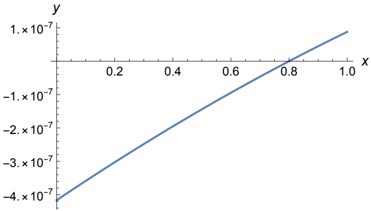
$$P_3(x) = 0.998537 + 0.0153524x - 0.554404x^2 + 0.0795056x^3$$

Example (3 of 4)

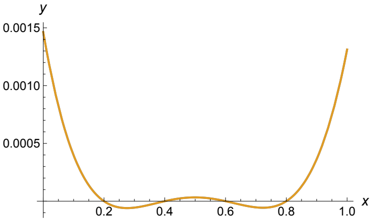


Example (4 of 4)

Difference between the forward and backward interpolation functions.



Error of the forward and backward interpolation functions.



Homework

- Read Section 3.3.
- Exercises: 1a, 3a, 5a, 13, 17