Double Integrals

MATH 375 *Numerical Analysis*

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Objectives

Now that we have discussed several methods for approximating definite integrals of the form

$$\int_a^b f(x)\,dx$$

we turn our attention to double integrals of the form

$$\iint_R f(x,y)\,dA.$$

In this lesson we will discuss quadrature methods which can be applied to the case when

- $R = \{(x,y)\,|\,a \le x \le b,\ c \le y \le d\}$, and
- $R = \{(x,y)\,|\,a \le x \le b,\ c(x) \le y \le d(x)\}$.

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In this lesson we will discuss quadrature methods which can be applied to the case when

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- $R = \{(x,y) \,|\, a \leq x \leq b,\ c(x) \leq y \leq d(x)\}$.

Generalizations to polar coordinates and to triple integrals can also be made.

Integrating Over a Rectangular Region (1 of 2)

Suppose $R = \{(x, y) \mid a \le x \le b,\ c \le y \le d\}$, then

$$\iint_R f(x, y)\, dA = \int_a^b \int_c^d f(x, y)\, dy\, dx = \int_a^b \left[\int_c^d f(x, y)\, dy \right] dx.$$

Integrating Over a Rectangular Region (1 of 2)

Suppose $R = \{(x, y) \,|\, a \leq x \leq b,\ c \leq y \leq d\}$, then

$$\iint_R f(x, y)\, dA = \int_a^b \int_c^d f(x, y)\, dy\, dx = \int_a^b \left[\int_c^d f(x, y)\, dy \right] dx.$$

Let m be an even integer and apply the Composite Simpson’s rule to the “inner” integral.

Integrating Over a Rectangular Region (2 of 2)

$$\begin{aligned}
\int_c^d f(x,y)\,dy \\
&= \frac{k}{3}\left[f(x,y_0) + 2\sum_{j=1}^{m/2-1} f(x,y_{2j}) + 4\sum_{j=1}^{m/2} f(x,y_{2j-1}) + f(x,y_m)\right] \\
&\quad - \frac{(d-c)}{180}k^4\frac{\partial^4 f}{\partial y^4}(x,\mu)
\end{aligned}$$

where

$$\begin{aligned}
k &= \frac{d-c}{m} \\
y_j &= c + j\,k \quad \text{for } j = 0, 1, \ldots, m \\
\mu &\in [c,d]
\end{aligned}$$

Outer Integration (1 of 2)

Now integrate with respect to x.

$$\begin{aligned}
&\int_a^b \int_c^d f(x,y)\,dy\,dx \\
&= \int_a^b \frac{k}{3}\left[f(x,y_0) + 2\sum_{j=1}^{m/2-1} f(x,y_{2j}) + 4\sum_{j=1}^{m/2} f(x,y_{2j-1}) + f(x,y_m)\right] dx \\
&\quad - \int_a^b \frac{(d-c)}{180} k^4 \frac{\partial^4 f}{\partial y^4}(x,\mu)\,dx
\end{aligned}$$

Outer Integration (2 of 2)

$$
\begin{aligned}
&\int_a^b \int_c^d f(x,y)\,dy\,dx \\
&= \frac{k}{3}\left[\int_a^b f(x,y_0)\,dx + 2\sum_{j=1}^{m/2-1}\int_a^b f(x,y_{2j})\,dx\right. \\
&\quad \left. + 4\sum_{j=1}^{m/2}\int_a^b f(x,y_{2j-1})\,dx + \int_a^b f(x,y_m)\,dx\right] \\
&\quad - \frac{(d-c)}{180}k^4\int_a^b \frac{\partial^4 f}{\partial y^4}(x,\mu)\,dx
\end{aligned}
$$

Outer Integration (2 of 2)

$$\begin{aligned}
\int_a^b \int_c^d f(x,y)\,dy\,dx
&= \frac{k}{3}\left[\int_a^b f(x,y_0)\,dx + 2\sum_{j=1}^{m/2-1}\int_a^b f(x,y_{2j})\,dx\right. \\
&\qquad \left. + 4\sum_{j=1}^{m/2}\int_a^b f(x,y_{2j-1})\,dx + \int_a^b f(x,y_m)\,dx\right] \\
&\qquad - \frac{(d-c)}{180}k^4 \int_a^b \frac{\partial^4 f}{\partial y^4}(x,\mu)\,dx
\end{aligned}$$

Fix a value of j and apply the Composite Simpson's rule to the integral $\int_a^b f(x,y_j)\,dx$.

Integration with Fixed j

If n is an even integer and

$$\begin{aligned} h &= \frac{b-a}{n} \\ x_i &= a + i\,h \quad \text{for } i = 0, 1, \ldots, n \\ \xi_j &\in [a, b] \end{aligned}$$

then

$$\begin{aligned} &\int_a^b f(x, y_j)\, dx \\ &= \frac{h}{3}\left[f(x_0, y_j) + 2 \sum_{i=1}^{n/2-1} f(x_{2i}, y_j) + 4 \sum_{i=1}^{n/2} f(x_{2i-1}, y_j) + f(x_n, y_j) \right] \\ &\quad - \frac{(b-a)}{180} h^4 \frac{\partial^4 f}{\partial x^4}(\xi_j, y_j) \end{aligned}$$

Quadrature Formula

$$
\begin{aligned}
&\int_a^b \int_c^d f(x,y)\,dy\,dx \\
&\approx \quad \frac{hk}{9}\Bigg[\Bigg(f(x_0,y_0) + 2\sum_{i=1}^{\frac{n}{2}-1} f(x_{2i},y_0) + 4\sum_{i=1}^{\frac{n}{2}} f(x_{2i-1},y_0) + f(x_n,y_0)\Bigg) \\
&\qquad + 2\Bigg(\sum_{j=1}^{\frac{m}{2}-1} f(x_0,y_{2j}) + 2\sum_{j=1}^{\frac{m}{2}-1}\sum_{i=1}^{\frac{n}{2}-1} f(x_{2i},y_{2j}) + 4\sum_{j=1}^{\frac{m}{2}-1}\sum_{i=1}^{\frac{n}{2}} f(x_{2i-1},y_{2j}) + \sum_{j=1}^{\frac{m}{2}-1} f(x_n,y_{2j})\Bigg) \\
&\qquad + 4\Bigg(\sum_{j=1}^{\frac{m}{2}} f(x_0,y_{2j-1}) + 2\sum_{j=1}^{\frac{m}{2}}\sum_{i=1}^{\frac{n}{2}-1} f(x_{2i},y_{2j-1}) + 4\sum_{j=1}^{\frac{m}{2}}\sum_{i=1}^{\frac{n}{2}} f(x_{2i-1},y_{2j-1}) + \sum_{j=1}^{\frac{m}{2}} f(x_n,y_{2j-1})\Bigg) \\
&\qquad + \Bigg(f(x_0,y_m) + 2\sum_{i=1}^{\frac{n}{2}-1} f(x_{2i},y_m) + 4\sum_{i=1}^{\frac{n}{2}} f(x_{2i-1},y_m) + f(x_n,y_m)\Bigg)\Bigg]
\end{aligned}
$$

Error

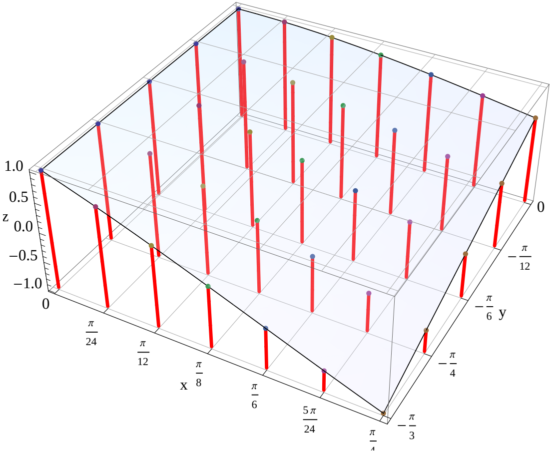
The error term for the double integral Composite Simpson's rule quadrature formula takes the form of

$$E(f) = -\frac{(d-c)(b-a)}{180}\left[h^4 \frac{\partial^4 f}{\partial x^4}(\bar{\xi}, \bar{\mu}) + k^4 \frac{\partial^4 f}{\partial y^4}(\hat{\xi}, \hat{\mu})\right]$$

Example (1 of 2)

Let $m = 4$ and $n = 6$ and use the Composite Simpson's rule to approximate the double integral:

$$\int_0^{\pi/4} \int_{-\pi/3}^{0} (2y \sin x + \cos^2 x)\, dy\, dx = \frac{\pi}{72}\left[6 + \pi(4\sqrt{2} - 5)\right] \approx 0.35184$$



Example (2 of 2)

$$\int_{0}^{\pi/4} \int_{-\pi/3}^{0} (2y \sin x + \cos^2 x)\, dy\, dx \approx 0.351846$$

Absolute error:

$$\left| \frac{\pi}{72} \left[6 + \pi(4\sqrt{2} - 5) \right] - 0.351846 \right| \approx 6.36354 \times 10^{-6}$$

Error bound:

$$\begin{aligned}
&|E(f)| \\
&\leq \max_{(\bar{\xi},\bar{\mu}),(\hat{\xi},\hat{\mu}) \in R} \left| -\frac{(d-c)(b-a)}{180} \left[h^4 \frac{\partial^4 f}{\partial x^4}(\bar{\xi},\bar{\mu}) + k^4 \frac{\partial^4 f}{\partial y^4}(\hat{\xi},\hat{\mu}) \right] \right| \\
&= \frac{(\pi/3)(\pi/4)}{180} \left(\frac{\pi}{24} \right)^4 \max_{(\bar{\xi},\bar{\mu}),(\hat{\xi},\hat{\mu}) \in R} \left| 8(\cos^2 x - \sin^2 x) + 2y \sin x \right| \\
&= \frac{\pi^6}{716636160} 8 \approx 1.07322 \times 10^{-5}
\end{aligned}$$

Double Integrals with Gaussian Quadrature

We may also use Gaussian quadrature formulas to approximate double integrals.

Consider the double integral:

$$\begin{aligned}
&\int_a^b \int_c^d f(x,y)\,dy\,dx \\
&\quad = \int_a^b \int_{-1}^1 f\left(x, \frac{(d-c)u+c+d}{2}\right) \frac{d-c}{2}\,du\,dx \\
&\quad = \int_{-1}^1 \int_{-1}^1 \left[f\left(\frac{(b-a)v+a+b}{2}, \frac{(d-c)u+c+d}{2}\right) \right. \\
&\qquad\qquad \left. \frac{(b-a)(d-c)}{4}\right] du\,dv
\end{aligned}$$

Gaussian Quadrature Formula

For the sake of convenience we will use the same precision, n, for each stage of the integration.

$$\begin{aligned}
&\int_a^b \int_c^d f(x,y)\,dy\,dx \\
&\quad \approx \quad \frac{(b-a)(d-c)}{4} \cdot \\
&\qquad \sum_{i=1}^{n} \sum_{j=1}^{n} c_{n,i} c_{n,j} f\left(\frac{(b-a)r_{n,i}+a+b}{2}, \frac{(d-c)r_{n,j}+c+d}{2} \right)
\end{aligned}$$

Example

Using the double integral form of Gaussian quadrature with $n = 4$ we may approximate

$$\int_0^{\pi/4} \int_{-\pi/3}^{0} (2y \sin x + \cos^2 x)\, dy\, dx \approx 0.35184$$

Absolute error:

$$\left| \frac{\pi}{72} \left[6 + \pi(4\sqrt{2} - 5) \right] - 0.35184 \right| \approx 5.94305 \times 10^{-9}$$

Integrating Over Non-rectangular Regions

Consider the double integral

$$\iint_R f(x,y)\,dA = \int_a^b \int_{c(x)}^{d(x)} f(x,y)\,dy\,dx.$$

Suppose we use the Composite Simpson's rule to approximate the "inner" integral.

Integrating Over Non-rectangular Regions

Consider the double integral

$$\iint_R f(x,y)\,dA = \int_a^b \int_{c(x)}^{d(x)} f(x,y)\,dy\,dx.$$

Suppose we use the Composite Simpson's rule to approximate the "inner" integral.

$$\begin{aligned}
&\int_{c(x)}^{d(x)} f(x,y)\,dy \\
&\approx \quad \frac{k(x)}{3}\left[f(x,c(x)) + 2\sum_{j=1}^{\frac{m}{2}-1} f(x,c(x)+2jk(x))\right. \\
&\qquad \left. + 4\sum_{j=1}^{\frac{m}{2}} f(x,c(x)+(2j-1)k(x)) + f(x,d(x))\right]
\end{aligned}$$

where m is an even integer and $k(x) = \dfrac{d(x)-c(x)}{m}$.

Simplification (1 of 2)

To make the notation of the next step simpler we will make the following change to the notation.

$$\begin{aligned} F(x) \quad = \quad & \frac{k(x)}{3}\left[f(x, c(x)) + 2\sum_{j=1}^{\frac{m}{2}-1} f(x, c(x) + 2jk(x)) \right. \\ & \left. +4\sum_{j=1}^{\frac{m}{2}} f(x, c(x) + (2j-1)k(x)) + f(x, d(x)) \right] \end{aligned}$$

Simplification (2 of 2)

This implies

$$\begin{aligned}
&\int_{c(x)}^{d(x)} f(x,y)\,dy \\
\approx\;& \frac{k(x)}{3}\Bigg[f(x,c(x)) + 2\sum_{j=1}^{\frac{m}{2}-1} f(x,c(x)+2jk(x)) \\
&\quad + 4\sum_{j=1}^{\frac{m}{2}} f(x,c(x)+(2j-1)k(x)) + f(x,d(x))\Bigg] \\
=\;& F(x)
\end{aligned}$$

Simplification (2 of 2)

This implies

$$\begin{aligned}
\int_{c(x)}^{d(x)} f(x,y)\,dy & \\
\approx \; & \frac{k(x)}{3}\left[f(x,c(x)) + 2\sum_{j=1}^{\frac{m}{2}-1} f(x,c(x)+2jk(x)) \right. \\
& \left. + 4\sum_{j=1}^{\frac{m}{2}} f(x,c(x)+(2j-1)k(x)) + f(x,d(x)) \right] \\
= \; & F(x)
\end{aligned}$$

Now we may approximate the outer integral.

Outer Integral

Now we choose n to be an even integer and apply the Composite Simpson's rule with $h = \dfrac{b-a}{n}$.

$$
\begin{aligned}
&\int_a^b \int_{c(x)}^{d(x)} f(x,y)\,dy\,dx \\
&\approx \int_a^b \frac{k(x)}{3}\left[f(x,c(x)) + 2\sum_{j=1}^{\frac{m}{2}-1} f(x,c(x)+2jk(x))\right. \\
&\qquad \left. +4\sum_{j=1}^{\frac{m}{2}} f(x,c(x)+(2j-1)k(x)) + f(x,d(x))\right] dx \\
&= \int_a^b F(x)\,dx \\
&\approx \frac{h}{3}\left[F(a) + 2\sum_{i=1}^{\frac{n}{2}-1} F(a+2ih) + 4\sum_{i=1}^{\frac{n}{2}} F(a+(2i-1)h) + F(b)\right]
\end{aligned}
$$

Example (1 of 2)

Consider the double integral,

$$\int_0^1 \int_x^{2x} (x^2 + y^3)\, dy\, dx = 1$$

1. Using the composite Simpson's rule with $n = m$ find the smallest values of n and m required to estimate the double integral to within 10^{-4} of its actual value.
2. Estimate the double integral.

Example (2 of 2)

Since we have no formula for the error term in this case we will use the composite Simpson's rule with $n = m = 2, 4, \ldots$ until successive approximations differ by less than 10^{-4}.

$n = m$	**Estimate**
2	1.03125
4	1.00195
6	1.00039
8	1.00012
10	1.00005

Gaussian Quadrature for General Regions (1 of 2)

Consider the double integral:

$$\int_a^b \int_{c(x)}^{d(x)} f(x, y)\, dy\, dx$$

We can apply Gaussian quadrature to the “inner” integral.

Gaussian Quadrature for General Regions (1 of 2)

Consider the double integral:

$$\int_a^b \int_{c(x)}^{d(x)} f(x,y)\,dy\,dx$$

We can apply Gaussian quadrature to the “inner” integral.

$$\begin{aligned}
&\int_{c(x)}^{d(x)} f(x,y)\,dy \\
&= \int_{-1}^{1} f\left(x, \frac{(d(x)-c(x))u + c(x) + d(x)}{2}\right) \frac{d(x)-c(x)}{2}\,du \\
&\approx \frac{d(x)-c(x)}{2} \sum_{j=1}^{n} c_{n,j} f\left(x, \frac{(d(x)-c(x))r_{n,j} + c(x) + d(x)}{2}\right)
\end{aligned}$$

Gaussian Quadrature for General Regions (2 of 2)

Therefore

$$\int_a^b \int_{c(x)}^{d(x)} f(x,y)\,dy\,dx$$

$$\approx \int_a^b \left[\frac{d(x)-c(x)}{2} \sum_{j=1}^{n} c_{n,j} f\left(x, \frac{(d(x)-c(x))r_{n,j}+c(x)+d(x)}{2}\right)\right] dx.$$

Now we apply Guassian quadrature to the remaining integral.

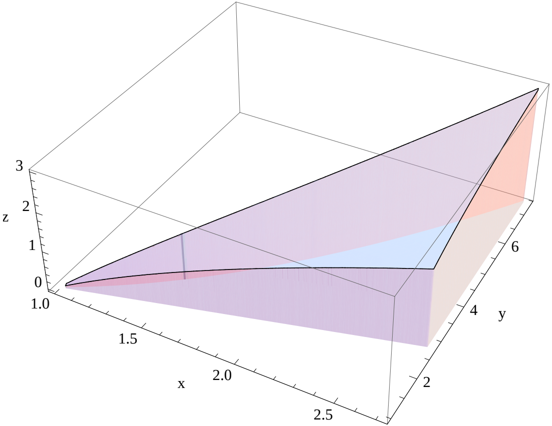
Algorithm

```
INPUT  $a$, $b$; positive integers $m$, $n$.
STEP 1 Set $h_1 = (b-a)/2$; $h_2 = (b+a)/2$; $J = 0$.
STEP 2 For $i = 1, 2, \ldots, m$ do STEPS 3–5.
STEP 3     Set $JX = 0$; $x = h_1 r_{m,j} + h_2$; $d_1 = d(x)$; $c_1 = c(x)$;
           $k_1 = (d_1 - c_1)/2$; $k_2 = (d_1 + c_1)/2$.
STEP 4     For $j = 1, 2, \ldots, n$ set $y = k_1 r_{n,j} + k_2$; $Q = f(x, y)$;
               $JX = JX + c_{n,j} Q$.
STEP 5     Set $J = J + c_{m,i} k_1 JX$.
STEP 6 Set $J = h_1 J$; OUTPUT $J$.
```

Example (1 of 2)

Use Gaussian quadrature with $n = 4$ to approximate the double integral

$$\int_{1}^{e} \int_{x}^{x^2} \ln(x\,y)\,dy\,dx.$$



Example (2 of 2)

$$\int_1^e \int_x^{x^2} \ln(x\,y)\,dy\,dx \approx 6.36185$$

Absolute error:

$$\left| \int_1^e \int_x^{x^2} \ln(x\,y)\,dy\,dx - 6.36185 \right| \approx 5.95798 \times 10^{-6}$$

Homework

- Read Section 4.8.
- Exercises: 1ab, 5ab