# Eigenvalues and Eigenvectors

## MATH 375 *Numerical Analysis*

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# Background

- Suppose $A \in \mathbb{R}^{n \times n}$ and $\mathbf{x} \in \mathbb{R}^n$ then multiplication by matrix $A$ defines a function $A : \mathbb{R}^n \to \mathbb{R}^n$.
- If there exists $\mathbf{x} \in \mathbb{R}^n$ with $\mathbf{x} \neq \mathbf{0}$ such that $A\mathbf{x}$ is a vector parallel to $\mathbf{x}$, then there exists $\lambda \in \mathbb{R}$ for which $A\mathbf{x} = \lambda\mathbf{x}$.

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- In this situation vector $\mathbf{x}$ is called and **eigenvector** and $\lambda$ is called the associated **eigenvalue**.

# Characteristic Polynomials

## Definition

If $A \in \mathbb{R}^{n \times n}$ the **characteristic polynomial of** $A$ is $p(\lambda) = \det(A - \lambda I_n)$.

**Remarks**:

- The characteristic polynomial is a polynomial of degree $n$.
- The characteristic polynomial has at most $n$ roots.
- If $\det(A - \lambda I_n) = 0$ then $(A - \lambda I_n)\mathbf{x} = \mathbf{0}$ has a non-zero solution $\mathbf{x}$.
- The roots of $p(\lambda) = 0$ are the **eigenvalues of** $A$.
- The non-zero solutions of $A\mathbf{x} = \lambda\mathbf{x}$ are the **eigenvectors of** $A$.

# Example

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Eigenvalues: $\lambda = -2, 3, 4$.

# Eigenvector Corresponding to $\lambda = -2$

Row-reduce $A - (-2)I_3$.

$$\begin{aligned} A - (-2)I_3 &= \begin{bmatrix} 3 & 2 & -1 \\ 1 & -2 & 3 \\ 2 & 0 & 4 \end{bmatrix} - \begin{bmatrix} -2 & 0 & 0 \\ 0 & -2 & 0 \\ 0 & 0 & -2 \end{bmatrix} \\ &= \begin{bmatrix} 5 & 2 & -1 \\ 1 & 0 & 3 \\ 2 & 0 & 6 \end{bmatrix} \\ &\mapsto \begin{bmatrix} 1 & 0 & 3 \\ 0 & 1 & -8 \\ 0 & 0 & 0 \end{bmatrix} \end{aligned}$$

We can choose $x_3 = 1$ which implies $x_2 = 8$ and $x_1 = -3$. Thus the eigenvector corresponding to $\lambda = -2$ is

$$\mathbf{x} = \begin{bmatrix} -3 \\ 8 \\ 1 \end{bmatrix}.$$

# Eigenvector Corresponding to $\lambda = 3$

Row-reduce $A - 3I_3$.

$$\begin{aligned}
A - 3I_3 &= \begin{bmatrix} 3 & 2 & -1 \\ 1 & -2 & 3 \\ 2 & 0 & 4 \end{bmatrix} - \begin{bmatrix} 3 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 3 \end{bmatrix} \\
&= \begin{bmatrix} 0 & 2 & -1 \\ 1 & -5 & 3 \\ 2 & 0 & 1 \end{bmatrix} \\
&\mapsto \begin{bmatrix} 2 & 0 & 1 \\ 0 & 2 & -1 \\ 0 & 0 & 0 \end{bmatrix}
\end{aligned}$$

We can choose $x_3 = 1$ which implies $x_2 = 1/2$ and $x_1 = -1/2$. Thus the eigenvector corresponding to $\lambda = 3$ is

$$\mathbf{x} = \begin{bmatrix} -1/2 \\ 1/2 \\ 1 \end{bmatrix}.$$

# Eigenvector Corresponding to $\lambda = 4$

Row-reduce $A - 4I_3$.

$$\begin{aligned} A - 4I_3 &= \begin{bmatrix} 3 & 2 & -1 \\ 1 & -2 & 3 \\ 2 & 0 & 4 \end{bmatrix} - \begin{bmatrix} 4 & 0 & 0 \\ 0 & 4 & 0 \\ 0 & 0 & 4 \end{bmatrix} \\ &= \begin{bmatrix} -1 & 2 & -1 \\ 1 & -6 & 3 \\ 2 & 0 & 0 \end{bmatrix} \\ &\mapsto \begin{bmatrix} 1 & 0 & 0 \\ 0 & 2 & -1 \\ 0 & 0 & 0 \end{bmatrix} \end{aligned}$$

We can choose $x_3 = 1$ which implies $x_2 = 1/2$ and $x_1 = 0$. Thus the eigenvector corresponding to $\lambda = 3$ is

$$\mathbf{x} = \begin{bmatrix} 0 \\ 1/2 \\ 1 \end{bmatrix}.$$

# Spectral Radius

## Definition

The **spectral radius** of $A \in \mathbb{R}^{n \times n}$ is denoted

$$\rho(A) = \max |\lambda|$$

where $\lambda$ is an eigenvalues of $A$.

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## Example

In the previous example we learned that the eigenvalues of
$A = \begin{bmatrix} 3 & 2 & -1 \\ 1 & -2 & 3 \\ 2 & 0 & 4 \end{bmatrix}$ are $\lambda = -2, 3, 4$, thus $\rho(A) = 4$.

# Computing the $l_2$-norm of a Matrix

## Theorem

*If $A \in \mathbb{R}^{n \times n}$ then*

1. $\|A\|_2 = \left[\rho(A^t A)\right]^{1/2}$,
2. $\rho(A) \leq \|A\|$ *for any induced norm* $\| \cdot \|$.

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## Proof.

(Of (2)):

- Suppose $\lambda$ is an eigenvalue of $A$ with corresponding eigenvector $\mathbf{x}$.
- Let $\mathbf{z} = \mathbf{x}/\|\mathbf{x}\|$ then

$$|\lambda| = |\lambda|\,\|\mathbf{z}\| = \|\lambda\,\mathbf{z}\| = \|A\,\mathbf{z}\| \le \|A\|\,\|\mathbf{z}\| = \|A\|.$$

- Hence $\rho(A) = \max|\lambda| \le \|A\|$.

$\square$

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|A^t A - \lambda I_3| &= \begin{vmatrix} 14-\lambda & 4 & 8 \\ 4 & 8-\lambda & -8 \\ 8 & -8 & 26-\lambda \end{vmatrix}
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Eigenvalues: $\lambda \approx 1.18927, 15.4378, 31.3729$ and thus

$$\|A\|_2 \approx \sqrt{31.3729} \approx 5.60115.$$

# Remarks

- If $A$ is symmetric then $\|A\|_2 = \rho(A)$.
- If $\epsilon > 0$ then there exists an induced norm such that $\rho(A) \leq \|A\| < \rho(A) + \epsilon$.

# Sequences of Matrices

## Definition

Matrix $A \in \mathbb{R}^{n \times n}$ is **convergent** if

$$\lim_{k \to \infty} (A^k)_{ij} = 0$$

for all $i, j = 1, 2, \ldots, n$.

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## Example

Show that matrix $A = \begin{bmatrix} 3/5 & 1 \\ 0 & 3/5 \end{bmatrix}$ is convergent.

# Proof

$$
\begin{aligned}
A &= \begin{bmatrix} 3/5 & 1 \\ 0 & 3/5 \end{bmatrix} \\
A^2 &= \begin{bmatrix} 9/25 & 6/5 \\ 0 & 9/25 \end{bmatrix} \\
A^3 &= \begin{bmatrix} 27/125 & 27/25 \\ 0 & 27/125 \end{bmatrix} \\
A^4 &= \begin{bmatrix} 81/625 & 108/125 \\ 0 & 81/625 \end{bmatrix} \\
&\vdots \\
A^n &= \begin{bmatrix} (3/5)^n & n(3/5)^{n-1} \\ 0 & (3/5)^n \end{bmatrix}
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\end{aligned}
$$

Thus $A^k \to 0^{2\times 2}$ as $k \to \infty$.

# Main Result

## Theorem

*The following statements are equivalent:*

1. *$A$ is a convergent matrix.*
2. $\lim_{n \to \infty} \|A^n\| = 0$ *for some induced norm.*
3. $\lim_{n \to \infty} \|A^n\| = 0$ *for all induced norms.*
4. $\rho(A) < 1$.
5. $\lim_{n \to \infty} A^n \mathbf{x} = \mathbf{0}$ *for all* $\mathbf{x} \in \mathbb{R}^n$.

# Homework

- Read Section 7.2.
- Exercises: 1, 5, 7, 9