

Matrix Factorization

MATH 375 *Numerical Analysis*

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Objective

Solve the linear system of equations $A\mathbf{x} = \mathbf{b}$

Remarks:

- ▶ The forms of Gaussian elimination we have studied require $O(n^3)$ operations to row reduce.
- ▶ Back substitution requires $O(n^2)$ operations to solve.

Matrix Factorization

Suppose we could write A as the product of two matrices, *i.e.*

$$\begin{aligned} A &= LU \\ &= \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{bmatrix} \\ &= \begin{bmatrix} l_{11} & 0 & \cdots & 0 \\ l_{21} & l_{22} & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ l_{n1} & l_{n2} & \cdots & l_{nn} \end{bmatrix} \begin{bmatrix} u_{11} & u_{12} & \cdots & u_{1n} \\ 0 & u_{22} & \cdots & u_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & u_{nn} \end{bmatrix} \end{aligned}$$

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Then we can solve the linear system using **forward** and **backward** substitution.

Forward and Backward Substitution

$$A\mathbf{x} = \mathbf{b}$$

$$LU\mathbf{x} = \mathbf{b}$$

$$L(U\mathbf{x}) = \mathbf{b} \quad (\text{let } U\mathbf{x} = \mathbf{y})$$

$$L\mathbf{y} = \mathbf{b} \quad (\text{solve for } \mathbf{y})$$

$$U\mathbf{x} = \mathbf{y} \quad (\text{solve for } \mathbf{x})$$

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- ▶ Forward and backward substitution require $O(n^2)$ operations.
- ▶ Not every square matrix A can be factored as LU .

Forward and Backward Substitution

$$\begin{aligned}A\mathbf{x} &= \mathbf{b} \\LU\mathbf{x} &= \mathbf{b} \\L(U\mathbf{x}) &= \mathbf{b} \quad (\text{let } U\mathbf{x} = \mathbf{y}) \\L\mathbf{y} &= \mathbf{b} \quad (\text{solve for } \mathbf{y}) \\U\mathbf{x} &= \mathbf{y} \quad (\text{solve for } \mathbf{x})\end{aligned}$$

Remarks:

- ▶ Forward and backward substitution require $O(n^2)$ operations.
- ▶ Not every square matrix A can be factored as LU .
- ▶ If A can be row reduced without pivoting, then A can be factored as LU .

Factoring a Matrix (1 of 12)

Let $A = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{bmatrix}$ and assume $a_{11} \neq 0$.

For $j = 2, 3, \dots, n$ define $m_{j1} = \frac{a_{j1}}{a_{11}}$ and define

$$M^{(1)} = \begin{bmatrix} 1 & 0 & \cdots & 0 \\ -m_{21} & 1 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ -m_{n1} & 0 & \cdots & 1 \end{bmatrix}$$

Factoring a Matrix (2 of 12)

Observe that

$$\begin{aligned} M^{(1)} A &= \begin{bmatrix} 1 & 0 & \cdots & 0 \\ -m_{21} & 1 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ -m_{n1} & 0 & \cdots & 1 \end{bmatrix} \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{bmatrix} \\ &= \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ 0 & a_{22}^{(2)} & \cdots & a_{2n}^{(2)} \\ \vdots & \vdots & \ddots & \vdots \\ 0 & a_{n2}^{(2)} & \cdots & a_{nn}^{(2)} \end{bmatrix}. \end{aligned}$$

Factoring a Matrix (3 of 12)

For $j = 3, 4, \dots, n$ define $m_{j2} = \frac{a_{j2}^{(2)}}{a_{22}^{(2)}}$ and define

$$M^{(2)} = \begin{bmatrix} 1 & 0 & 0 & \cdots & 0 \\ 0 & 1 & 0 & \cdots & 0 \\ 0 & -m_{32} & 1 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & -m_{n2} & 0 & \cdots & 1 \end{bmatrix}$$

Factoring a Matrix (4 of 12)

Observe that

$$\begin{aligned} & M^{(2)} M^{(1)} A \\ &= \begin{bmatrix} 1 & 0 & 0 & \cdots & 0 \\ 0 & 1 & 0 & \cdots & 0 \\ 0 & -m_{32} & 1 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & -m_{n2} & 0 & \cdots & 1 \end{bmatrix} \begin{bmatrix} a_{11} & a_{12} & a_{13} & \cdots & a_{1n} \\ 0 & a_{22}^{(2)} & a_{23}^{(2)} & \cdots & a_{2n}^{(2)} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & a_{n2}^{(2)} & a_{n3}^{(2)} & \cdots & a_{nn}^{(2)} \end{bmatrix} \\ &= \begin{bmatrix} a_{11} & a_{12} & a_{13} & \cdots & a_{1n} \\ 0 & a_{22}^{(2)} & a_{23}^{(2)} & \cdots & a_{2n}^{(2)} \\ 0 & 0 & a_{33}^{(3)} & \cdots & a_{3n}^{(3)} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & a_{n3}^{(3)} & \cdots & a_{nn}^{(3)} \end{bmatrix}. \end{aligned}$$

Factoring a Matrix (5 of 12)

After $i - 1$ repetitions define $m_{ji} = \frac{a_{ji}^{(i)}}{a_{ii}^{(i)}}$ for $j = i + 1, \dots, n$ and define

$$M^{(i)} = \begin{bmatrix} 1 & 0 & \cdots & 0 & 0 & 0 & \cdots & 0 \\ 0 & 1 & \cdots & 0 & 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 1 & 0 & 0 & \cdots & 0 \\ 0 & 0 & \cdots & 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & \cdots & 0 & -m_{i+1,i} & 1 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 0 & -m_{n,i} & 0 & \cdots & 1 \end{bmatrix}.$$

Factoring a Matrix (7 of 12)

Multiplying by $n - 1$ such matrices yields

$$M^{(n-1)} M^{(n-2)} \dots M^{(1)} A = U$$

an upper triangular matrix.

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If $M^{(i)}$ is non-singular for each $i = 1, 2, \dots, n - 1$ then

$$\begin{aligned} A &= (M^{(1)})^{-1} \dots (M^{(n-2)})^{-1} (M^{(n-1)})^{-1} M^{(n-1)} M^{(n-2)} \dots M^{(1)} A \\ &= (M^{(1)})^{-1} \dots (M^{(n-2)})^{-1} (M^{(n-1)})^{-1} U \\ &= LU. \end{aligned}$$

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Questions:

1. Is $M^{(i)}$ non-singular?
2. Is $(M^{(1)})^{-1} \dots (M^{(n-2)})^{-1} (M^{(n-1)})^{-1}$ lower triangular?

Factoring a Matrix (8 of 12)

$$[M^{(i)}]^{-1} = \begin{bmatrix} 1 & 0 & \cdots & 0 & 0 & 0 & \cdots & 0 \\ 0 & 1 & \cdots & 0 & 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 1 & 0 & 0 & \cdots & 0 \\ 0 & 0 & \cdots & 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & \cdots & 0 & m_{i+1,i} & 1 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 0 & m_{n,i} & 0 & \cdots & 1 \end{bmatrix} .$$

Factoring a Matrix (8 of 12)

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Remarks:

- ▶ This matrix is lower triangular.
- ▶ The product of lower triangular matrices is lower triangular.

Factoring a Matrix (9 of 12)

We can also think of the proposed inverse matrix in block form.

$$[M^{(i)}]^{-1} = \left[\begin{array}{ccc|ccc} & & & 0 & & \\ & I_{i-1} & & \vdots & & 0^{(i-1) \times (n-i)} \\ & & & 0 & & \\ \hline 0 & \dots & 0 & 1 & 0 & \dots & 0 \\ \hline & & & m_{i+1,i} & & & \\ & 0^{(n-i) \times (i-1)} & & \vdots & & I_{n-i} & \\ & & & m_{n,i} & & & \end{array} \right].$$

Factoring a Matrix (10 of 12)

Question: can we confirm $[M^{(i)}]^{-1}$ is the inverse of $M^{(i)}$?

$$M^{(i)} [M^{(i)}]^{-1} = \begin{bmatrix} 1 & 0 & \cdots & 0 & 0 & 0 & \cdots & 0 \\ 0 & 1 & \cdots & 0 & 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 1 & 0 & 0 & \cdots & 0 \\ 0 & 0 & \cdots & 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & \cdots & 0 & -m_{i+1,i} & 1 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 0 & -m_{n,i} & 0 & \cdots & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & \cdots & 0 & 0 & 0 & \cdots & 0 \\ 0 & 1 & \cdots & 0 & 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 1 & 0 & 0 & \cdots & 0 \\ 0 & 0 & \cdots & 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & \cdots & 0 & m_{i+1,i} & 1 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 0 & m_{n,i} & 0 & \cdots & 1 \end{bmatrix}$$

Factoring a Matrix (10 of 12)

Consider the dot product of the j th row of $M^{(i)}$ with $i < j \leq n$ and the k th column of $[M^{(i)}]^{-1}$ with $i \leq k \leq n$.

If $k \neq i$

$$\langle 0, \dots, 0, -m_{ji}, \dots, 1_{jj}, \dots, 0 \rangle \cdot \langle 0, \dots, 0, 1_{kk}, 0, \dots, 0 \rangle = \delta_{jk}.$$

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If $k = i$

$$\begin{aligned} \langle 0, \dots, 0, -m_{ji}, \dots, 1_{jj}, \dots, 0 \rangle &\cdot \langle 0, \dots, 0, 1_{ii}, m_{i+1,i}, \dots, m_{n,i} \rangle \\ &= (-m_{ji})(1_{ii}) + (1_{jj})(m_{ji}) \\ &= 0. \end{aligned}$$

Factoring a Matrix (12 of 12)

Define $L^{(i)} = [M^{(i)}]^{-1}$ for $i = 1, 2, \dots, n - 1$ and then

$$L = L^{(1)} L^{(2)} \dots L^{(n-1)}.$$

At last we have factored $A = LU$ and proved the following theorem.

Theorem

If Gaussian elimination can be performed on matrix A without row interchanges, then A can be factored as $A = LU$ where L is a lower triangular matrix and U is an upper triangular matrix.

Example

If $A = \begin{bmatrix} 1 & 2 \\ 2 & 1 \end{bmatrix}$ then

$$M^{(1)} = \begin{bmatrix} 1 & 0 \\ -2 & 1 \end{bmatrix} \quad \text{and} \quad L^{(1)} = [M^{(1)}]^{-1} = \begin{bmatrix} 1 & 0 \\ 2 & 1 \end{bmatrix} = L.$$

We can check that

$$M^{(1)} A = \begin{bmatrix} 1 & 2 \\ 0 & -3 \end{bmatrix} = U$$

and

$$\begin{bmatrix} 1 & 2 \\ 2 & 1 \end{bmatrix} = A = LU = \begin{bmatrix} 1 & 0 \\ 2 & 1 \end{bmatrix} \begin{bmatrix} 1 & 2 \\ 0 & -3 \end{bmatrix}$$

Remarks

- ▶ We must select u_{ij} and l_{ij} so that $u_{ij}l_{ij} = a_{ij}^{(i)}$.
- ▶ For convenience we will choose $l_{ij} = 1$ so that $u_{ij} = a_{ij}^{(i)}$.
- ▶ We are assuming that for $i = 1, 2, \dots, n$ we have $a_{ii}^{(i)} \neq 0$.

Algorithm: LU Factorization

Given matrix $A = [a_{ij}]_{i=1, \dots, n; j=1, \dots, n}$

STEP 1 Set $l_{11} = 1$; $u_{11} = a_{11}$.

STEP 2 For $j = 2, \dots, n$ set $u_{1j} = a_{1j}$; $l_{j1} = \frac{a_{j1}}{u_{11}}$.

STEP 3 For $i = 2, \dots, n$ do STEPS 4–5.

STEP 4 Set $l_{ii} = 1$; $u_{ii} = a_{ii} - \sum_{k=1}^{i-1} l_{ik} u_{ki}$.

STEP 5 For $j = i + 1, \dots, n$ set

$$u_{ij} = a_{ij} - \sum_{k=1}^{i-1} l_{ik} u_{kj};$$

$$l_{ji} = \frac{1}{u_{ii}} \left[a_{ji} - \sum_{k=1}^{i-1} l_{jk} u_{ki} \right].$$

STEP 6 Set $l_{nn} = 1$; $u_{nn} = a_{nn} - \sum_{k=1}^{n-1} l_{nk} u_{kn}$.

OUTPUT $L = [l_{ij}]_{i=1, \dots, n; j=1, \dots, n}$ and $U = [u_{ij}]_{i=1, \dots, n; j=1, \dots, n}$

Remarks

- ▶ We have assumed throughout our discussion that A can be LU decomposed without row interchanges.
- ▶ In our discussion of pivoting strategies we saw that pivoting can reduce round-off error.
- ▶ Fortunately, every non-singular matrix can be LU decomposed, though some will require pivoting.

Permutation Matrices

Definition

An $n \times n$ permutation matrix contains exactly one entry in every row and column whose value is 1. All other entries are 0.

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Example

$$P = \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Multiplication by Permutation Matrices

Note the effect produced by multiplying a 3×3 matrix by P .

$$\begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix} = \begin{bmatrix} a_{21} & a_{22} & a_{23} \\ a_{11} & a_{12} & a_{13} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$$
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- ▶ Left multiplying by P swaps rows 1 and 2.
- ▶ Right multiplying by P swaps columns 1 and 2.

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- ▶ For every non-singular matrix A there exists a permutation matrix P such that PA can be LU decomposed without row interchanges.

$$\begin{aligned}PA &= LU \\ A &= P^t LU\end{aligned}$$

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$$\begin{aligned}PA &= LU \\ A &= P^t LU\end{aligned}$$

If $P \neq I$ then $P^t L$ is not lower triangular.

Example

Find a permutation matrix P so that the following matrix can be factored as $P^t L U$.

$$A = \begin{bmatrix} 1 & 2 & -1 \\ 2 & 4 & 7 \\ -1 & 2 & 5 \end{bmatrix}$$

Solution (1 of 4)

If we row reduce A then after the first stage we have

$$\begin{bmatrix} 1 & 2 & -1 \\ 2 & 4 & 7 \\ -1 & 2 & 5 \end{bmatrix} \mapsto \begin{bmatrix} 1 & 2 & -1 \\ 0 & 0 & 9 \\ 0 & 4 & 4 \end{bmatrix}.$$

Solution (1 of 4)

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We must interchange rows 2 and 3.

$$P = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix} = P^t$$

and

$$PA = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} 1 & 2 & -1 \\ 2 & 4 & 7 \\ -1 & 2 & 5 \end{bmatrix} = \begin{bmatrix} 1 & 2 & -1 \\ -1 & 2 & 5 \\ 2 & 4 & 7 \end{bmatrix}$$

Solution (2 of 4)

$$\text{If } PA = \begin{bmatrix} 1 & 2 & -1 \\ -1 & 2 & 5 \\ 2 & 4 & 7 \end{bmatrix} \text{ then}$$

$$M^{(1)} = \begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ -2 & 0 & 1 \end{bmatrix} \quad \text{and} \quad L^{(1)} = \begin{bmatrix} 1 & 0 & 0 \\ -1 & 1 & 0 \\ 2 & 0 & 1 \end{bmatrix}$$

Solution (2 of 4)

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So far we have

$$M^{(1)} PA = \begin{bmatrix} 1 & 2 & -1 \\ 0 & 4 & 4 \\ 0 & 0 & 9 \end{bmatrix}.$$

Solution (3 of 4)

$$\text{If } M^{(1)}PA = \begin{bmatrix} 1 & 2 & -1 \\ 0 & 4 & 4 \\ 0 & 0 & 9 \end{bmatrix} \text{ then}$$

$$M^{(2)} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = I_3 \quad \text{and} \quad L^{(2)} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = I_3.$$

Finally we have

$$M^{(2)}M^{(1)}PA = U = \begin{bmatrix} 1 & 2 & -1 \\ 0 & 4 & 4 \\ 0 & 0 & 9 \end{bmatrix}.$$

Solution (4 of 4)

$$\begin{aligned} A &= P^t L^{(1)} L^{(2)} U \\ &= P^t L U \\ \begin{bmatrix} 1 & 2 & -1 \\ 2 & 4 & 7 \\ -1 & 2 & 5 \end{bmatrix} &= \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ -1 & 1 & 0 \\ 2 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 2 & -1 \\ 0 & 4 & 4 \\ 0 & 0 & 9 \end{bmatrix} \end{aligned}$$

Homework

- ▶ Read Section 6.5.
- ▶ Exercises: 1, 3ab, 5ab