Linear Systems of Equations

MATH 375

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Introduction

Linear systems are equations of the form:

$$\begin{aligned} a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n &= b_1 \\ a_{21}x_1 + a_{22}x_2 + \cdots + a_{2n}x_n &= b_2 \\ &\vdots \\ a_{n1}x_1 + a_{n2}x_2 + \cdots + a_{nn}x_n &= b_n \end{aligned}$$

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- The coefficients a_{ij} for $i, j = 1, 2, \ldots, n$ are constants.
- The expressions b_i for $i = 1, 2, \ldots, n$ are constant.
- The unknowns are x_i for $i = 1, 2, \ldots, n$.

Direct Methods

We will study **direct methods** for solving such systems of linear equations.

Direct methods can solve a linear system in a **predictable**, **fixed** number of steps.

We will employ the following operations to solve linear systems:

- An equation can be replaced by a nonzero multiple of itself.
- An equation can be replaced by the sum of itself and another equation.
- Any two equations can be swapped.

Solution Strategy

Use of these elementary operations will enable us to convert a linear system into an equivalent linear system in **reduced** or **triangular** form. Then **back-substitution** can be used to solve the equivalent system.

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Example

Derive the equivalent linear system in triangular form.

$$\begin{aligned} 2x_1 + 4x_2 - x_3 &= -5 \\ x_1 + x_2 - 3x_3 &= -9 \\ 4x_1 + x_2 + 2x_3 &= 9 \end{aligned}$$

Solution (1 of 2)

$$\begin{aligned} 2x_1 + 4x_2 - x_3 &= -5 \\ x_1 + x_2 - 3x_3 &= -9 \quad \text{(mult. 1st by } \tfrac{1}{2}\text{, subt.)} \\ 4x_1 + x_2 + 2x_3 &= 9 \quad \text{(mult. 1st by 2, subt.)} \end{aligned}$$

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2x_1 + 4x_2 - x_3 &= -5 \\
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4x_1 + x_2 + 2x_3 &= 9 \quad \text{(mult. 1st by 2, subt.)}
\end{aligned}$$

$$\begin{aligned}
2x_1 + 4x_2 - x_3 &= -5 \\
-x_2 - \frac{5}{2}x_3 &= -\frac{13}{2} \\
-7x_2 + 4x_3 &= 19 \quad \text{(mult. 2nd by 7, subt.)}
\end{aligned}$$

Solution (1 of 2)

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$$\begin{aligned} 2x_1 + 4x_2 - x_3 &= -5 \\ -x_2 - \frac{5}{2}x_3 &= -\frac{13}{2} \quad \text{(mult. by } -1\text{)} \\ \frac{43}{2}x_3 &= \frac{129}{2} \end{aligned}$$

Solution (2 of 2)

$$\begin{aligned} 2x_1 + 4x_2 - x_3 &= -5 \\ x_2 + \frac{5}{2}x_3 &= \frac{13}{2} \quad \text{(mult. by 2)} \\ \frac{43}{2}x_3 &= \frac{129}{2} \quad \text{(mult. by 2)} \end{aligned}$$

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The final linear system is in triangular (reduced) form.

Solving for (x_1, x_2, x_3)

$$\begin{aligned} 2x_1 + 4x_2 - x_3 &= -5 \\ 2x_2 + 5x_3 &= 13 \\ 43x_3 &= 129 \end{aligned}$$

1. We may solve the 3rd equation for x_3 and substitute this result into the 1st and 2nd equations.
2. We may solve the 2nd equation for x_2 and substitute this result into the 1st equation.
3. We may solve the 1st equation for x_1.

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$$\begin{aligned} x_3 &= \frac{129}{43} = 3 \\ 2x_2 + 5(3) &= 13 \quad \Longrightarrow \quad x_2 = -1 \\ 2x_1 + 4(-1) - 3 &= -5 \quad \Longrightarrow \quad x_1 = 1 \end{aligned}$$

Matrix Notation

Since we only manipulated the constants and coefficients of the linear system, we only need keep track of them and can suppress the unknowns.

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Notation:

$$A = [a_{ij}] = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1m} \\ a_{21} & a_{22} & \cdots & a_{2m} \\ \vdots & \vdots & \ddots & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nm} \end{bmatrix}$$

is called an $n \times m$ **matrix.**

Vectors

A $1 \times n$ matrix

$$\begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \end{bmatrix}$$

is called a **row vector**.

An $n \times 1$ matrix

$$\begin{bmatrix} a_{11} \\ a_{21} \\ \vdots \\ a_{n1} \end{bmatrix}$$

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is called a **column vector**.

Remark: we can represent any linear system with an $n \times n$ matrix and an $n \times 1$ column vector.

Augmented Matrix

The linear system of the form:

$$\begin{aligned} a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n &= b_1 \\ a_{21}x_1 + a_{22}x_2 + \cdots + a_{2n}x_n &= b_2 \\ &\vdots \\ a_{n1}x_1 + a_{n2}x_2 + \cdots + a_{nn}x_n &= b_n \end{aligned}$$

will often be represented by an $n \times (n+1)$ **augmented matrix**:

$$\left[\begin{array}{c|c} A & \mathbf{b} \end{array}\right] = \left[\begin{array}{cccc|c} a_{11} & a_{12} & \cdots & a_{1n} & b_1 \\ a_{21} & a_{22} & \cdots & a_{2n} & b_2 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} & b_n \end{array}\right] = \tilde{A}.$$

Gaussian Elimination with Back Substitution

Reducing the A portion of an augmented matrix to triangular form and then using back substitution to solve the linear system is called **Gaussian elimination with back substitution**.

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Assuming that for some $i \in \{1, 2, \ldots, n\}$ we have $a_{i1} \neq 0$ the reduced form of $\tilde{A}$ resembles the following.

$$\tilde{\tilde{A}} = \left[\begin{array}{cccc|c} \tilde{a}_{11} & \tilde{a}_{12} & \cdots & \tilde{a}_{1n} & \tilde{b}_1 \\ 0 & \tilde{a}_{22} & \cdots & \tilde{a}_{2n} & \tilde{b}_2 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & \tilde{a}_{nn} & \tilde{b}_n \end{array}\right]$$

Back Substitution

Given

$$\left[\begin{array}{cccc|c} \tilde{a}_{11} & \tilde{a}_{12} & \cdots & \tilde{a}_{1n} & \tilde{b}_1 \\ 0 & \tilde{a}_{22} & \cdots & \tilde{a}_{2n} & \tilde{b}_2 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & \tilde{a}_{nn} & \tilde{b}_n \end{array}\right],$$

we have the solution:

$$\begin{aligned} x_n &= \frac{\tilde{b}_n}{\tilde{a}_{nn}} \\ &\vdots \\ x_i &= \frac{\tilde{b}_i - \tilde{a}_{in}x_n - \tilde{a}_{i,n-1}x_{n-1} - \cdots - \tilde{a}_{i,i+1}x_{i+1}}{\tilde{a}_{ii}} = \frac{\tilde{b}_i - \sum_{j=i+1}^{n} \tilde{a}_{ij}x_j}{\tilde{a}_{ii}} \\ &\vdots \\ x_1 &= \frac{\tilde{b}_1 - \tilde{a}_{1n}x_n - \tilde{a}_{1,n-1}x_{n-1} - \cdots - \tilde{a}_{12}x_2}{\tilde{a}_{11}} = \frac{\tilde{b}_1 - \sum_{j=2}^{n} \tilde{a}_{1j}x_j}{\tilde{a}_{11}} \end{aligned}$$

Pivoting

Comment: at each stage of the reduction process we have assumed that $\tilde{a}_{ii} \neq 0$. If we encounter $\tilde{a}_{ii} = 0$ then we look for $j > i$ (a lower row) such that $\tilde{a}_{ji} \neq 0$ and swap rows i and j.

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This operation is called **pivoting**.

Use Gaussian elimination with back substitution to solve the following linear system given in augmented matrix form.

$$\left[\begin{array}{ccc|c} 1 & -1 & 3 & 2 \\ 3 & -3 & 1 & -1 \\ 1 & 1 & 0 & 3 \end{array}\right]$$

Solution

Gaussian elimination:

$$\left[\begin{array}{ccc|c} 1 & -1 & 3 & 2 \\ 3 & -3 & 1 & -1 \\ 1 & 1 & 0 & 3 \end{array}\right] \mapsto \left[\begin{array}{ccc|c} 1 & -1 & 3 & 2 \\ 0 & 0 & -8 & -7 \\ 0 & 2 & -3 & 1 \end{array}\right]$$

$$\mapsto \left[\begin{array}{ccc|c} 1 & -1 & 3 & 2 \\ 0 & 2 & -3 & 1 \\ 0 & 0 & -8 & -7 \end{array}\right]$$

Back substitution:

$$\begin{aligned} x_3 &= \frac{7}{8} \\ x_2 &= \frac{1-(-3)(7/8)}{2} = \frac{29}{16} \\ x_1 &= \frac{2-3(7/8)-(-1)(29/16)}{1} = \frac{19}{16} \end{aligned}$$

Operation Count (1 of 8)

Question: how many multiplications/divisions and how many additions/subtractions are necessary to reduce the augmented matrix?

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Suppose that at the beginning of the *i*th stage of the reduction the augmented matrix resembles the following.

$$\tilde{A} = \left[\begin{array}{cccccc|c} a_{11} & a_{12} & \cdots & a_{1i} & \cdots & a_{1n} & a_{1,n+1} \\ 0 & a_{22} & \cdots & a_{2i} & \cdots & a_{2n} & a_{2,n+1} \\ \vdots & \vdots & \ddots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & a_{ii} & \cdots & a_{in} & a_{i,n+1} \\ \vdots & \vdots & \ddots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & a_{ni} & \cdots & a_{nn} & a_{n,n+1} \end{array}\right]$$

Operation Count (2 of 8)

To row reduce below the *i*th row we must

- Find row multipliers requiring multiplications/divisions.
- Multiply the *i*th row by each of the row multipliers requiring multiplications/divisions.
- Add a multiple of the *i*th row to the rows beneath requiring additions/subtractions.

Operation Count (2 of 8)

To row reduce below the ith row we must

- Find $n - i$ row multipliers requiring $n - i$ multiplications/divisions.
- Multiply the ith row by each of the $n - i$ row multipliers requiring multiplications/divisions.
- Add a multiple of the ith row to the $n - i$ rows beneath requiring additions/subtractions.

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To row reduce below the *i*th row we must

- Find $n - i$ row multipliers requiring $n - i$ multiplications/divisions.
- Multiply the *i*th row by each of the $n - i$ row multipliers requiring $(n - i)(n - i + 1)$ multiplications/divisions.
- Add a multiple of the *i*th row to the $n - i$ rows beneath requiring additions/subtractions.

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- Multiply the ith row by each of the $n - i$ row multipliers requiring $(n - i)(n - i + 1)$ multiplications/divisions.
- Add a multiple of the ith row to the $n - i$ rows beneath requiring $(n - i)(n - i + 1)$ additions/subtractions.

Summary: to row reduce below the ith row requires

$$(n-i)+(n-i)(n-i+1) = (n-i)(n-i+2) \quad \text{multiplications/divisions}$$

and $(n - i)(n - i + 1)$ additions/subtractions.

Operation Count (2 of 8)

To row reduce below the ith row we must

- Find $n - i$ row multipliers requiring $n - i$ multiplications/divisions.
- Multiply the ith row by each of the $n - i$ row multipliers requiring $(n - i)(n - i + 1)$ multiplications/divisions.
- Add a multiple of the ith row to the $n - i$ rows beneath requiring $(n - i)(n - i + 1)$ additions/subtractions.

Summary: to row reduce below the ith row requires

$$(n-i)+(n-i)(n-i+1) = (n-i)(n-i+2) \quad \text{multiplications/divisions}$$

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Starting from the original matrix, this set of row reductions must be carried out times.

Operation Count (2 of 8)

To row reduce below the ith row we must

- Find $n - i$ row multipliers requiring $n - i$ multiplications/divisions.
- Multiply the ith row by each of the $n - i$ row multipliers requiring $(n - i)(n - i + 1)$ multiplications/divisions.
- Add a multiple of the ith row to the $n - i$ rows beneath requiring $(n - i)(n - i + 1)$ additions/subtractions.

Summary: to row reduce below the ith row requires

$$(n-i)+(n-i)(n-i+1) = (n-i)(n-i+2) \quad \text{multiplications/divisions}$$

and $(n - i)(n - i + 1)$ additions/subtractions.

Starting from the original matrix, this set of row reductions must be carried out $n - 1$ times.

Operation Count (3 of 8)

Total multiplications/divisions for matrix reduction:

$$\sum_{i=1}^{n-1}(n-i)(n-i+2)$$

$$= \sum_{i=1}^{n-1}(n^2+2n) - (2n+2)\sum_{i=1}^{n-1} i + \sum_{i=1}^{n-1} i^2$$

$$= (n^2+2n)(n-1) - (2n+2)\frac{n(n-1)}{2} + \frac{(n-1)(n)(2n-1)}{6}$$

$$= \frac{2n^3+3n^2-5n}{6}$$

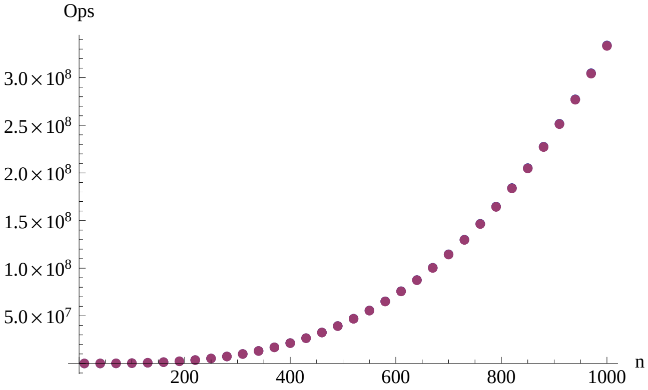
Operation Count (4 of 8)

Total additions/subtractions for matrix reduction:

$$\begin{aligned}
&\sum_{i=1}^{n-1}(n-i)(n-i+1) \\
&= \sum_{i=1}^{n-1}(n^2+n) - (2n+1)\sum_{i=1}^{n-1} i + \sum_{i=1}^{n-1} i^2 \\
&= (n^2+n)(n-1) - (2n+1)\frac{n(n-1)}{2} + \frac{(n-1)(n)(2n-1)}{6} \\
&= \frac{n^3-n}{3}
\end{aligned}$$

Operation Counts (5 of 8)

Growth of operation counts required for row reduction:



Operation Counts (6 of 8)

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$$x_i = \frac{\tilde{b}_i - \tilde{a}_{in}x_n - \tilde{a}_{i,n-1}x_{n-1} - \cdots - \tilde{a}_{i,i+1}x_{i+1}}{\tilde{a}_{ii}}$$

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multiplications/divisions: $n - i + 1$ (if $i \neq n$) or 1 (if $i = n$)

additions/subtractions: $n - i$

Operation Counts (6 of 8)

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multiplications/divisions: $n - i + 1$ (if $i \neq n$) or 1 (if $i = n$)

additions/subtractions: $n - i$

This operation must be carried out $n - 1$ times.

Operation Counts (7 of 8)

Total multiplications/divisions for back substitution:

$$\begin{aligned}
1 + \sum_{i=1}^{n-1}(n-i+1) &= 1 + \sum_{i=1}^{n-1}(n+1) - \sum_{i=1}^{n-1} i \\
&= 1 + (n-1)(n+1) - \frac{n(n-1)}{2} \\
&= \frac{n^2+n}{2}
\end{aligned}$$

Total additions/subtractions for back substitution:

$$\begin{aligned}
\sum_{i=1}^{n-1}(n-i) &= \sum_{i=1}^{n-1} n - \sum_{i=1}^{n-1} i \\
&= n(n-1) - \frac{n(n-1)}{2} \\
&= \frac{n^2-n}{2}
\end{aligned}$$

Operation Counts (8 of 8)

Thus the total operation counts for solving a linear system via Gaussian elimination and back substitution are

multiplications/divisions: $\dfrac{n^3 + 3n^2 - n}{3}$

additions/subtractions: $\dfrac{2n^3 + 3n^2 - 5n}{6}$

Algorithm: Gaussian Elimination with Back Substitution

Given the augmented matrix $A = [a_{ij}]_{i=1,\dots,n,\, j=1,\dots,n+1}$:

STEP 1 For $i = 1, 2, \dots, n-1$ set

$$p = \min_{j \in \{i, i+1, \dots, n\}} \{j \mid a_{ji} \neq 0\}$$

- If $p \neq i$ then transpose rows i and p.
- For $j = i+1, i+2, \dots, n$ replace row j by the sum of row j and $-\dfrac{a_{ji}}{a_{ii}}$ times row i.

STEP 2 Set $x_n = \dfrac{a_{n,n+1}}{a_{nn}}$.

STEP 3 For $i = n-1, n-2, \dots, 1$ set

$$x_i = \frac{a_{i,n+1} - \sum_{j=i+1}^{n} a_{ij} x_j}{a_{ii}}$$

Comments

- If no value for p (the pivot) can be found, then no unique solution to the linear system exists.
- If $a_{nn} = 0$ then no unique solution exists.

Example

Solve the following linear system using Gaussian elimination with back substitution and 3-digit chopping arithmetic.

$$\begin{aligned} 3.33x_1 + 15900x_2 - 10.3x_3 &= 15900 \\ 2.22x_1 + 16.7x_2 + 9.61x_3 &= 28.5 \\ 1.56x_1 + 5.17x_2 + 16.8x_3 &= 8.42 \end{aligned}$$

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Solve the following linear system using Gaussian elimination with back substitution and 3-digit chopping arithmetic.

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For comparison purposes, the exact solution is nearly:

$$\begin{aligned} x_1 &= 7.5073 \\ x_2 &= 0.998102 \\ x_3 &= -0.50307 \end{aligned}$$

Solution (1 of 4)

Augmented matrix:

$$\left[\begin{array}{ccc|c} 3.33 & 15900 & -10.3 & 15900 \\ 2.22 & 16.7 & 9.61 & 28.5 \\ 1.56 & 5.17 & 16.8 & 8.42 \end{array}\right]$$

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Multiply row 1 by 0.666 and subtract from row 2.

$$\left[\begin{array}{ccc|c} 2.21 & 10500 & -6.85 & 10500 \end{array}\right]$$

Solution (1 of 4)

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Multiply row 1 by 0.666 and subtract from row 2.

$$\left[\begin{array}{ccc|c} 2.21 & 10500 & -6.85 & 10500 \end{array}\right]$$

$$\left[\begin{array}{ccc|c} 3.33 & 15900 & -10.3 & 15900 \\ 0.01 & -10500 & 16.4 & -10500 \\ 1.56 & 5.17 & 16.8 & 8.42 \end{array}\right]$$

Solution (2 of 4)

Augmented matrix:

$$\left[\begin{array}{ccc|c} 3.33 & 15900 & -10.3 & 15900 \\ 0.01 & -10500 & 16.4 & -10500 \\ 1.56 & 5.17 & 16.8 & 8.42 \end{array}\right]$$

Solution (2 of 4)

Augmented matrix:

$$\left[\begin{array}{ccc|c} 3.33 & 15900 & -10.3 & 15900 \\ 0.01 & -10500 & 16.4 & -10500 \\ 1.56 & 5.17 & 16.8 & 8.42 \end{array}\right]$$

Multiply row 1 by 0.468 and subtract from row 3.

$$\left[\begin{array}{ccc|c} 1.55 & 7440 & -4.82 & 7440 \end{array}\right]$$

Solution (2 of 4)

Augmented matrix:

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$$\left[\begin{array}{ccc|c} 3.33 & 15900 & -10.3 & 15900 \\ 0.01 & -10500 & 16.4 & -10500 \\ 0.01 & -7440 & 21.6 & -7440 \end{array}\right]$$

Solution (3 of 4)

Partially reduced matrix:

$$\left[\begin{array}{ccc|c} 3.33 & 15900 & -10.3 & 15900 \\ 0.01 & -10500 & 16.4 & -10500 \\ 0.01 & -7440 & 21.6 & -7440 \end{array}\right]$$

Multiply row 2 by 0.708 and subtract from row 3.

Solution (3 of 4)

Partially reduced matrix:

$$\left[\begin{array}{ccc|c} 3.33 & 15900 & -10.3 & 15900 \\ 0.01 & -10500 & 16.4 & -10500 \\ 0.01 & -7440 & 21.6 & -7440 \end{array}\right]$$

Multiply row 2 by 0.708 and subtract from row 3.

$$\left[\begin{array}{ccc|c} 0.01 & -7430 & 11.6 & -7430 \end{array}\right]$$

Solution (3 of 4)

Partially reduced matrix:

$$\left[\begin{array}{ccc|c} 3.33 & 15900 & -10.3 & 15900 \\ 0.01 & -10500 & 16.4 & -10500 \\ 0.01 & -7440 & 21.6 & -7440 \end{array}\right]$$

Multiply row 2 by 0.708 and subtract from row 3.

$$\left[\begin{array}{ccc|c} 0.01 & -7430 & 11.6 & -7430 \end{array}\right]$$

$$\left[\begin{array}{ccc|c} 3.33 & 15900 & -10.3 & 15900 \\ 0.01 & -10500 & 16.4 & -10500 \\ 0.01 & -10.0 & 10.0 & -10.0 \end{array}\right]$$

Solution (4 of 4)

Given the reduced matrix:

$$\left[\begin{array}{ccc|c} 3.33 & 15900 & -10.3 & 15900 \\ 0.01 & -10500 & 16.4 & -10500 \\ 0.01 & -10.0 & 10.0 & -10.0 \end{array}\right]$$

Begin back substitution.

Solution (4 of 4)

Given the reduced matrix:

$$\left[\begin{array}{ccc|c} 3.33 & 15900 & -10.3 & 15900 \\ 0.01 & -10500 & 16.4 & -10500 \\ 0.01 & -10.0 & 10.0 & -10.0 \end{array}\right]$$

Begin back substitution.

$$\begin{aligned} x_3 &= -1.00 \\ x_2 &= \frac{-10500 - 16.4(-1.00)}{-10500} = 1.00 \\ x_1 &= \frac{15900 - (-10.3)(-1.00) - (15900)(1.00)}{3.33} = 0.00 \end{aligned}$$

Solution (4 of 4)

Given the reduced matrix:

$$\left[\begin{array}{ccc|c} 3.33 & 15900 & -10.3 & 15900 \\ 0.01 & -10500 & 16.4 & -10500 \\ 0.01 & -10.0 & 10.0 & -10.0 \end{array}\right]$$

Begin back substitution.

$$\begin{aligned} x_3 &= -1.00 \\ x_2 &= \frac{-10500 - 16.4(-1.00)}{-10500} = 1.00 \\ x_1 &= \frac{15900 - (-10.3)(-1.00) - (15900)(1.00)}{3.33} = 0.00 \end{aligned}$$

This does not compare very well with the “exact” solution.

$$\begin{aligned} x_3 &= -0.50307 \\ x_2 &= 0.998102 \\ x_1 &= 7.5073 \end{aligned}$$

Homework

- Read Section 6.1.
- Exercises: 1ad, 3, 5ad, 9, 11