Gaussian Quadrature

MATH 375

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Introduction

Consider the definite integral:

$$\int_a^b f(x)\,dx$$

The Newton-Cotes formulas discussed so far have used equally spaced nodes in the interval $[a, b]$ of the form $x_j = a + jh$ for $j \in \mathbb{Z}$ where

$$h = \frac{b-a}{n}$$

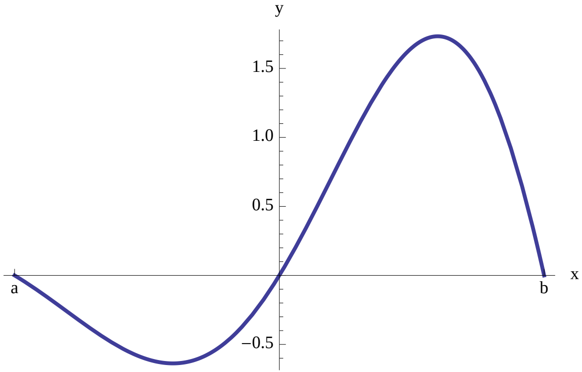
for some $n \in \mathbb{N}$.

The approximation method known as **Gaussian Quadrature** makes an adaptive choice of nodes that minimizes the error in the approximation.

Motivation (1 of 2)

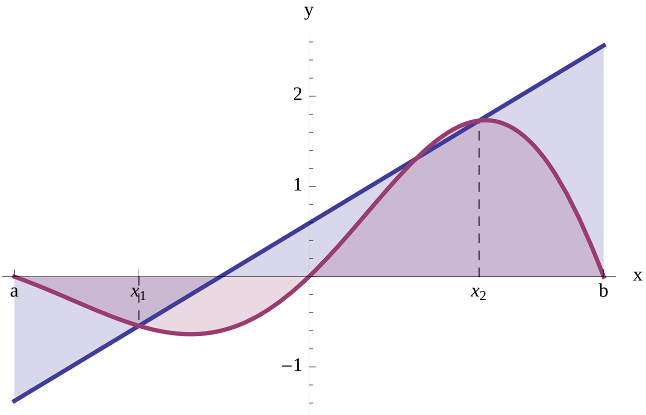
Note that the Trapezoidal rule uses the nodes at $x = a$ and $x = b$ to approximate the integrand by a linear function. For the function below, the Trapezoidal rule would approximate

$$\int_a^b f(x)\,dx \approx 0.$$



Motivation (2 of 2)

If we chose the nodes for the linear interpolation more appropriately we could get a better approximation.



Basic Idea

Choose x_i and c_i for $i = 1, 2, \ldots, n$ such that

$$\int_a^b f(x)\,dx \approx \sum_{i=1}^{n} c_i f(x_i).$$

Note: The constants c_i are arbitrary, but $x_i \in [a, b]$ for $i = 1, 2, \ldots, n$.

There are $2n$ values to be selected.

Objective

- The efficacy of this approach will be judged by the considering the size of the class of polynomials for which this approximation formula gives exact results.
- With $2n$ values to be determined, we should expect the upper limit of the **precision** of this method to be $2n - 1$.
- In other words, this method should be exact for polynomials of degree $2n - 1$ or less.

Example: $n = 2$ (1 of 3)

- Let $[a, b] = [-1, 1]$ and $n = 2$.
- We want to choose x_1, x_2, c_1, and c_2 so that

$$\int_{-1}^{1} f(x)\,dx \approx c_1 f(x_1) + c_2 f(x_2).$$

- The approximation should be exact for any polynomial of degree 3 or less.

Example: $n = 2$ (2 of 3)

If we let

$$P(x) = a_0 + a_1 x + a_2 x^2 + a_3 x^3$$

then

$$\int_{-1}^{1} P(x)\,dx = a_0 \int_{-1}^{1} 1\,dx + a_1 \int_{-1}^{1} x\,dx + a_2 \int_{-1}^{1} x^2\,dx + a_3 \int_{-1}^{1} x^3\,dx$$

Each of the definite integrals on the right-hand side has an integrand of degree 3 or less. Gaussian Quadrature should be exact for each of these.

Example: $n = 2$ (3 of 3)

Applying Gaussian Quadrature to each remaining integral yields:

$$
\begin{aligned}
\int_{-1}^{1} 1\, dx &= 2 = c_1 + c_2 \\
\int_{-1}^{1} x\, dx &= 0 = c_1 x_1 + c_2 x_2 \\
\int_{-1}^{1} x^2\, dx &= \frac{2}{3} = c_1 x_1^2 + c_2 x_2^2 \\
\int_{-1}^{1} x^3\, dx &= 0 = c_1 x_1^3 + c_2 x_2^3
\end{aligned}
$$

Remark: if each one of these integrals can be made exact, then the overall method will have a precision of 3.

We must solve this system of 4 nonlinear equations in 4 unknowns.

Solution to Nonlinear System

Solving the nonlinear system gives us

$$c_1 = 1, \quad c_2 = 1, \quad x_1 = -\frac{\sqrt{3}}{3}, \quad x_2 = \frac{\sqrt{3}}{3}.$$

Therefore

$$\int_{-1}^{1} f(x)\,dx \approx f\left(-\frac{\sqrt{3}}{3}\right) + f\left(\frac{\sqrt{3}}{3}\right).$$

Test Case

$$\begin{aligned}\int_{-1}^{1} e^x \sin \pi x \, dx &= \frac{\pi(e^2-1)}{e(\pi^2+1)} \approx 0.679326 \text{ (exact value)} \\ &\approx 1.18409 \text{ (Gaussian Quadrature)} \\ &\approx 0 \text{ (Trapezoidal rule)} \\ &\approx 0 \text{ (Simpson's rule)}\end{aligned}$$

Generalizing the Method for $n > 2$

We must introduce the **Legendre Polynomials**, $\{P_0(x), P_1(x), \ldots\}$, which have the properties:

- $\deg P_n = n$
- If $P(x)$ is a polynomial of degree less than n, then

$$\int_{-1}^{1} P(x)P_n(x)\,dx = 0.$$

Legendre's Differential Equation

The Legendre equation is

$$(1 - x^2)y'' - 2x\,y' + \alpha(\alpha + 1)y = 0$$

where α is a constant.

- We would like to find two linearly independent solutions to this ODE.
- In general the solutions are power series in x which converge for $-1 < x < 1$.

Solving Legendre's Equation (1 of 6)

Assume $y(x) = \sum_{n=0}^{\infty} a_n x^n$ and substitute this into Legendre's equation.

$$\begin{aligned}
0 &= (1-x^2)y'' - 2xy' + \alpha(\alpha+1)y \\
&= (1-x^2)\sum_{n=2}^{\infty} n(n-1)a_n x^{n-2} - 2x\sum_{n=1}^{\infty} n a_n x^{n-1} \\
&\quad + \alpha(\alpha+1)\sum_{n=0}^{\infty} a_n x^n \\
&= \sum_{n=2}^{\infty} n(n-1)a_n x^{n-2} - \sum_{n=2}^{\infty} n(n-1)a_n x^n - \sum_{n=1}^{\infty} 2n a_n x^n \\
&\quad + \alpha(\alpha+1)\sum_{n=0}^{\infty} a_n x^n
\end{aligned}$$

Solving Legendre's Equation (2 of 6)

$$
\begin{aligned}
0 &= (1-x^2)y'' - 2xy' + \alpha(\alpha+1)y \\
&= \sum_{n=0}^{\infty}(n+2)(n+1)a_{n+2}x^n - \sum_{n=0}^{\infty} n(n-1)a_n x^n - \sum_{n=0}^{\infty} 2na_n x^n \\
&\quad + \alpha(\alpha+1)\sum_{n=0}^{\infty} a_n x^n \\
&= \sum_{n=0}^{\infty}\left[(n+2)(n+1)a_{n+2} - (n(n+1)-\alpha(\alpha+1))a_n\right]x^n
\end{aligned}
$$

Solving Legendre's Equation (3 of 6)

$$0 = \sum_{n=0}^{\infty} \left[(n+2)(n+1)a_{n+2} - (n(n+1) - \alpha(\alpha+1))a_n\right] x^n$$

If we equate coefficients of powers of x on both sides of the equation we see that

$$\begin{aligned} 0 &= (n+2)(n+1)a_{n+2} - (n(n+1) - \alpha(\alpha+1))a_n \\ a_{n+2} &= \frac{(n(n+1) - \alpha(\alpha+1))a_n}{(n+2)(n+1)}, \end{aligned}$$

for $n = 0, 1, \ldots$.

Solving Legendre's Equation (3 of 6)

$$0 = \sum_{n=0}^{\infty} \left[(n+2)(n+1)a_{n+2} - (n(n+1) - \alpha(\alpha+1))a_n\right] x^n$$

If we equate coefficients of powers of x on both sides of the equation we see that

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for $n = 0, 1, \ldots$.

Remarks:

- The last equation is called a **recurrence relation**.
- The recurrence relation generates the $(n+2)$nd coefficient of the power series from the nth coefficient of the power series.

Solving Legendre's Equation (4 of 6)

$$a_{n+2} = \frac{(n(n+1) - \alpha(\alpha+1))a_n}{(n+2)(n+1)}$$

Questions:

1. Suppose $a_0 = 0$, what is a_{2n} for $n \in \mathbb{N}$?

Solving Legendre's Equation (4 of 6)

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Questions:

1. Suppose $a_0 = 0$, what is a_{2n} for $n \in \mathbb{N}$?
2. Suppose $a_1 = 0$, what is a_{2n+1} for $n \in \mathbb{N}$?

Solving Legendre's Equation (4 of 6)

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Questions:

1. Suppose $a_0 = 0$, what is a_{2n} for $n \in \mathbb{N}$?
2. Suppose $a_1 = 0$, what is a_{2n+1} for $n \in \mathbb{N}$?
3. If $a_0 = 0$, $a_1 = 1$ and $\alpha = 2m + 1$ for some fixed $m \in \mathbb{N}$, what type of function is the solution to Legendre's equation?

Solving Legendre's Equation (4 of 6)

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3. If $a_0 = 0$, $a_1 = 1$ and $\alpha = 2m + 1$ for some fixed $m \in \mathbb{N}$, what type of function is the solution to Legendre's equation?
4. If $a_0 = 1$, $a_1 = 0$ and $\alpha = 2m$ for some fixed $m \in \mathbb{N}$, what type of function is the solution to Legendre's equation?

Solving Legendre's Equation (5 of 6)

$$a_{n+2} = \frac{(n(n+1) - \alpha(\alpha+1))a_n}{(n+2)(n+1)}$$

for $n = 0, 1, \ldots$.

Suppose $a_0 = 1$ and $a_1 = 0$, then

- if $\alpha = 0$ then $y_0(x) = 1$ a polynomial of degree 0,
- if $\alpha = 2$ then $y_2(x) =$
- if $\alpha = 4$ then $y_4(x) =$

Solving Legendre's Equation (5 of 6)

$$a_{n+2} = \frac{(n(n+1) - \alpha(\alpha+1))a_n}{(n+2)(n+1)}$$

for $n = 0, 1, \ldots$.

Suppose $a_0 = 1$ and $a_1 = 0$, then

- if $\alpha = 0$ then $y_0(x) = 1$ a polynomial of degree 0,
- if $\alpha = 2$ then $y_2(x) = 1 - 3x^2$ a polynomial of degree 2 containing only even powers of x,
- if $\alpha = 4$ then $y_4(x) =$

Solving Legendre's Equation (5 of 6)

$$a_{n+2} = \frac{(n(n+1) - \alpha(\alpha+1))a_n}{(n+2)(n+1)}$$

for $n = 0, 1, \ldots$.

Suppose $a_0 = 1$ and $a_1 = 0$, then

- if $\alpha = 0$ then $y_0(x) = 1$ a polynomial of degree 0,
- if $\alpha = 2$ then $y_2(x) = 1 - 3x^2$ a polynomial of degree 2 containing only even powers of x,
- if $\alpha = 4$ then $y_4(x) = 1 - 10x^2 + \frac{35}{3}x^4$ a polynomial of degree 4 containing only even powers of x.

Solving Legendre's Equation (6 of 6)

$$a_{n+2} = \frac{(n(n+1) - \alpha(\alpha+1))a_n}{(n+2)(n+1)}$$

for $n = 0, 1, \ldots$.

Suppose $a_0 = 0$ and $a_1 = 1$, then

- if $\alpha = 1$ then $y_1(x) = x$ a polynomial of degree 1,
- if $\alpha = 3$ then $y_3(x) =$
- if $\alpha = 5$ then $y_5(x) =$

Solving Legendre's Equation (6 of 6)

$$a_{n+2} = \frac{(n(n+1) - \alpha(\alpha+1))a_n}{(n+2)(n+1)}$$

for $n = 0, 1, \ldots$.

Suppose $a_0 = 0$ and $a_1 = 1$, then

- if $\alpha = 1$ then $y_1(x) = x$ a polynomial of degree 1,
- if $\alpha = 3$ then $y_3(x) = x - \frac{5}{3}x^3$ a polynomial of degree 3 containing only odd powers of x,
- if $\alpha = 5$ then $y_5(x) =$

Solving Legendre's Equation (6 of 6)

$$a_{n+2} = \frac{(n(n+1) - \alpha(\alpha+1))a_n}{(n+2)(n+1)}$$

for $n = 0, 1, \ldots$.

Suppose $a_0 = 0$ and $a_1 = 1$, then

- if $\alpha = 1$ then $y_1(x) = x$ a polynomial of degree 1,
- if $\alpha = 3$ then $y_3(x) = x - \frac{5}{3}x^3$ a polynomial of degree 3 containing only odd powers of x,
- if $\alpha = 5$ then $y_5(x) = x - \frac{14}{3}x^3 + \frac{21}{5}x^5$ a polynomial of degree 5 containing only odd powers of x.

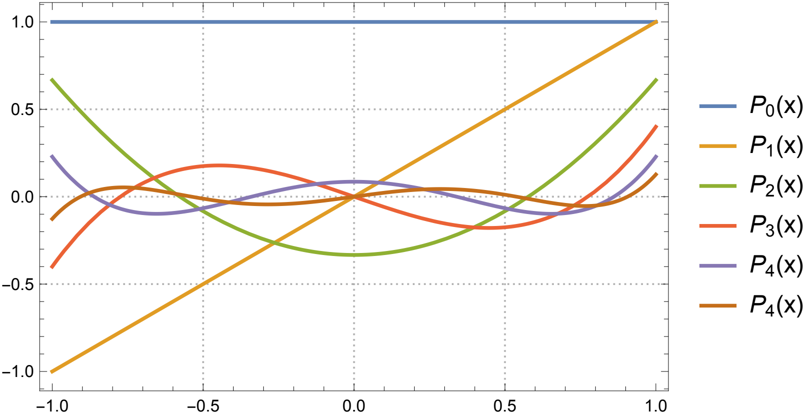
The Legendre Polynomials

Definition

The **Legendre polynomial** $P_n(x)$ is the monic polynomial solution of the Legendre equation with $\alpha = n$.

$$
\begin{aligned}
P_0(x) &= 1 \\
P_1(x) &= x \\
P_2(x) &= x^2 - \frac{1}{3} \\
P_3(x) &= x^3 - \frac{3}{5}x \\
P_4(x) &= x^4 - \frac{6}{7}x^2 + \frac{3}{35} \\
P_5(x) &= x^5 - \frac{10}{9}x^3 + \frac{5}{21}x
\end{aligned}
$$

Graphs



Orthogonality of Legendre Polynomials (1 of 7)

Recall Legendre's differential equation:

$$\begin{aligned} 0 &= (1-x^2)y'' - 2xy' + \alpha(\alpha+1)y \\ &= (1-x^2)(y')' + (1-x^2)'y' + \alpha(\alpha+1)y \\ -\alpha(\alpha+1)y &= [(1-x^2)y']' \quad \text{(product rule).} \end{aligned}$$

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-\alpha(\alpha+1)y &= [(1-x^2)y']' \quad \text{(product rule).}
\end{aligned}$$

Thus when $y(x) = P_n(x)$ we have

$$[(1-x^2)P_n'(x)]' = -n(n+1)P_n(x).$$

Suppose we multiply this equation by $P_m(x)$ where $m \neq n$, then

$$\begin{aligned}
[(1-x^2)P_n'(x)]'P_m(x) &= -n(n+1)P_n(x)P_m(x) \\
\int_{-1}^{1} [(1-x^2)P_n'(x)]'P_m(x)\,dx &= -n(n+1)\int_{-1}^{1} P_n(x)P_m(x)\,dx
\end{aligned}$$

Orthogonality of Legendre Polynomials (2 of 7)

$$\int_{-1}^{1} [(1-x^2)P_n'(x)]' P_m(x)\,dx = -n(n+1)\int_{-1}^{1} P_n(x)P_m(x)\,dx$$

We can apply integration by parts to the integral on the left-hand side of this equation.

$$\begin{aligned} u &= P_m(x) & v &= (1-x^2)P_n'(x) \\ du &= P_m'(x)\,dx & dv &= [(1-x^2)P_n'(x)]'\,dx \end{aligned}$$

Orthogonality of Legendre Polynomials (3 of 7)

$$\begin{aligned}
&\int_{-1}^{1} [(1-x^2)P_n'(x)]' P_m(x)\,dx \\
&= \quad \left[P_m(x)[(1-x^2)P_n'(x)]\right]_{x=-1}^{x=1} - \int_{-1}^{1} (1-x^2)P_n'(x)P_m'(x)\,dx \\
&= \quad -\int_{-1}^{1} (1-x^2)P_n'(x)P_m'(x)\,dx.
\end{aligned}$$

Orthogonality of Legendre Polynomials (4 of 7)

Similarly when $y(x) = P_m(x)$ we have

$$[(1 - x^2)P_m'(x)]' = -m(m+1)P_m(x).$$

Suppose we multiply this equation by $P_n(x)$, then

$$\begin{aligned}
[(1 - x^2)P_m'(x)]'P_n(x) &= -m(m+1)P_m(x)P_n(x) \\
\int_{-1}^{1} [(1 - x^2)P_m'(x)]'P_n(x)\,dx &= -m(m+1)\int_{-1}^{1} P_m(x)P_n(x)\,dx
\end{aligned}$$

We can apply integration by parts to the integral on the left-hand side of this equation.

$$\begin{aligned}
u &= P_n(x) & \qquad v &= (1 - x^2)P_m'(x) \\
du &= P_n'(x)\,dx & \qquad dv &= [(1 - x^2)P_m'(x)]'\,dx
\end{aligned}$$

Orthogonality of Legendre Polynomials (5 of 7)

Thus

$$\begin{aligned}
&\int_{-1}^{1} [(1-x^2)P_m'(x)]' P_n(x)\,dx \\
&= \left[P_n(x)[(1-x^2)P_m'(x)]\right]_{x=-1}^{x=1} - \int_{-1}^{1} (1-x^2)P_m'(x)P_n'(x)\,dx \\
&= -\int_{-1}^{1} (1-x^2)P_m'(x)P_n'(x)\,dx.
\end{aligned}$$

Orthogonality of Legendre Polynomials (6 of 7)

Combining the equations we see that

$$\begin{aligned}\int_{-1}^{1}(1-x^2)P_m'(x)P_n'(x)\,dx &= \int_{-1}^{1}(1-x^2)P_n'(x)P_m'(x)\,dx \\ m(m+1)\int_{-1}^{1}P_m(x)P_n(x)\,dx &= n(n+1)\int_{-1}^{1}P_n(x)P_m(x)\,dx\end{aligned}$$

Re-arranging terms in the last equation yields

$$[m(m+1)-n(n+1)]\int_{-1}^{1}P_m(x)P_n(x)\,dx = 0$$

Orthogonality of Legendre Polynomials (7 of 7)

Since

$$[m(m+1) - n(n+1)] \int_{-1}^{1} P_m(x) P_n(x)\, dx = 0$$

either

$$\int_{-1}^{1} P_m(x) P_n(x)\, dx = 0$$

which is the orthogonality property, or

$$0 = [m(m+1) - n(n+1)] = (m-n)(m+n+1)$$

Since $m, n \in \mathbb{N}$ and we have assumed $m \neq n$ then this equation is not satisfied.

Comment

As a corollary to the orthogonality property of the Legendre polynomials we have the following result.

Corollary

If $P(x)$ is a polynomial of degree less than n, then

$$\int_{-1}^{1} P(x)\, P_n(x)\, dx = 0.$$

Properties of Legendre Polynomials (1 of 2)

- The set of Legendre Polynomials forms an infinite-dimensional inner product space.
- First five Legendre polynomials:

$$
\begin{aligned}
P_0(x) &= 1 \\
P_1(x) &= x \\
P_2(x) &= x^2 - \frac{1}{3} \\
P_3(x) &= x^3 - \frac{3}{5}x \\
P_4(x) &= x^4 - \frac{6}{7}x^2 + \frac{3}{35}
\end{aligned}
$$

- Higher degree Legendre polynomials can be generated using the **Gram-Schmidt Orthogonalization Process**.

Properties of Legendre Polynomials (2 of 2)

- Roots of the polynomials lie in the interval $(-1, 1)$.
- Roots are symmetric about the origin.
- Roots are distinct.
- For $P_n(x)$ the roots $\{x_1, x_2, \ldots, x_n\}$ are the nodes needed by Gaussian Quadrature. That is, the quadrature formula

$$\int_{-1}^{1} f(x)\, dx = \sum_{k=1}^{n} c_k f(x_k)$$

will be exact for any polynomial for degree $2n - 1$ or less.

Example

Use the Gram-Schmidt Orthogonalization process with the inner product

$$\langle P(x), Q(x) \rangle = \int_{-1}^{1} P(x)Q(x)\, dx$$

and the first three Legendre polynomials:

$$\begin{aligned} P_0(x) &= 1 \\ P_1(x) &= x \\ P_2(x) &= x^2 - \frac{1}{3} \end{aligned}$$

to find $P_3(x)$.

Solution (1 of 2)

$$P_3(x) = x^3 - \frac{\langle x^3, 1\rangle}{\langle 1, 1\rangle}(1) - \frac{\langle x^3, x\rangle}{\langle x, x\rangle}(x) - \frac{\langle x^3, x^2 - 1/3\rangle}{\langle x^2 - 1/3, x^2 - 1/3\rangle}\left(x^2 - \frac{1}{3}\right)$$

Evaluate the denominators.

Solution (1 of 2)

$$P_3(x) = x^3 - \frac{\langle x^3, 1\rangle}{\langle 1, 1\rangle}(1) - \frac{\langle x^3, x\rangle}{\langle x, x\rangle}(x) - \frac{\langle x^3, x^2 - 1/3\rangle}{\langle x^2 - 1/3, x^2 - 1/3\rangle}\left(x^2 - \frac{1}{3}\right)$$

Evaluate the denominators.

$$\begin{aligned}
\langle 1, 1\rangle &= \int_{-1}^{1} (1)^2\,dx = 2 \\
\langle x, x\rangle &= \int_{-1}^{1} x^2\,dx = \frac{2}{3} \\
\left\langle x^2 - \frac{1}{3}, x^2 - \frac{1}{3}\right\rangle &= \int_{-1}^{1} \left(x^2 - \frac{1}{3}\right)^2\,dx = \frac{8}{45}
\end{aligned}$$

Solution (2 of 2)

$$P_3(x) = x^3 - \frac{\langle x^3, 1\rangle}{2}(1) - \frac{\langle x^3, x\rangle}{2/3}(x) - \frac{\langle x^3, x^2 - 1/3\rangle}{8/45}\left(x^2 - \frac{1}{3}\right)$$

Evaluate the numerators.

Solution (2 of 2)

$$P_3(x) = x^3 - \frac{\langle x^3, 1\rangle}{2}(1) - \frac{\langle x^3, x\rangle}{2/3}(x) - \frac{\langle x^3, x^2 - 1/3\rangle}{8/45}\left(x^2 - \frac{1}{3}\right)$$

Evaluate the numerators.

$$\begin{aligned}
\langle x^3, 1\rangle &= \int_{-1}^{1}(1)x^3\,dx = 0 \\
\langle x^3, x\rangle &= \int_{-1}^{1} x^3 x\,dx = \frac{2}{5} \\
\left\langle x^3, x^2 - \frac{1}{3}\right\rangle &= \int_{-1}^{1} x^3\left(x^2 - \frac{1}{3}\right)dx = 0
\end{aligned}$$

Solution (2 of 2)

$$P_3(x) = x^3 - \frac{\langle x^3, 1\rangle}{2}(1) - \frac{\langle x^3, x\rangle}{2/3}(x) - \frac{\langle x^3, x^2 - 1/3\rangle}{8/45}\left(x^2 - \frac{1}{3}\right)$$

Evaluate the numerators.

$$\begin{aligned}
\langle x^3, 1\rangle &= \int_{-1}^{1} (1)x^3\,dx = 0 \\
\langle x^3, x\rangle &= \int_{-1}^{1} x^3 x\,dx = \frac{2}{5} \\
\left\langle x^3, x^2 - \frac{1}{3}\right\rangle &= \int_{-1}^{1} x^3\left(x^2 - \frac{1}{3}\right)\,dx = 0
\end{aligned}$$

$$P_3 = x^3 - \frac{2/5}{2/3}x = x^3 - \frac{3}{5}x$$

Result

Theorem

Suppose $x_1, x_2, \ldots, x_n$ are the roots of the nth Legendre polynomial $P_n(x)$ and that for each $i = 1, 2, \ldots, n$, the numbers c_i are defined by

$$c_i = \int_{-1}^{1} \prod_{j=1,\, j\neq i}^{n} \frac{x - x_j}{x_i - x_j}\, dx.$$

If $P(x)$ is any polynomial of degree less than $2n$, then

$$\int_{-1}^{1} P(x)\, dx = \sum_{i=1}^{n} c_i P(x_i).$$

Proof (1 of 4)

Case 1: $P(x)$ has degree less than n.

We can re-write $P(x)$ in terms of the Lagrange basis polynomials of degree $(n-1)$ with nodes chosen to be the n distinct roots of the Legendre polynomial $P_n(x)$.

$$P(x) = \sum_{i=1}^{n} L_i(x)\, P(x_i)$$

Note: the error term is absent since $\frac{d^n}{dx^n} P(x) = 0$.

Now integrate both sides of the equation.

Proof (2 of 4)

$$
\begin{aligned}
P(x) &= \sum_{i=1}^{n} L_i(x)\, P(x_i) = \sum_{i=1}^{n} \left[\prod_{j=1\, j\neq i}^{n} \frac{x - x_j}{x_i - x_j} P(x_i) \right] \\
\int_{-1}^{1} P(x)\, dx &= \int_{-1}^{1} \sum_{i=1}^{n} \left[\prod_{j=1\, j\neq i}^{n} \frac{x - x_j}{x_i - x_j} P(x_i) \right] dx \\
&= \sum_{i=1}^{n} \left[\int_{-1}^{1} \prod_{j=1\, j\neq i}^{n} \frac{x - x_j}{x_i - x_j}\, dx \right] P(x_i) \\
&= \sum_{i=1}^{n} c_i P(x_i)
\end{aligned}
$$

Result is established for polynomials of degree less than n.

Proof (3 of 4)

Case 2: $P(x)$ has degree greater than $n-1$ and less than $2n$.

Using polynomial division we can write

$$\frac{P(x)}{P_n(x)} = Q(x) + \frac{R(x)}{P_n(x)}.$$

Proof (3 of 4)

Case 2: $P(x)$ has degree greater than $n - 1$ and less than $2n$.

Using polynomial division we can write

$$\frac{P(x)}{P_n(x)} = Q(x) + \frac{R(x)}{P_n(x)}.$$

Question: what are the degrees of $Q(x)$ and $R(x)$?

Proof (3 of 4)

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$$P(x) = Q(x)\,P_n(x) + R(x)$$

where the degrees of $Q(x)$ and $R(x)$ are each less than n.

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Note:

$$P(x_i) = Q(x_i)\,P_n(x_i) + R(x_i) = R(x_i)$$

for each root x_i of the Legendre polynomial $P_n(x)$.

Proof (4 of 4)

$$P(x) = Q(x)P_n(x) + R(x)$$

Integrate both sides over $[-1, 1]$.

$$\begin{aligned}
\int_{-1}^{1} P(x)\,dx &= \int_{-1}^{1} Q(x)P_n(x) + R(x)\,dx \\
&= \int_{-1}^{1} Q(x)P_n(x)\,dx + \int_{-1}^{1} R(x)\,dx \\
&= \int_{-1}^{1} R(x)\,dx \quad \text{(by corollary)} \\
&= \sum_{i=1}^{n} c_i R(x_i) \quad \text{(by first case)} \\
&= \sum_{i=1}^{n} c_i P(x_i)
\end{aligned}$$

Thus the result is established.

Roots of the Legendre Polynomials

n	$r_{n,i}$	$\tilde{r}_{n,i}$
2	$\pm\frac{\sqrt{3}}{3}$	± 0.5773502692
3	0	0
	$\pm\frac{\sqrt{15}}{5}$	± 0.7745966692
4	$\pm\frac{1}{35}\sqrt{525-70\sqrt{30}}$	± 0.3399810436
	$\pm\frac{1}{35}\sqrt{525+70\sqrt{30}}$	± 0.8611363116
5	0	0
	$\pm\frac{1}{21}\sqrt{245-14\sqrt{70}}$	± 0.5384693101
	$\pm\frac{1}{21}\sqrt{245+14\sqrt{70}}$	± 0.9061798459

Weights

For a fixed n the weights used in Gaussian Quadrature of order n are the values of

$$c_{n,i} = \int_{-1}^{1} \prod_{j=1,\, j\neq i}^{n} \frac{x - x_j}{x_i - x_j}\, dx = \int_{-1}^{1} L_{n,i}(x)\, dx,$$

where $L_{n,i}(x)$ is the ith Lagrange Basis Polynomial of degree $n-1$.

Nodes and Coefficients

n	$c_{n,i}$	$\tilde{c}_{n,i}$
2	1	1
3	$\frac{8}{9}$	0.8888888889
	$\frac{5}{9}$	0.5555555556
4	$\frac{1}{36}(18+\sqrt{30})$	0.6521451549
	$\frac{1}{36}(18-\sqrt{30})$	0.3478548451
5	$\frac{128}{225}$	0.5688888889
	$\frac{1}{900}(322+13\sqrt{70})$	0.4786286705
	$\frac{1}{900}(322-13\sqrt{70})$	0.2369268850

Summary

n	$\tilde{r}_{n,i}$	$\tilde{c}_{n,i}$
2	± 0.5773502692	1
3	0	0.8888888889
	± 0.7745966692	0.5555555556
4	± 0.3399810436	0.6521451549
	± 0.8611363116	0.3478548451
5	0	0.5688888889
	± 0.5384693101	0.4786286705
	± 0.9061798459	0.2369268850

Example

Approximate the following definite integral using Gaussian quadrature with $n = 5$.

$$\int_{-1}^{1} e^x \sin \pi x \, dx = \frac{\pi(e^2 - 1)}{e(\pi^2 + 1)} \approx 0.679326$$

Solution

Let $f(x) = e^x \sin \pi x$, then

$$\begin{aligned}\int_{-1}^{1} e^x \sin \pi x \, dx &\approx 0.23693 f(-0.90618) + 0.47863 f(-0.53847) \\ &\quad + 0.56889 f(0) + 0.47863 f(0.53847) \\ &\quad + 0.23693 f(0.90618) \\ &= 0.679307\end{aligned}$$

Absolute error:

$$|0.679307 - 0.678326| = 0.0000194196.$$

When $[a, b] \neq [-1, 1]$ (1 of 2)

The previous theorem is valid only for integrals of the form

$$\int_{-1}^{1} f(x)\, dx.$$

For the case of

$$\int_{a}^{b} f(x)\, dx$$

we integrate by substitution using $x = mt + n$ with constants m and n chosen so that $[-1, 1]$ maps to $[a, b]$.

$$\begin{aligned} a &= -m + n \\ b &= m + n \end{aligned}$$

Solve for m and n.

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$$\begin{aligned} a &= -m + n \\ b &= m + n \end{aligned}$$

Solve for m and n.

$$\begin{aligned} m &= \frac{b - a}{2} \\ n &= \frac{a + b}{2} \end{aligned}$$

When $[a, b] \neq [-1, 1]$ (2 of 2)

$$\begin{aligned} x &= \frac{(b-a)}{2}t + \frac{(a+b)}{2} \\ dx &= \frac{(b-a)}{2}\,dt. \end{aligned}$$

Under this change of variables

$$\int_a^b f(x)\,dx = \int_{-1}^1 f\left(\frac{(b-a)}{2}t + \frac{(a+b)}{2}\right)\frac{(b-a)}{2}\,dt.$$

Example One

Using Gaussian Quadrature with $n = 2$,

$$\int_1^{1.5} x^2 \ln x \, dx = \int_{-1}^{1} \left(\frac{t}{4} + \frac{5}{4}\right)^2 \ln\left(\frac{t}{4} + \frac{5}{4}\right) \frac{1}{4} \, dt$$

Let $f(t) = \frac{1}{4}\left(\frac{t}{4} + \frac{5}{4}\right)^2 \ln\left(\frac{t}{4} + \frac{5}{4}\right)$, then

$$\int_1^{1.5} x^2 \ln x \, dx$$

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Let $f(t) = \frac{1}{4}\left(\frac{t}{4} + \frac{5}{4}\right)^2 \ln\left(\frac{t}{4} + \frac{5}{4}\right)$, then

$$\begin{aligned}
\int_1^{1.5} x^2 \ln x \, dx &\approx f\left(-\frac{\sqrt{3}}{3}\right) + f\left(\frac{\sqrt{3}}{3}\right) \\
&\approx 0.0306981 + 0.161571 \\
&\approx 0.192269 \\
\text{Absolute error} &\approx 9.35 \times 10^{-6}
\end{aligned}$$

Example Two

Using Gaussian Quadrature with $n = 3$,

$$\int_1^{1.6} \frac{2x}{x^2 - 4}\, dx = \int_{-1}^{1} \frac{2(0.3t + 1.3)}{(0.3t + 1.3)^2 - 4} \cdot 0.3\, dt$$

Let $f(t) = \dfrac{0.6(0.3t + 1.3)}{(0.3t + 1.3)^2 - 4}$ then

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Let $f(t) = \dfrac{0.6(0.3t+1.3)}{(0.3t+1.3)^2-4}$ then

$$\begin{aligned}
\int_1^{1.6} \frac{2x}{x^2-4}\,dx &\approx \frac{5}{9}f(-\sqrt{15}/5) + \frac{8}{9}f(0) + \frac{5}{9}f(\sqrt{15}/5) \\
&\approx -0.124423 - 0.300144 - 0.309231 \\
&\approx -0.733799 \\
\text{Absolute error} &\approx 1.70 \times 10^{-4}
\end{aligned}$$

Homework

- Read Section 4.7.
- Exercises: 1ad, 3ad, 11