Gaussian Quadrature
MATH 375

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Consider the definite integral:

$$\int_{a}^{b} f(x) \, dx$$

The Newton-Cotes formulas discussed so far have used equally spaced nodes in the interval \([a, b]\) of the form \(x_j = a + jh\) for \(j \in \mathbb{Z}\) where

$$h = \frac{b - a}{n}$$

for some \(n \in \mathbb{N}\).

The approximation method known as **Gaussian Quadrature** makes an adaptive choice of nodes that minimizes the error in the approximation.
Motivation (1 of 2)

Note that the Trapezoidal rule uses the nodes at $x = a$ and $x = b$ to approximate the integrand by a linear function. For the function below, the Trapezoidal rule would approximate

\[
\int_{a}^{b} f(x) \, dx \approx 0.
\]
If we chose the nodes for the linear interpolation more appropriately we could get a better approximation.
Basic Idea

Choose $x_i$ and $c_i$ for $i = 1, 2, \ldots, n$ such that

$$\int_a^b f(x) \, dx \approx \sum_{i=1}^{n} c_i f(x_i).$$

**Note:** The constants $c_i$ are arbitrary, but $x_i \in [a, b]$ for $i = 1, 2, \ldots, n$. There are $2n$ values to be selected.
Objective

- The efficacy of this approach will be judged by the considering the size of the class of polynomials for which this approximation formula gives exact results.
- With $2n$ values to be determined, we should expect the upper limit of the precision of this method to be $2n - 1$.
- In other words, this method should be exact for polynomials of degree $2n - 1$ or less.
Example: \( n = 2 \) (1 of 3)

- Let \([a, b] = [-1, 1]\) and \( n = 2 \).
- We want to choose \( x_1, x_2, c_1, \) and \( c_2 \) so that
  \[
  \int_{-1}^{1} f(x) \, dx \approx c_1 f(x_1) + c_2 f(x_2).
  \]
- The approximation should be exact for any polynomial of degree 3 or less.
Example: $n = 2$ (2 of 3)

If we let

$$P(x) = a_0 + a_1 x + a_2 x^2 + a_3 x^3$$

then

$$\int_{-1}^{1} P(x) \, dx = a_0 \int_{-1}^{1} 1 \, dx + a_1 \int_{-1}^{1} x \, dx + a_2 \int_{-1}^{1} x^2 \, dx + a_3 \int_{-1}^{1} x^3 \, dx$$

Each of the definite integrals on the right-hand side has an integrand of degree 3 or less. Gaussian Quadrature should be exact for each of these.
Example: \( n = 2 \) (3 of 3)

Applying Gaussian Quadrature to each remaining integral yields:

\[
\begin{align*}
\int_{-1}^{1} 1 \, dx &= 2 = c_1 + c_2 \\
\int_{-1}^{1} x \, dx &= 0 = c_1 x_1 + c_2 x_2 \\
\int_{-1}^{1} x^2 \, dx &= \frac{2}{3} = c_1 x_1^2 + c_2 x_2^2 \\
\int_{-1}^{1} x^3 \, dx &= 0 = c_1 x_1^3 + c_2 x_2^3
\end{align*}
\]

**Remark:** if each one of these integrals can be made exact, then the overall method will have a precision of 3.

We must solve this system of 4 nonlinear equations in 4 unknowns.
Solution to Nonlinear System

Solving the nonlinear system gives us

\[ c_1 = 1, \quad c_2 = 1, \quad x_1 = -\frac{\sqrt{3}}{3}, \quad x_2 = \frac{\sqrt{3}}{3}. \]

Therefore

\[
\int_{-1}^{1} f(x) \, dx \approx f \left( -\frac{\sqrt{3}}{3} \right) + f \left( \frac{\sqrt{3}}{3} \right).
\]
Test Case

\[ \int_{-1}^{1} e^x \sin \pi x \, dx = \frac{\pi(e^2 - 1)}{e(\pi^2 + 1)} \approx 0.679326 \text{ (exact value)} \]

\[ \approx 1.18409 \text{ (Gaussian Quadrature)} \]

\[ \approx 0 \text{ (Trapezoidal rule)} \]

\[ \approx 0 \text{ (Simpson’s rule)} \]
We must introduce the **Legendre Polynomials**, \( \{ P_0(x), P_1(x), \ldots \} \), which have the properties:

- \( \text{deg } P_n = n \)
- If \( P(x) \) is a polynomial of degree less than \( n \), then

\[
\int_{-1}^{1} P(x)P_n(x) \, dx = 0.
\]
Legendre’s Differential Equation

The Legendre equation is

\[(1 - x^2)y'' - 2xy' + \alpha(\alpha + 1)y = 0\]

where \(\alpha\) is a constant.

- We would like to find two linearly independent solutions to this ODE.
- In general the solutions are power series in \(x\) which converge for \(-1 < x < 1\).
Solving Legendre’s Equation (1 of 6)

Assume \( y(x) = \sum_{n=0}^{\infty} a_n x^n \) and substitute this into Legendre’s equation.

\[
0 = (1 - x^2)y'' - 2xy' + \alpha(\alpha + 1)y
\]

\[
= (1 - x^2) \sum_{n=2}^{\infty} n(n-1)a_n x^{n-2} - 2x \sum_{n=1}^{\infty} na_n x^{n-1}
\]

\[
+ \alpha(\alpha + 1) \sum_{n=0}^{\infty} a_n x^n
\]

\[
= \sum_{n=2}^{\infty} n(n-1)a_n x^{n-2} - \sum_{n=2}^{\infty} n(n-1)a_n x^n - \sum_{n=1}^{\infty} 2na_n x^n
\]

\[
+ \alpha(\alpha + 1) \sum_{n=0}^{\infty} a_n x^n
\]
Solving Legendre’s Equation (2 of 6)

\[ 0 = (1 - x^2)y'' - 2xy' + \alpha(\alpha + 1)y \]

\[ = \sum_{n=0}^{\infty} (n + 2)(n + 1)a_{n+2}x^n - \sum_{n=0}^{\infty} n(n - 1)a_n x^n - \sum_{n=0}^{\infty} 2na_n x^n \]

\[ + \alpha(\alpha + 1) \sum_{n=0}^{\infty} a_n x^n \]

\[ = \sum_{n=0}^{\infty} [(n + 2)(n + 1)a_{n+2} - (n(n + 1) - \alpha(\alpha + 1))a_n] x^n \]
Solving Legendre’s Equation (3 of 6)

\[ 0 = \sum_{n=0}^{\infty} \left[ (n + 2)(n + 1)a_{n+2} - (n(n + 1) - \alpha(\alpha + 1))a_n \right] x^n \]

If we equate coefficients of powers of \(x\) on both sides of the equation we see that

\[ 0 = (n + 2)(n + 1)a_{n+2} - (n(n + 1) - \alpha(\alpha + 1))a_n \]

\[ a_{n+2} = \frac{(n(n + 1) - \alpha(\alpha + 1))a_n}{(n + 2)(n + 1)}, \]

for \( n = 0, 1, \ldots \).
Solving Legendre’s Equation (3 of 6)

\[ 0 = \sum_{n=0}^{\infty} [(n + 2)(n + 1)a_{n+2} \ - \ (n(n + 1) - \alpha(\alpha + 1))a_n] \ x^n \]

If we equate coefficients of powers of \( x \) on both sides of the equation we see that

\[ 0 \ = \ (n + 2)(n + 1)a_{n+2} \ - \ (n(n + 1) - \alpha(\alpha + 1))a_n \]

\[ a_{n+2} \ = \ \frac{(n(n + 1) - \alpha(\alpha + 1))a_n}{(n + 2)(n + 1)} , \]

for \( n = 0, 1, \ldots \)

Remarks:

▶ The last equation is called a recurrence relation.

▶ The recurrence relation generates the \((n + 2)\)nd coefficient of the power series from the \(n\)th coefficient of the power series.
Solving Legendre’s Equation (4 of 6)

\[ a_{n+2} = \frac{(n(n+1) - \alpha(\alpha + 1))a_n}{(n+2)(n+1)} \]

Questions:
1. Suppose \( a_0 = 0 \), what is \( a_{2n} \) for \( n \in \mathbb{N} \)?
Solving Legendre’s Equation (4 of 6)

\[ a_{n+2} = \frac{(n(n + 1) - \alpha(\alpha + 1))a_n}{(n + 2)(n + 1)} \]

Questions:

1. Suppose \( a_0 = 0 \), what is \( a_{2n} \) for \( n \in \mathbb{N} \)?

2. Suppose \( a_1 = 0 \), what is \( a_{2n+1} \) for \( n \in \mathbb{N} \)?
Solving Legendre’s Equation (4 of 6)

\[ a_{n+2} = \frac{(n(n + 1) - \alpha(\alpha + 1))a_n}{(n + 2)(n + 1)} \]

Questions:
1. Suppose \( a_0 = 0 \), what is \( a_{2n} \) for \( n \in \mathbb{N} \)?
2. Suppose \( a_1 = 0 \), what is \( a_{2n+1} \) for \( n \in \mathbb{N} \)?
3. If \( a_0 = 0 \), \( a_1 = 1 \) and \( \alpha = 2m + 1 \) for some fixed \( m \in \mathbb{N} \), what type of function is the solution to Legendre’s equation?
Solving Legendre’s Equation (4 of 6)

\[ a_{n+2} = \frac{(n(n + 1) - \alpha(\alpha + 1))a_n}{(n + 2)(n + 1)} \]

Questions:

1. Suppose \( a_0 = 0 \), what is \( a_{2n} \) for \( n \in \mathbb{N} \)?

2. Suppose \( a_1 = 0 \), what is \( a_{2n+1} \) for \( n \in \mathbb{N} \)?

3. If \( a_0 = 0 \), \( a_1 = 1 \) and \( \alpha = 2m + 1 \) for some fixed \( m \in \mathbb{N} \), what type of function is the solution to Legendre’s equation?

4. If \( a_0 = 1 \), \( a_1 = 0 \) and \( \alpha = 2m \) for some fixed \( m \in \mathbb{N} \), what type of function is the solution to Legendre’s equation?
Solving Legendre’s Equation (5 of 6)

\[ a_{n+2} = \frac{(n(n + 1) - \alpha(\alpha + 1))a_n}{(n + 2)(n + 1)} \]

for \( n = 0, 1, \ldots \).

Suppose \( a_0 = 1 \) and \( a_1 = 0 \), then

- if \( \alpha = 0 \) then \( y_0(x) = 1 \) a polynomial of degree 0,
- if \( \alpha = 2 \) then \( y_2(x) = \)
- if \( \alpha = 4 \) then \( y_4(x) = \)
Solving Legendre’s Equation (5 of 6)

\[ a_{n+2} = \frac{(n(n + 1) - \alpha(\alpha + 1))a_n}{(n + 2)(n + 1)} \]

for \( n = 0, 1, \ldots \).

Suppose \( a_0 = 1 \) and \( a_1 = 0 \), then

- if \( \alpha = 0 \) then \( y_0(x) = 1 \) a polynomial of degree 0,
- if \( \alpha = 2 \) then \( y_2(x) = 1 - 3x^2 \) a polynomial of degree 2 containing only even powers of \( x \),
- if \( \alpha = 4 \) then \( y_4(x) = \)

\[ a_{n+2} = \frac{(n(n + 1) - \alpha(\alpha + 1))a_n}{(n + 2)(n + 1)} \]
Solving Legendre’s Equation (5 of 6)

\[ a_{n+2} = \frac{(n(n+1) - \alpha(\alpha + 1))a_n}{(n+2)(n+1)} \]

for \( n = 0, 1, \ldots \).

Suppose \( a_0 = 1 \) and \( a_1 = 0 \), then

- if \( \alpha = 0 \) then \( y_0(x) = 1 \) a polynomial of degree 0,
- if \( \alpha = 2 \) then \( y_2(x) = 1 - 3x^2 \) a polynomial of degree 2 containing only even powers of \( x \),
- if \( \alpha = 4 \) then \( y_4(x) = 1 - 10x^2 + \frac{35}{3}x^4 \) a polynomial of degree 4 containing only even powers of \( x \).
Solving Legendre’s Equation (6 of 6)

\[ a_{n+2} = \frac{(n(n + 1) - \alpha(\alpha + 1))a_n}{(n + 2)(n + 1)} \]

for \( n = 0, 1, \ldots \).

Suppose \( a_0 = 0 \) and \( a_1 = 1 \), then

- if \( \alpha = 1 \) then \( y_1(x) = x \) a polynomial of degree 1,
- if \( \alpha = 3 \) then \( y_3(x) = \)
- if \( \alpha = 5 \) then \( y_5(x) = \)
Solving Legendre’s Equation (6 of 6)

\[ a_{n+2} = \frac{(n(n+1) - \alpha(\alpha + 1))a_n}{(n+2)(n+1)} \]

for \( n = 0, 1, \ldots \).

Suppose \( a_0 = 0 \) and \( a_1 = 1 \), then

- if \( \alpha = 1 \) then \( y_1(x) = x \) a polynomial of degree 1,
- if \( \alpha = 3 \) then \( y_3(x) = x - \frac{5}{3}x^3 \) a polynomial of degree 3 containing only odd powers of \( x \),
- if \( \alpha = 5 \) then \( y_5(x) = \)
Solving Legendre’s Equation (6 of 6)

\[ a_{n+2} = \frac{(n(n+1) - \alpha(\alpha + 1))a_n}{(n+2)(n+1)} \]

for \( n = 0, 1, \ldots \).

Suppose \( a_0 = 0 \) and \( a_1 = 1 \), then

- if \( \alpha = 1 \) then \( y_1(x) = x \) a polynomial of degree 1,
- if \( \alpha = 3 \) then \( y_3(x) = x - \frac{5}{3}x^3 \) a polynomial of degree 3 containing only odd powers of \( x \),
- if \( \alpha = 5 \) then \( y_5(x) = x - \frac{14}{3}x^3 + \frac{21}{5}x^5 \) a polynomial of degree 5 containing only odd powers of \( x \).
The Legendre Polynomials

Definition
The **Legendre polynomial** $P_n(x)$ is the monic polynomial solution of the Legendre equation with $\alpha = n$.

\[
\begin{align*}
P_0(x) &= 1 \\
P_1(x) &= x \\
P_2(x) &= x^2 - \frac{1}{3} \\
P_3(x) &= x^3 - \frac{3}{5}x \\
P_4(x) &= x^4 - \frac{6}{7}x^2 + \frac{3}{35} \\
P_5(x) &= x^5 - \frac{10}{9}x^3 + \frac{5}{21}x
\end{align*}
\]
Orthogonality of Legendre Polynomials (1 of 7)

Recall Legendre’s differential equation:

\[ 0 = (1 - x^2)y'' - 2xy' + \alpha(\alpha + 1)y \]
\[ = (1 - x^2)(y')' + (1 - x^2)'y' + \alpha(\alpha + 1)y \]
\[ -\alpha(\alpha + 1)y = [(1 - x^2)y']' \quad \text{(product rule)}. \]
Recall Legendre’s differential equation:

\[
0 = (1 - x^2)y''' - 2xy' + \alpha(\alpha + 1)y \\
= (1 - x^2)(y')' + (1 - x^2)'y' + \alpha(\alpha + 1)y \\
-\alpha(\alpha + 1)y = [(1 - x^2)y']' \quad \text{(product rule)}.
\]

Thus when \( y(x) = P_n(x) \) we have

\[
[(1 - x^2)P_n'(x)]' = -n(n + 1)P_n(x).
\]

Suppose we multiply this equation by \( P_m(x) \) where \( m \neq n \), then

\[
[(1 - x^2)P_n'(x)]' P_m(x) = -n(n + 1)P_n(x)P_m(x) \\
\int_{-1}^{1} [(1 - x^2)P_n'(x)]' P_m(x) \, dx = -n(n + 1) \int_{-1}^{1} P_n(x)P_m(x) \, dx
\]
Orthogonality of Legendre Polynomials (2 of 7)

\[ \int_{-1}^{1} \left[ (1 - x^2)P'_n(x) \right]'P_m(x) \, dx = -n(n + 1) \int_{-1}^{1} P_n(x)P_m(x) \, dx \]

We can apply integration by parts to the integral on the left-hand side of this equation.

\[
\begin{align*}
  u & = P_m(x) & v & = (1 - x^2)P'_n(x) \\
  du & = P'_m(x) \, dx & dv & = \left[(1 - x^2)P'_n(x)\right]' \, dx
\end{align*}
\]
\[
\int_{-1}^{1} [(1 - x^2) P'_n(x)]' P_m(x) \, dx \\
= \left[ P_m(x) [(1 - x^2) P'_n(x)] \right]_{x=-1}^{x=1} - \int_{-1}^{1} (1 - x^2) P'_n(x) P'_m(x) \, dx \\
= - \int_{-1}^{1} (1 - x^2) P'_n(x) P'_m(x) \, dx.
\]
Similarly when $y(x) = P_m(x)$ we have

$$[(1 - x^2)P'_m(x)]' = -m(m+1)P_m(x).$$

Suppose we multiply this equation by $P_n(x)$, then

$$[(1 - x^2)P'_m(x)]'P_n(x) = -m(m+1)P_m(x)P_n(x)$$

$$\int_{-1}^{1} [(1 - x^2)P'_m(x)]'P_n(x) \, dx = -m(m+1) \int_{-1}^{1} P_m(x)P_n(x) \, dx$$

We can apply integration by parts to the integral on the left-hand side of this equation.

$$u = P_n(x) \quad v = (1 - x^2)P'_m(x)$$
$$du = P'_n(x) \, dx \quad dv = [(1 - x^2)P'_m(x)]' \, dx$$
Thus

\[
\int_{-1}^{1} \left[ (1 - x^2)P'_m(x) \right]' P_n(x) \, dx = P_n(x) \left[ (1 - x^2)P'_m(x) \right]_{x=1} - \int_{-1}^{1} (1 - x^2)P'_m(x)P'_n(x) \, dx
\]

\[
= \int_{-1}^{1} (1 - x^2)P'_m(x)P'_n(x) \, dx.
\]
Combining the equations we see that

\[
\int_{-1}^{1} (1 - x^2) P'_m(x) P'_n(x) \, dx = \int_{-1}^{1} (1 - x^2) P'_n(x) P'_m(x) \, dx
\]

\[
m(m+1) \int_{-1}^{1} P_m(x) P_n(x) \, dx = n(n+1) \int_{-1}^{1} P_n(x) P_m(x) \, dx
\]

Re-arranging terms in the last equation yields

\[
[m(m+1) - n(n+1)] \int_{-1}^{1} P_m(x) P_n(x) \, dx = 0
\]
Orthogonality of Legendre Polynomials (7 of 7)

Since

$$[m(m + 1) - n(n + 1)] \int_{-1}^{1} P_m(x)P_n(x) \, dx = 0$$

either

$$\int_{-1}^{1} P_m(x)P_n(x) \, dx = 0$$

which is the orthogonality property, or

$$0 = [m(m + 1) - n(n + 1)] = (m - n)(m + n + 1)$$

Since $m, n \in \mathbb{N}$ and we have assumed $m \neq n$ then this equation is not satisfied.
As a corollary to the orthogonality property of the Legendre polynomials we have the following result.

**Corollary**

*If* $P(x)$ *is a polynomial of degree less than* $n$, *then*

$$\int_{-1}^{1} P(x) \, P_n(x) \, dx = 0.$$
Properties of Legendre Polynomials (1 of 2)

- The set of Legendre Polynomials forms an infinite-dimensional inner product space.
- First five Legendre polynomials:

\[
\begin{align*}
P_0(x) &= 1 \\
P_1(x) &= x \\
P_2(x) &= x^2 - \frac{1}{3} \\
P_3(x) &= x^3 - \frac{3}{5}x \\
P_4(x) &= x^4 - \frac{6}{7}x^2 + \frac{3}{35}
\end{align*}
\]

- Higher degree Legendre polynomials can be generated using the Gram-Schmidt Orthogonalization Process.
Properties of Legendre Polynomials (2 of 2)

- Roots of the polynomials lie in the interval $(-1, 1)$.
- Roots are symmetric about the origin.
- Roots are distinct.
- For $P_n(x)$ the roots $\{x_1, x_2, \ldots, x_n\}$ are the nodes needed by Gaussian Quadrature. That is, the quadrature formula

$$\int_{-1}^{1} f(x) \, dx = \sum_{k=1}^{n} c_k f(x_k)$$

will be exact for any polynomial for degree $2n - 1$ or less.
Example

Use the Gram-Schmidt Orthogonalization process with the inner product

\[ \langle P(x), Q(x) \rangle = \int_{-1}^{1} P(x)Q(x) \, dx \]

and the first three Legendre polynomials:

\[
\begin{align*}
P_0(x) &= 1 \\
P_1(x) &= x \\
P_2(x) &= x^2 - \frac{1}{3}
\end{align*}
\]

to find \( P_3(x) \).
Solution (1 of 2)

\[ P_3(x) = x^3 - \frac{\langle x^3, 1 \rangle}{\langle 1, 1 \rangle}(1) - \frac{\langle x^3, x \rangle}{\langle x, x \rangle}(x) - \frac{\langle x^3, x^2 - 1/3 \rangle}{\langle x^2 - 1/3, x^2 - 1/3 \rangle} \left( x^2 - \frac{1}{3} \right) \]

Evaluate the denominators.
Solution (1 of 2)

\[ P_3(x) = x^3 - \frac{\langle x^3, 1 \rangle}{\langle 1, 1 \rangle}(1) - \frac{\langle x^3, x \rangle}{\langle x, x \rangle}(x) - \frac{\langle x^3, x^2 - 1/3 \rangle}{\langle x^2 - 1/3, x^2 - 1/3 \rangle} \left( x^2 - \frac{1}{3} \right) \]

Evaluate the denominators.

\[
\langle 1, 1 \rangle = \int_{-1}^{1} (1)^2 \, dx = 2
\]

\[
\langle x, x \rangle = \int_{-1}^{1} x^2 \, dx = \frac{2}{3}
\]

\[
\langle x^2 - \frac{1}{3}, x^2 - \frac{1}{3} \rangle = \int_{-1}^{1} \left( x^2 - \frac{1}{3} \right)^2 \, dx = \frac{8}{45}
\]
Solution (2 of 2)

\[ P_3(x) = x^3 - \frac{\langle x^3, 1 \rangle}{2}(1) - \frac{\langle x^3, x \rangle}{2/3}(x) - \frac{\langle x^3, x^2 - 1/3 \rangle}{8/45} \left( x^2 - \frac{1}{3} \right) \]

Evaluate the numerators.
Solution (2 of 2)

\[ P_3(x) = x^3 - \frac{\langle x^3, 1 \rangle}{2}(1) - \frac{\langle x^3, x \rangle}{2/3}(x) - \frac{\langle x^3, x^2 - 1/3 \rangle}{8/45}
\left(x^2 - \frac{1}{3}\right) \]

Evaluate the numerators.

\[ \langle x^3, 1 \rangle = \int_{-1}^{1} (1)x^3 \, dx = 0 \]

\[ \langle x^3, x \rangle = \int_{-1}^{1} x^3 x \, dx = \frac{2}{5} \]

\[ \langle x^3, x^2 - \frac{1}{3} \rangle = \int_{-1}^{1} x^3 \left(x^2 - \frac{1}{3}\right) \, dx = 0 \]
Solution (2 of 2)

\[ P_3(x) = x^3 - \frac{\langle x^3, 1 \rangle}{2} (1) - \frac{\langle x^3, x \rangle}{2/3} (x) - \frac{\langle x^3, x^2 - 1/3 \rangle}{8/45} \left( x^2 - \frac{1}{3} \right) \]

Evaluate the numerators.

\[ \langle x^3, 1 \rangle = \int_{-1}^{1} (1) x^3 \, dx = 0 \]

\[ \langle x^3, x \rangle = \int_{-1}^{1} x^3 x \, dx = \frac{2}{5} \]

\[ \langle x^3, x^2 - \frac{1}{3} \rangle = \int_{-1}^{1} x^3 \left( x^2 - \frac{1}{3} \right) \, dx = 0 \]

\[ P_3 = x^3 - \frac{2/5}{2/3} x = x^3 - \frac{3}{5} x \]
Theorem
Suppose $x_1, x_2, \ldots, x_n$ are the roots of the $n$th Legendre polynomial $P_n(x)$ and that for each $i = 1, 2, \ldots, n$, the numbers $c_i$ are defined by

$$c_i = \int_{-1}^{1} \prod_{j=1, j \neq i}^{n} \frac{x - x_j}{x_i - x_j} \, dx.$$ 

If $P(x)$ is any polynomial of degree less than $2n$, then

$$\int_{-1}^{1} P(x) \, dx = \sum_{i=1}^{n} c_i P(x_i).$$
Case 1: $P(x)$ has degree less than $n$.

We can re-write $P(x)$ in terms of the Lagrange basis polynomials of degree $(n - 1)$ with nodes chosen to be the $n$ distinct roots of the Legendre polynomial $P_n(x)$.

$$P(x) = \sum_{i=1}^{n} L_i(x) P(x_i)$$

Note: the error term is absent since $\frac{d^n}{dx^n} P(x) = 0$.

Now integrate both sides of the equation.
Proof (2 of 4)

\[ P(x) = \sum_{i=1}^{n} L_i(x) P(x_i) = \sum_{i=1}^{n} \left[ \prod_{j=1 \atop j \neq i}^{n} \frac{x - x_j}{x_i - x_j} P(x_i) \right] \]

\[ \int_{-1}^{1} P(x) \, dx = \int_{-1}^{1} \sum_{i=1}^{n} \left[ \prod_{j=1 \atop j \neq i}^{n} \frac{x - x_j}{x_i - x_j} P(x_i) \right] \, dx \]

\[ = \sum_{i=1}^{n} \left[ \int_{-1}^{1} \prod_{j=1 \atop j \neq i}^{n} \frac{x - x_j}{x_i - x_j} \, dx \right] P(x_i) \]

\[ = \sum_{i=1}^{n} c_i P(x_i) \]

Result is established for polynomials of degree less than \( n \).
**Case 2:** $P(x)$ has degree greater than $n - 1$ and less than $2n$.

Using polynomial division we can write

$$\frac{P(x)}{P_n(x)} = Q(x) + \frac{R(x)}{P_n(x)}.$$
Case 2: $P(x)$ has degree greater than $n - 1$ and less than $2n$.

Using polynomial division we can write

$$\frac{P(x)}{P_n(x)} = Q(x) + \frac{R(x)}{P_n(x)}.$$ 

Question: what are the degrees of $Q(x)$ and $R(x)$?
Proof (3 of 4)

Case 2: \( P(x) \) has degree greater than \( n - 1 \) and less than \( 2n \).

Using polynomial division we can write

\[
\frac{P(x)}{P_n(x)} = Q(x) + \frac{R(x)}{P_n(x)}.
\]

**Question:** what are the degrees of \( Q(x) \) and \( R(x) \)?

\[
P(x) = Q(x) P_n(x) + R(x)
\]

where the degrees of \( Q(x) \) and \( R(x) \) are each less than \( n \).
Case 2: $P(x)$ has degree greater than $n - 1$ and less than $2n$.

Using polynomial division we can write

$$\frac{P(x)}{P_n(x)} = Q(x) + \frac{R(x)}{P_n(x)}.$$

**Question:** what are the degrees of $Q(x)$ and $R(x)$?

$$P(x) = Q(x) P_n(x) + R(x)$$

where the degrees of $Q(x)$ and $R(x)$ are each less than $n$.

**Note:**

$$P(x_i) = Q(x_i) P_n(x_i) + R(x_i) = R(x_i)$$

for each root $x_i$ of the Legendre polynomial $P_n(x)$. 
Proof (4 of 4)

\[ P(x) = Q(x)P_n(x) + R(x) \]

Integrate both sides over \([-1, 1]\).

\[
\int_{-1}^{1} P(x) \, dx = \int_{-1}^{1} Q(x)P_n(x) + R(x) \, dx \\
= \int_{-1}^{1} Q(x)P_n(x) \, dx + \int_{-1}^{1} R(x) \, dx \\
= \int_{-1}^{1} R(x) \, dx \quad \text{(by corollary)} \\
= \sum_{i=1}^{n} c_i R(x_i) \quad \text{(by first case)} \\
= \sum_{i=1}^{n} c_i P(x_i)
\]

Thus the result is established.
### Roots of the Legendre Polynomials

<table>
<thead>
<tr>
<th>$n$</th>
<th>$r_{n,i}$</th>
<th>$\tilde{r}_{n,i}$</th>
</tr>
</thead>
<tbody>
<tr>
<td>2</td>
<td>$\pm \frac{\sqrt{3}}{3}$</td>
<td>$\pm 0.5773502692$</td>
</tr>
<tr>
<td>3</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td></td>
<td>$\pm \frac{\sqrt{15}}{5}$</td>
<td>$\pm 0.7745966692$</td>
</tr>
<tr>
<td>4</td>
<td>$\pm \frac{1}{35} \sqrt{525 - 70\sqrt{30}}$</td>
<td>$\pm 0.3399810436$</td>
</tr>
<tr>
<td></td>
<td>$\pm \frac{1}{35} \sqrt{525 + 70\sqrt{30}}$</td>
<td>$\pm 0.8611363116$</td>
</tr>
<tr>
<td>5</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td></td>
<td>$\pm \frac{1}{21} \sqrt{245 - 14\sqrt{70}}$</td>
<td>$\pm 0.5384693101$</td>
</tr>
<tr>
<td></td>
<td>$\pm \frac{1}{21} \sqrt{245 + 14\sqrt{70}}$</td>
<td>$\pm 0.9061798459$</td>
</tr>
</tbody>
</table>
Weights

For a fixed $n$ the weights used in Gaussian Quadrature of order $n$ are the values of

$$c_{n,i} = \int_{-1}^{1} \prod_{j=1, j\neq i}^{n} \frac{x - x_j}{x_i - x_j} \, dx = \int_{-1}^{1} L_{n,i}(x) \, dx,$$

where $L_{n,i}(x)$ is the $i$th Lagrange Basis Polynomial of degree $n - 1$. 
## Nodes and Coefficients

<table>
<thead>
<tr>
<th>$n$</th>
<th>$c_{n,i}$</th>
<th>$\tilde{c}_{n,i}$</th>
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<tbody>
<tr>
<td>2</td>
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<td>1</td>
</tr>
<tr>
<td>3</td>
<td>$\frac{8}{9}$</td>
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<tr>
<td></td>
<td>$\frac{5}{9}$</td>
<td>0.5555555556</td>
</tr>
<tr>
<td>4</td>
<td>$\frac{1}{36}(18 + \sqrt{30})$</td>
<td>0.6521451549</td>
</tr>
<tr>
<td></td>
<td>$\frac{1}{36}(18 - \sqrt{30})$</td>
<td>0.3478548451</td>
</tr>
<tr>
<td>5</td>
<td>$\frac{128}{225}$</td>
<td>0.5688888889</td>
</tr>
<tr>
<td></td>
<td>$\frac{1}{900}(322 + 13\sqrt{70})$</td>
<td>0.4786286705</td>
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<tr>
<td></td>
<td>$\frac{1}{900}(322 - 13\sqrt{70})$</td>
<td>0.2369268850</td>
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### Summary

<table>
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<tr>
<th>$n$</th>
<th>$\tilde{r}_{n,i}$</th>
<th>$\tilde{c}_{n,i}$</th>
</tr>
</thead>
<tbody>
<tr>
<td>2</td>
<td>$\pm 0.5773502692$</td>
<td>1</td>
</tr>
<tr>
<td>3</td>
<td>$0$</td>
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<td>4</td>
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<td>$0.6521451549$</td>
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<td>$\pm 0.8611363116$</td>
<td>$0.3478548451$</td>
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<td>5</td>
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<td>$0.5688888889$</td>
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<td>$\pm 0.5384693101$</td>
<td>$0.4786286705$</td>
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<tr>
<td></td>
<td>$\pm 0.9061798459$</td>
<td>$0.2369268850$</td>
</tr>
</tbody>
</table>
Example

Approximate the following definite integral using Gaussian quadrature with $n = 5$.

$$\int_{-1}^{1} e^x \sin \pi x \, dx = \frac{\pi(e^2 - 1)}{e(\pi^2 + 1)} \approx 0.679326$$
Let \( f(x) = e^x \sin \pi x \), then

\[
\int_{-1}^{1} e^x \sin \pi x \, dx \approx 0.23693f(-0.90618) + 0.47863f(-0.53847) \\
+ 0.56889f(0) + 0.47863f(0.53847) \\
+ 0.23693f(0.90618) \\
= 0.679307
\]

Absolute error:

\[
|0.679307 - 0.678326| = 0.0000194196.
\]
When \([a, b] \neq [-1, 1]\) (1 of 2)

The previous theorem is valid only for integrals of the form

\[
\int_{-1}^{1} f(x) \, dx.
\]

For the case of

\[
\int_{a}^{b} f(x) \, dx
\]

we integrate by substitution using \(x = mt + n\) with constants \(m\) and \(n\) chosen so that \([-1, 1]\) maps to \([a, b]\).

\[
\begin{align*}
a &= -m + n \\
b &= m + n
\end{align*}
\]

Solve for \(m\) and \(n\).
When $[a, b] \neq [-1, 1]$ (1 of 2)

The previous theorem is valid only for integrals of the form

$$\int_{-1}^{1} f(x) \, dx.$$ 

For the case of

$$\int_{a}^{b} f(x) \, dx$$

we integrate by substitution using $x = mt + n$ with constants $m$ and $n$ chosen so that $[-1, 1]$ maps to $[a, b]$.

$$a = -m + n$$
$$b = m + n$$

Solve for $m$ and $n$.

$$m = \frac{b - a}{2}$$
$$n = \frac{a + b}{2}$$
When \([a, b] \neq [-1, 1]\) (2 of 2)

\[
x = \frac{(b - a)}{2} t + \frac{(a + b)}{2}
\]

\[
dx = \frac{(b - a)}{2} \, dt.
\]

Under this change of variables

\[
\int_{a}^{b} f(x) \, dx = \int_{-1}^{1} f \left( \frac{(b - a)}{2} t + \frac{(a + b)}{2} \right) \frac{(b - a)}{2} \, dt.
\]
Example One

Using Gaussian Quadrature with $n = 2$,

$$
\int_{1}^{1.5} x^2 \ln x \, dx = \int_{-1}^{1} \left( \frac{t}{4} + \frac{5}{4} \right)^2 \ln \left( \frac{t}{4} + \frac{5}{4} \right) \frac{1}{4} \, dt
$$

Let $f(t) = \frac{1}{4} \left( \frac{t}{4} + \frac{5}{4} \right)^2 \ln \left( \frac{t}{4} + \frac{5}{4} \right)$, then

$$
\int_{1}^{1.5} x^2 \ln x \, dx
$$
Example One

Using Gaussian Quadrature with \( n = 2 \),

\[
\int_{1}^{1.5} x^2 \ln x \, dx = \int_{-1}^{1} \left( \frac{t}{4} + \frac{5}{4} \right)^2 \ln \left( \frac{t}{4} + \frac{5}{4} \right) \frac{1}{4} \, dt
\]

Let \( f(t) = \frac{1}{4} \left( \frac{t}{4} + \frac{5}{4} \right)^2 \ln \left( \frac{t}{4} + \frac{5}{4} \right) \), then

\[
\int_{1}^{1.5} x^2 \ln x \, dx \approx f \left( -\frac{\sqrt{3}}{3} \right) + f \left( \frac{\sqrt{3}}{3} \right)
\]

\[
\approx 0.0306981 + 0.161571
\]

\[
\approx 0.192269
\]

Absolute error \( \approx 9.35 \times 10^{-6} \)
Example Two

Using Gaussian Quadrature with \( n = 3 \),

\[
\int_{1}^{1.6} \frac{2x}{x^2 - 4} \, dx = \int_{-1}^{1} \frac{2(0.3t + 1.3)}{(0.3t + 1.3)^2 - 4} \cdot 0.3 \, dt
\]

Let \( f(t) = \frac{0.6(0.3t + 1.3)}{(0.3t + 1.3)^2 - 4} \) then

\[
\int_{1}^{1.6} \frac{2x}{x^2 - 4} \, dx
\]
Example Two

Using Gaussian Quadrature with $n = 3$,

\[
\int_{1}^{1.6} \frac{2x}{x^2 - 4} \, dx = \int_{-1}^{1} \frac{2(0.3t + 1.3)}{(0.3t + 1.3)^2 - 4} \cdot 0.3 \, dt
\]

Let $f(t) = \frac{0.6(0.3t + 1.3)}{(0.3t + 1.3)^2 - 4}$ then

\[
\int_{1}^{1.6} \frac{2x}{x^2 - 4} \, dx \approx \frac{5}{9} f(-\sqrt{15}/5) + \frac{8}{9} f(0) + \frac{5}{9} f(\sqrt{15}/5)
\]

\[
\approx -0.124423 - 0.300144 - 0.309231
\]

\[
\approx -0.733799
\]

Absolute error \approx 1.70 \times 10^{-4}
Homework

▶ Read Section 4.7.
▶ Exercises: 1ad, 3ad, 11