Hermite Interpolation

MATH 375 *Numerical Analysis*

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Objectives

- We have encountered the Taylor polynomial and Lagrange interpolating polynomial for approximating functions.
- In this lesson we will generalize both types of polynomials to develop a polynomial which agrees with a given function and its derivatives at a set of points.

Osculating Polynomials

Given $n+1$ numbers $\{x_0, x_1, \ldots, x_n\} \in [a, b]$ and $n+1$ nonnegative integers $\{m_0, m_1, \ldots, m_n\}$:

- let $m = \max\{m_0, m_1, \ldots, m_n\}$, and
- consider the set of functions $f \in \mathcal{C}^m[a, b]$.

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Definition

The **osculating polynomial** approximating f is the polynomial $P(x)$ of least degree such that

$$\frac{d^k P(x_i)}{dx^k} = \frac{d^k f(x_i)}{dx^k}$$

for each $i = 0, 1, \ldots, n$ and for $k = 0, 1, \ldots, m_i$.

Remarks

- If $n = 0$ then we have one node $\{x_0\}$ and $P(x)$ is the polynomial of least degree such that

$$\frac{d^k P(x_0)}{dx^k} = \frac{d^k f(x_0)}{dx^k} \text{ for } k = 0, 1, \ldots, m_0.$$

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- If $m_i = 0$ for $i = 0, 1, \ldots, n$ then $P(x)$ is the polynomial of least degree such that

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- Thus we see the osculating polynomial is a generalization of the Taylor and Lagrange interpolating polynomials.

Hermite Polynomials

Definition

The **Hermite polynomial** approximating f is the polynomial $H(x)$ of least degree such that

$$H(x_i) = f(x_i)$$
$$H'(x_i) = f'(x_i)$$

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Remarks:

- The Hermite polynomials $H(x)$ agree with $f(x)$ and the derivatives of the Hermite polynomials $H'(x)$ agree with $f'(x)$.
- The degree of the Hermite polynomial is $2n + 1$ since $2n + 2$ conditions must be met ($n + 1$ points and $n + 1$ derivatives).

Main Result

Theorem

If $f \in \mathcal{C}^1[a,b]$ and $x_0, x_1, \ldots, x_n \in [a,b]$ are distinct points, the unique polynomial of least degree agreeing with f and f' at $x_0, x_1, \ldots, x_n$ is the Hermite polynomial of degree at most $2n+1$ given by

$$H_{2n+1}(x) = \sum_{j=0}^{n} f(x_j) H_{n,j}(x) + \sum_{j=0}^{n} f'(x_j) \widehat{H}_{n,j}(x)$$

where

$$H_{n,j}(x) = \left[1 - 2(x - x_j) L'_{n,j}(x_j)\right] L^2_{n,j}(x)$$
$$\widehat{H}_{n,j}(x) = (x - x_j) L^2_{n,j}(x)$$

and $L_{n,j}(x)$ is jth Lagrange basis polynomial of degree n.

Proof (1 of 4)

The question to be answered is “does $H_{2n+1}(x_i) = f(x_i)$ for $i = 0, 1, \ldots, n$?”

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- Recall the property of the Lagrange basis function

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- Suppose $i \neq j$, then

$$H_{n,j}(x_i) = \left[1 - 2(x_i - x_j)L'_{n,j}(x_j)\right] L^2_{n,j}(x_i) = 0$$
$$\widehat{H}_{n,j}(x_i) = (x_i - x_j)L^2_{n,j}(x_i) = 0.$$

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- If $i = j$, then

$$H_{n,j}(x_j) = \left[1 - 2(x_j - x_j)L'_{n,j}(x_j)\right] L^2_{n,j}(x_j) = L^2_{n,j}(x_j) = 1$$
$$\widehat{H}_{n,j}(x_i) = (x_j - x_j)L^2_{n,j}(x_i) = 0.$$

Proof (2 of 4)

$$
\begin{aligned}
H_{2n+1}(x_i) &= \sum_{j=0}^{n} f(x_j) H_{n,j}(x_i) + \sum_{j=0}^{n} f'(x_j) \widehat{H}_{n,j}(x_i) \\
&= \sum_{j=0, j\neq i}^{n} f(x_j) \cdot 0 + f(x_i) \cdot 1 + \sum_{j=0}^{n} f'(x_j) 0 \\
&= f(x_i)
\end{aligned}
$$

Proof (3 of 4)

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- Note that

$$H'_{n,j}(x) = -2L'_{n,j}(x_j)L^2_{n,j}(x) + 2\left[1 - 2(x - x_j)L'_{n,j}(x_j)\right] L_{n,j}(x)L'_{n,j}(x)$$
$$\widehat{H}'_{n,j}(x) = L^2_{n,j}(x) + 2(x - x_j)L_{n,j}(x)L'_{n,j}(x).$$

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$$\widehat{H}'_{n,j}(x) = L^2_{n,j}(x) + 2(x - x_j)L_{n,j}(x)L'_{n,j}(x).$$

- If $i \neq j$ then $H'_{n,j}(x_i) = 0$ and $\widehat{H}'_{n,j}(x_i) = 0$.

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- If $i \neq j$ then $H'_{n,j}(x_i) = 0$ and $\widehat{H}'_{n,j}(x_i) = 0$.
- If $i = j$ then $H'_{n,j}(x_j) = -2L'_{n,j}(x_j)(1)^2 + 2(1)L'_{n,j}(x_j) = 0$ and $\widehat{H}'_{n,j}(x_j) = (1)^2 = 1$.

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Note: $H'_{n,j}(x_i) = 0$ for all $i = 0, 1, \ldots, n$.

Proof (4 of 4)

Consequently

$$\begin{aligned} H'_{2n+1}(x_i) &= \sum_{j=0}^{n} f(x_j) H'_{n,j}(x_i) + \sum_{j=0}^{n} f'(x_j) \widehat{H}'_{n,j}(x_i) \\ &= \sum_{j=0}^{n} f(x_j) \cdot 0 + \sum_{j=0, j \neq i}^{n} f'(x_j) \widehat{H}'_{n,j}(x_i) + f'(x_i) \widehat{H}'_{n,i}(x_i) \\ &= \sum_{j=0, j \neq i}^{n} f'(x_j) \cdot 0 + f'(x_i) \cdot 1 \\ &= f'(x_i). \end{aligned}$$

Uniqueness of Hermite Polynomial

- Suppose $P(x)$ is another polynomial of degree at most $2n+1$ for which $P(x_i) = f(x_i)$ and $P'(x_i) = f'(x_i)$ for $i = 0, 1, \ldots, n$.
- Define function $D(x) = H_{2n+1}(x) - P(x)$. The polynomial $D(x)$ has degree at most $2n+1$.
- Note that for $i = 0, 1, \ldots, n$,

  $$D(x_i) = H_{2n+1}(x_i) - P(x_i) = f(x_i) - f(x_i) = 0$$
  $$D'(x_i) = H'_{2n+1}(x_i) - P'(x_i) = f'(x_i) - f'(x_i) = 0$$

  and thus $D(x)$ has roots of multiplicity 2 at the distinct points x_0, $x_1, \ldots, x_n$.

  $$D(x) = (x - x_0)^2 (x - x_1)^2 \cdots (x - x_n)^2 Q(x)$$

- Unless $Q(x) = 0$ then $D(x)$ has degree $2n+2$ or higher which is a contradiction.

Error Formula

Under the assumptions of the previous theorem, if $f \in \mathcal{C}^{2n+2}[a, b]$ then

$$f(x) = H_{2n+1}(x) + \frac{(x - x_0)^2 \cdots (x - x_n)^2}{(2n + 2)!} f^{(2n+2)}(z(x))$$

for some $z(x) \in [a, b]$.

Proof (1 of 3)

$$f(x) = H_{2n+1}(x) + \frac{(x - x_0)^2 \cdots (x - x_n)^2}{(2n+2)!} f^{(2n+2)}(z(x))$$

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- If $x \neq x_i$ for all $i = 0, 1, \ldots, n$, then define

$$g(t) = f(t) - H_{2n+1}(t) - \frac{(t-x_0)^2 \cdots (t-x_n)^2}{(x-x_0)^2 \cdots (x-x_n)^2} [f(x) - H_{2n+1}(x)]$$

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- The last equation holds since $(x_i - x_i)^2$ appears in the numerator.

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- The last equation holds since $(x_i - x_i)^2$ appears in the numerator.
- We see that function $g(t)$ has $n+2$ distinct zeros in $[a, b]$. By Rolle's Theorem, $g'(t)$ has $n+1$ distinct zeros $\xi_0, \xi_1, \ldots, \xi_n$ interspersed between the numbers $x_0, x_1, \ldots, x_n$, and x.

Proof (2 of 3)

$$g'(t) = f'(t) - H'_{2n+1}(t) - \frac{2\left[f(x) - H_{2n+1}(x)\right]}{(x - x_0)^2 \cdots (x - x_n)^2} \sum_{k=0}^{n} (t - x_k) \prod_{j=0, j \neq k}^{n} (t - x_j)^2$$

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for $i = 0, 1, \ldots, n$.

Proof (2 of 3)

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for $i = 0, 1, \ldots, n$.

- Thus $g(t)$ has $2n+2$ distinct zeros in the interval $[a, b]$.
- Since $g'(t)$ is $2n+1$ times differentiable (because $f(x)$ is $2n+2$ times differentiable, then by the Generalized Rolle's Theorem, there exists $z \in [a, b]$ such that $g^{(2n+2)}(z) = 0$.

$$g(t) = f(t) - H_{2n+1}(t) - \frac{(t-x_0)^2 \cdots (t-x_n)^2}{(x-x_0)^2 \cdots (x-x_n)^2} [f(x) - H_{2n+1}(x)]$$

$$g^{(2n+2)}(t) = f^{(2n+2)}(t) - H_{2n+1}^{(2n+2)}(t) - \frac{[f(x) - H_{2n+1}(x)]}{(x-x_0)^2 \cdots (x-x_n)^2} \frac{d^{2n+2}}{dx^{2n+2}} \left[(t-x_0)^2 \cdots (t-x_n)^2\right]$$

$$= f^{(2n+2)}(t) - \frac{[f(x) - H_{2n+1}(x)]\,(2n+2)!}{(x-x_0)^2 \cdots (x-x_n)^2}$$

Proof (3 of 3)

$$g^{(2n+2)}(t) = f^{(2n+2)}(t) - \frac{[f(x) - H_{2n+1}(x)]\,(2n+2)!}{(x-x_0)^2 \cdots (x-x_n)^2}$$

$$0 = f^{(2n+2)}(z) - \frac{[f(x) - H_{2n+1}(x)]\,(2n+2)!}{(x-x_0)^2 \cdots (x-x_n)^2}$$

$$f(x) = H_{2n+1}(x) + \frac{(x-x_0)^2 \cdots (x-x_n)^2}{(2n+2)!} f^{(2n+2)}(z)$$

Example

Construct a Hermite interpolating polynomial for the following data.

x	$f(x)$	$f'(x)$
0.1	-0.29004996	-2.8019975
0.2	-0.56079734	-2.6159201
0.3	-0.81401972	-2.9734038

Solution (1 of 3)

Let $x_0 = 0.1$, $x_1 = 0.2$, and $x_2 = 0.3$ and make a list of the Lagrange basis polynomials.

$$L_{2,0}(x) = \frac{(x - x_1)(x - x_2)}{(x_0 - x_1)(x_0 - x_2)} = 50x^2 - 25x + 3$$

$$L'_{2,0}(x) = 100x - 25$$

$$L_{2,1}(x) = \frac{(x - x_0)(x - x_2)}{(x_1 - x_0)(x_1 - x_2)} = -100x^2 + 40x - 3$$

$$L'_{2,1}(x) = -200x + 40$$

$$L_{2,2}(x) = \frac{(x - x_0)(x - x_1)}{(x_2 - x_0)(x_2 - x_1)} = 50x^2 - 15x + 1$$

$$L'_{2,2}(x) = 100x - 15$$

Solution (2 of 3)

Second, list the Hermite polynomials $H_{2,j}(x)$ and $\widehat{H}_{2,j}(x)$.

$$\begin{aligned}
H_{2,0}(x) &= \left[1 - 2(x - x_0)L'_{2,0}(x_0)\right] L^2_{2,0}(x) \\
&= 75000x^5 - 80000x^4 + 32750x^3 - 6350x^2 + 570x - 18 \\
\widehat{H}_{2,0}(x) &= (x - x_0)L^2_{2,0}(x) \\
&= 2500x^5 - 2750x^4 + 1175x^3 - 242.5x^2 + 24x - 0.9 \\
H_{2,1}(x) &= \left[1 - 2(x - x_1)L'_{2,1}(x_1)\right] L^2_{2,1}(x) \\
&= 10000x^4 - 8000x^3 + 2200x^2 - 240x + 9 \\
\widehat{H}_{2,1}(x) &= (x - x_1)L^2_{2,1}(x) \\
&= 10000x^5 - 10000x^4 + 3800x^3 - 680x^2 + 57x - 1.8 \\
H_{2,2}(x) &= \left[1 - 2(x - x_2)L'_{2,2}(x_2)\right] L^2_{2,2}(x) \\
&= -75000x^5 + 70000x^4 - 24750x^3 + 4150x^2 - 330x + 10 \\
\widehat{H}_{2,2}(x) &= (x - x_2)L^2_{2,2}(x) \\
&= 2500x^5 - 2250x^4 + 775x^3 - 127.5x^2 + 10x - 0.3
\end{aligned}$$

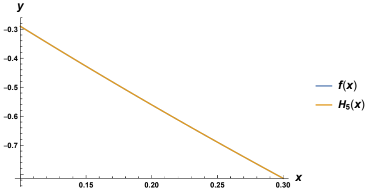
Solution (3 of 3)

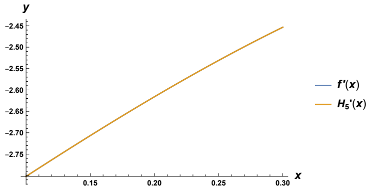
Lastly, the Hermite interpolating polynomial is

$$
\begin{aligned}
H_5(x) = {} & f(x_0)H_{2,0}(x) + f'(x_0)\widehat{H}_{2,0}(x) \\
& + f(x_1)H_{2,1}(x) + f'(x_1)\widehat{H}_{2,1}(x) \\
& + f(x_2)H_{2,2}(x) + f'(x_2)\widehat{H}_{2,2}(x) \\
= {} & -0.29004996H_{2,0}(x) - 2.8019975\widehat{H}_{2,0}(x) \\
& - 0.56079734H_{2,1}(x) - 2.6159201\widehat{H}_{2,1}(x) \\
& - 0.81401972H_{2,2}(x) - 2.9734038\widehat{H}_{2,2}(x).
\end{aligned}
$$

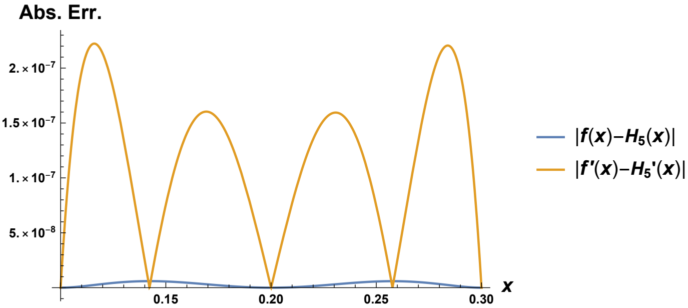
Graphs of Function and Approximation

The function approximated in the previous example is
$f(x) = x^2 \cos x - 3x$.





Graph of Absolute Error



$$f(0.18) = -0.50812346435$$

$$H_5(0.18) = -0.50812346583$$

$$|f(0.18) - H_5(0.18)| = 1.48 \times 10^{-9}$$

Error Analysis

A bound for the error in the previous approximation can be found.

$$
\begin{aligned}
&|f(0.18) - H_5(0.18)| \\
&= \left| \frac{(0.18-0.1)^2(0.18-0.2)^2(0.18-0.3)^2}{6!} f^{(6)}(z) \right| \\
&\leq (5.12 \times 10^{-11}) \max_{0.1<z<0.3} |f^{(6)}(z)| \\
&= (5.12 \times 10^{-11}) f^{(6)}(0.1) \\
&= 1.52168 \times 10^{-9}
\end{aligned}
$$

Divided Differences

- Suppose we are given $\{(x_0, f(x_0), f'(x_0)), (x_1, f(x_1), f'(x_1)), \ldots, (x_n, f(x_n), f'(x_n))\}$.
- Define a new sequence $z_0, z_1, \ldots, z_{2n+1}$ by

$$z_{2i} = x_i \text{ for } i = 0, 1, \ldots, n$$
$$z_{2i+1} = x_i \text{ for } i = 0, 1, \ldots, n.$$

- Create the divided difference table using $z_0, z_1, \ldots, z_{2n+1}$.

Divided Differences

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- Define a new sequence $z_0, z_1, \ldots, z_{2n+1}$ by

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\begin{aligned}
z_{2i} &= x_i \text{ for } i = 0, 1, \ldots, n \\
z_{2i+1} &= x_i \text{ for } i = 0, 1, \ldots, n.
\end{aligned}
$$

- Create the divided difference table using $z_0, z_1, \ldots, z_{2n+1}$.

Note: use $f'(x_i)$ in place of the first divided difference $f[z_{2i}, z_{2i+1}]$ (since otherwise this divided difference would be undefined).

Divided Difference Table

z	$f[z]$	First divided differences	Second divided differences
$z_0 = x_0$	$f[z_0] = f(x_0)$		
		$f[z_0, z_1] = f'(x_0)$	
$z_1 = x_0$	$f[z_1] = f(x_0)$		$f[z_0, z_1, z_2] = \frac{f[z_1,z_2]-f[z_0,z_1]}{z_2-z_0}$
		$f[z_1, z_2] = \frac{f[z_2]-f[z_1]}{z_2-z_1}$	
$z_2 = x_1$	$f[z_2] = f(x_1)$		$f[z_1, z_2, z_3] = \frac{f[z_2,z_3]-f[z_1,z_2]}{z_3-z_1}$
		$f[z_2, z_3] = f'(x_1)$	
$z_3 = x_1$	$f[z_3] = f(x_1)$		$f[z_2, z_3, z_4] = \frac{f[z_3,z_4]-f[z_2,z_3]}{z_4-z_2}$
		$f[z_3, z_4] = \frac{f[z_4]-f[z_3]}{z_4-z_3}$	
$z_4 = x_2$	$f[z_4] = f(x_2)$		$f[z_3, z_4, z_5] = \frac{f[z_4,z_5]-f[z_3,z_4]}{z_5-z_3}$
		$f[z_4, z_5] = f'(x_2)$	
$z_5 = x_2$	$f[z_5] = f(x_2)$		

$$H_{2n+1}(x) = f[z_0] + \sum_{k=1}^{2n+1} f[z_0, \ldots, z_k](x - z_0)(x - z_1)\cdots(x - z_{k-1})$$

Example

Use the divided difference approach to approximate $f(0.18)$ given the following data.

x	$f(x)$	$f'(x)$
0.1	−0.29004996	−2.8019975
0.2	−0.56079734	−2.6159201
0.3	−0.81401972	−2.9734038

Solution (1 of 2)

Original data is shown in blue.

0.1	−0.29004996					
		−2.8019975				
0.1	−0.29004996		0.94523716			
		−2.7074738		−0.29700724		
0.2	−0.56079734		0.91553643		−0.47928682	
		−2.6159201		−0.39286461		0.04933582
0.2	−0.56079734		0.83696351		−0.46941966	
		−2.5322238		−0.48674854		
0.3	−0.81401972		0.78828866			
		−2.9734038				
0.3	−0.81401972					

The values at the top of each column are the coefficients used to construct the Hermite interpolating polynomial.

Solution (2 of 2)

Using the results from the table of divided differences yields

$$\begin{aligned}
H_5(x) = &f[z_0] + f[z_0, z_1](x - z_0) + f[z_0, z_1, z_2](x - z_0)(x - z_1) \\
&+ f[z_0, z_1, z_2, z_3](x - z_0)(x - z_1)(x - z_2) \\
&+ f[z_0, z_1, z_2, z_3, z_4](x - z_0)(x - z_1)(x - z_2)(x - z_3) \\
&+ f[z_0, z_1, z_2, z_3, z_4, z_5](x - z_0)(x - z_1)(x - z_2)(x - z_3)(x - z_4) \\
H_5(0.18) = &-0.29004996 - 2.8019975(0.08) \\
&+ 0.94523716(0.08)^2 - 0.29700724(0.08)^2(-0.02) \\
&- 0.47928682(0.08)^2(-0.02)^2 \\
&+ 0.04933582(0.08)^2(-0.02)^2(-0.12) \\
= &-0.50812346583
\end{aligned}$$

Homework

- Read Section 3.4.
- Exercises: 1a, 3a, 8