Numerical Integration

MATH 375

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Objectives

In this lesson we will learn to

- approximate definite integrals of the form $\int_a^b f(x)\,dx$, and
- bound the error in the approximation.

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- bound the error in the approximation.

For integrands $f(x)$ for which no explicit antiderivative is known we will use a sum of the form

$$\int_a^b f(x)\,dx \approx \sum_{i=0}^{n} a_i f(x_i).$$

Background

Throughout this presentation we will use the following definite integral to evaluate the performance of the various approximations.

$$\begin{aligned}\int_0^{\pi/4} e^{3x} \sin 2x \, dx &= \left[\frac{1}{13} e^{3x} \left(3 \sin 2x - 2 \cos 2x \right) \right]_{x=0}^{x=\pi/4} \\ &= \frac{2 + 3e^{3\pi/4}}{13} \\ &\approx 2.58863\end{aligned}$$

Integration and Lagrange Interpolating Polynomials (1 of 3)

If we can create the Lagrange interpolating polynomial for $f(x)$ on $[a, b]$:

$$P_n(x) = \sum_{i=0}^{n} f(x_i) L_i(x)$$

$$f(x) = P_n(x) + \frac{f^{(n+1)}(z(x))}{(n+1)!} \prod_{i=0}^{n} (x - x_i)$$

we can create a **quadrature formula** for the definite integral.

Integration and Lagrange Interpolating Polynomials (2 of 3)

$$\begin{aligned}
\int_a^b f(x)\,dx &= \int_a^b P_n(x)\,dx + \int_a^b \frac{f^{(n+1)}(z(x))}{(n+1)!}\prod_{i=0}^{n}(x-x_i)\,dx \\
&= \int_a^b \sum_{i=0}^{n} f(x_i)L_i(x)\,dx \\
&\quad + \int_a^b \frac{f^{(n+1)}(z(x))}{(n+1)!}\prod_{i=0}^{n}(x-x_i)\,dx \\
&= \sum_{i=0}^{n} f(x_i)\int_a^b L_i(x)\,dx \\
&\quad + \frac{1}{(n+1)!}\int_a^b f^{(n+1)}(z(x))\prod_{i=0}^{n}(x-x_i)\,dx
\end{aligned}$$

Integration and Lagrange Interpolating Polynomials (3 of 3)

Thus

$$\int_a^b f(x)\,dx = \sum_{i=0}^{n} a_i f(x_i) + E(f)$$

where

$$\int_a^b f(x)\,dx \approx \sum_{i=0}^{n} a_i f(x_i)$$

and the error term is

$$E(f) = \frac{1}{(n+1)!} \int_a^b f^{(n+1)}(z(x)) \prod_{i=0}^{n} (x - x_i)\,dx.$$

Integration and Lagrange Interpolating Polynomials (3 of 3)

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where

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$$E(f) = \frac{1}{(n+1)!} \int_a^b f^{(n+1)}(z(x)) \prod_{i=0}^{n} (x - x_i)\,dx.$$

We will start with some elementary quadrature formulas which use degree one and two Lagrange interpolating polynomials with equally-spaced nodes.

Trapezoidal Rule (1 of 5)

- Two nodes $x_0 = a$ and $x_1 = b$.
- Spacing between nodes $h = x_1 - x_0$.
- Lagrange interpolating polynomial of degree one for $f(x)$.

$$P_1(x) = \frac{(x - x_1)f(x_0)}{x_0 - x_1} + \frac{(x - x_0)f(x_1)}{x_1 - x_0}$$

Trapezoidal Rule (2 of 5)

Now integrate:

$$
\begin{aligned}
\int_a^b f(x)\,dx &= \int_{x_0}^{x_1} \left[\frac{(x-x_1)f(x_0)}{x_0-x_1} + \frac{(x-x_0)f(x_1)}{x_1-x_0} \right] dx \\
&\quad + \frac{1}{2}\int_{x_0}^{x_1} f''(z(x))(x-x_0)(x-x_1)\,dx \\
&= \left[\frac{(x-x_1)^2 f(x_0)}{2(x_0-x_1)} + \frac{(x-x_0)^2 f(x_1)}{2(x_1-x_0)} \right]_{x=x_0}^{x=x_1} + E(f) \\
&= -\frac{(x_0-x_1)f(x_0)}{2} + \frac{(x_1-x_0)f(x_1)}{2} + E(f) \\
&= \frac{h}{2}[f(x_0)+f(x_1)] + E(f)
\end{aligned}
$$

Trapezoidal Rule (3 of 5)

Consider the error term:

$$E(f) = \frac{1}{2}\int_{x_0}^{x_1} f''(z(x))\,\underbrace{(x - x_0)(x - x_1)}_{\leq 0 \text{ on } [x_0, x_1]}\,dx$$

Trapezoidal Rule (3 of 5)

Consider the error term:

$$E(f) = \frac{1}{2} \int_{x_0}^{x_1} f''(z(x)) \underbrace{(x - x_0)(x - x_1)}_{\leq 0 \text{ on } [x_0, x_1]} \, dx$$

Since $(x - x_0)(x - x_1)$ does not change sign on $[x_0, x_1]$ we may apply the Weighted Mean Value Theorem.

Theorem

Suppose function $f \in C[a, b]$ and suppose function g is Riemann integrable on $[a, b]$ and that $g(x)$ does not change sign on $[a, b]$. There exists $c \in (a, b)$ such that

$$\int_a^b f(x)\, g(x)\, dx = f(c) \int_a^b g(x)\, dx.$$

Trapezoidal Rule (4 of 5)

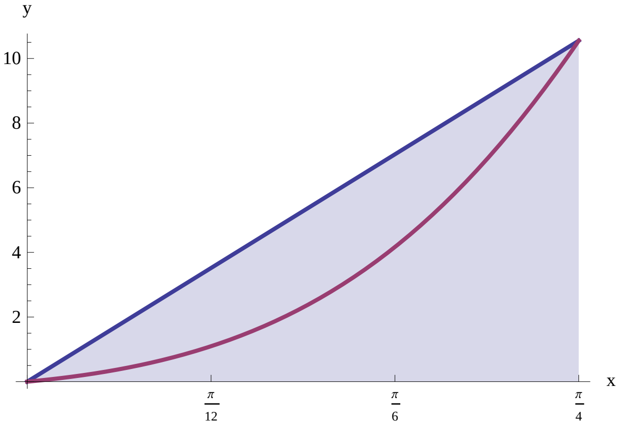
$$
\begin{aligned}
&\frac{1}{2}\int_{x_0}^{x_1} f''(z(x))(x-x_0)(x-x_1)\,dx \\
&= \frac{1}{2}f''(z)\int_{x_0}^{x_1}(x-x_0)(x-x_1)\,dx \\
&= \frac{1}{2}f''(z)\int_{x_0}^{x_1} x^2-(x_0+x_1)x+x_0x_1\,dx \\
&= \frac{1}{2}f''(z)\left[\frac{x^3}{3}-\frac{(x_0+x_1)}{2}x^2+x_0x_1x\right]_{x=x_0}^{x=x_1} \\
&= \frac{1}{2}f''(z)\left[\frac{x_1^3-x_0^3}{3}-\frac{(x_0+x_1)}{2}(x_1^2-x_0^2)+x_0x_1(x_1-x_0)\right] \\
&= -\frac{h^3}{12}f''(z)
\end{aligned}
$$

for some z between x_0 and x_1.

Trapezoidal Rule (5 of 5)

The Trapezoidal Rule can be stated as

$$\int_a^b f(x)\,dx = \frac{h}{2}[f(x_0) + f(x_1)] - \frac{h^3}{12} f''(z).$$



Example: Trapezoidal Rule

$$\int_{x_0}^{x_1} f(x)\,dx = \frac{h}{2}\left[f(x_0) + f(x_1)\right] - \frac{h^3}{12} f''(z)$$

where $x_0 < z < x_1$.

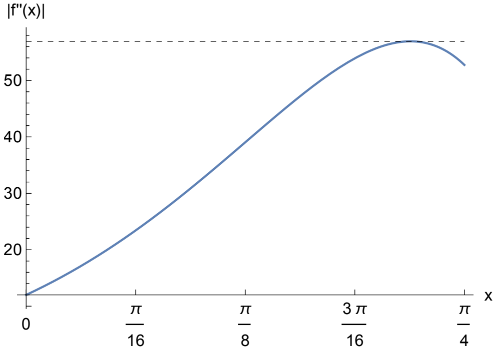
$$\int_0^{\pi/4} e^{3x} \sin 2x\,dx = \frac{2 + e^{3\pi/4}}{13} \approx 4.14326$$

$$\text{Absolute error} \approx 1.55463$$

$$\text{Error bound} \approx 2.29883$$

Determination of Error Bound

$$f(x) = e^{3x}\sin(2x)$$
$$f''(x) = e^{3x}(12\cos(2x) + 5\sin(2x))$$



$$\text{error} \le \max_{0\le z\le \pi/4} |f''(z)|\,\frac{h^3}{12} = (56.9401)\frac{(\pi/4)^3}{12} \approx 2.29883$$

Simpson's Rule (1 of 5)

- Let $h = \dfrac{b-a}{2}$
- Use three equally-spaced nodes $x_0 = a$, $x_1 = a + h$, and $x_2 = b$.
- Assuming $f \in \mathcal{C}^4[a, b]$ form the Taylor polynomial of degree three about x_1.

$$\begin{aligned} f(x) = {} & f(x_1) + f'(x_1)(x - x_1) + \frac{f''(x_1)}{2}(x - x_1)^2 \\ & + \frac{f'''(x_1)}{6}(x - x_1)^3 + \frac{f^{(4)}(z(x))}{24}(x - x_1)^4 \end{aligned}$$

Now integrate.

Simpson's Rule (2 of 5)

$$\begin{aligned}\int_a^b f(x)\,dx = & \int_{x_0}^{x_2}\left[f(x_1) + f'(x_1)(x - x_1) + \frac{f''(x_1)}{2}(x - x_1)^2\right.\\ & \left. + \frac{f'''(x_1)}{6}(x - x_1)^3\right]\,dx\\ & + \frac{1}{24}\int_{x_0}^{x_2} f^{(4)}(z(x))(x - x_1)^4\,dx\end{aligned}$$

Simpson's Rule (2 of 5)

$$\begin{aligned}
\int_a^b f(x)\,dx &= \int_{x_0}^{x_2} \left[f(x_1) + f'(x_1)(x-x_1) + \frac{f''(x_1)}{2}(x-x_1)^2 \right. \\
&\quad \left. + \frac{f'''(x_1)}{6}(x-x_1)^3 \right] dx \\
&\quad + \frac{1}{24} \int_{x_0}^{x_2} f^{(4)}(z(x))(x-x_1)^4\,dx \\
&= \left[f(x_1)x + \frac{f'(x_1)}{2}(x-x_1)^2 + \frac{f''(x_1)}{6}(x-x_1)^3 \right. \\
&\quad \left. + \frac{f'''(x_1)}{24}(x-x_1)^4 \right]_{x=x_0}^{x=x_2} + E(f)
\end{aligned}$$

Simpson's Rule (3 of 5)

$$\int_a^b f(x)\,dx = 2hf(x_1) + \frac{h^3}{3} f''(x_1) + E(f)$$

Simpson's Rule (3 of 5)

$$\begin{aligned}\int_a^b f(x)\,dx &= 2hf(x_1) + \frac{h^3}{3}f''(x_1) + E(f)\\ &= 2hf(x_1) + \frac{h^3}{3}\left(\frac{1}{h^2}[f(x_0) - 2f(x_1) + f(x_2)] - \frac{h^2}{12}f^{(4)}(w)\right)\\ &\quad + E(f)\end{aligned}$$

Simpson's Rule (3 of 5)

$$\begin{aligned}
\int_a^b f(x)\,dx &= 2hf(x_1) + \frac{h^3}{3} f''(x_1) + E(f) \\
&= 2hf(x_1) + \frac{h^3}{3}\left(\frac{1}{h^2}[f(x_0) - 2f(x_1) + f(x_2)] - \frac{h^2}{12} f^{(4)}(w)\right) \\
&\quad + E(f) \\
&= 2hf(x_1) + \frac{h}{3}[f(x_0) - 2f(x_1) + f(x_2)] - \frac{h^5}{36} f^{(4)}(w) \\
&\quad + E(f) \\
&= \frac{h}{3}[f(x_0) + 4f(x_1) + f(x_2)] - \frac{h^5}{36} f^{(4)}(w) + E(f)
\end{aligned}$$

Simpson's Rule (3 of 5)

$$\begin{aligned}
\int_a^b f(x)\,dx &= 2hf(x_1) + \frac{h^3}{3}f''(x_1) + E(f) \\
&= 2hf(x_1) + \frac{h^3}{3}\left(\frac{1}{h^2}[f(x_0) - 2f(x_1) + f(x_2)] - \frac{h^2}{12}f^{(4)}(w)\right) \\
&\quad + E(f) \\
&= 2hf(x_1) + \frac{h}{3}[f(x_0) - 2f(x_1) + f(x_2)] - \frac{h^5}{36}f^{(4)}(w) \\
&\quad + E(f) \\
&= \frac{h}{3}[f(x_0) + 4f(x_1) + f(x_2)] - \frac{h^5}{36}f^{(4)}(w) + E(f)
\end{aligned}$$

Now we must consider the error term.

Simpson's Rule (4 of 5)

$$\begin{aligned}
E(f) &= \frac{1}{24}\int_{x_0}^{x_2} f^{(4)}(z(x))\underbrace{(x-x_1)^4}_{\geq 0}\,dx \\
&= \frac{1}{24} f^{(4)}(v)\int_{x_0}^{x_2} (x-x_1)^4\,dx \text{ (by the WMVT)} \\
&= \left[\frac{1}{120} f^{(4)}(v)(x-x_1)^5\right]_{x=x_0}^{x=x_2} \\
&= \frac{1}{60} f^{(4)}(v)h^5
\end{aligned}$$

Simpson's Rule (5 of 5)

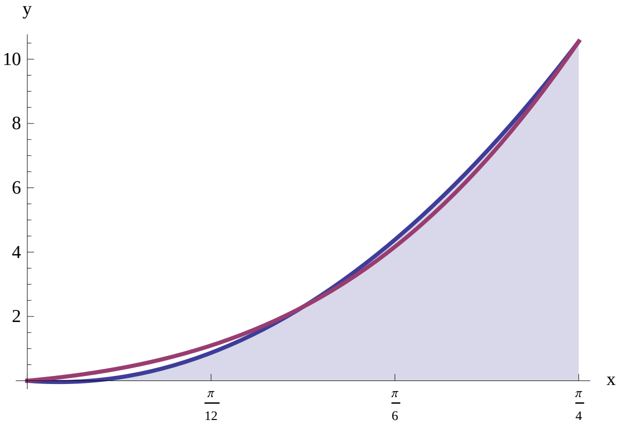
Thus the total error in Simpson's Rule can be expressed as

$$\begin{aligned} -\frac{h^5}{36} f^{(4)}(w) + E(f) &= -\frac{h^5}{36} f^{(4)}(w) + \frac{1}{60} f^{(4)}(v) h^5 \\ &= -\frac{h^5}{12} \left[\frac{1}{3} f^{(4)}(w) - \frac{1}{5} f^{(4)}(v) \right] \\ &= -\frac{h^5}{90} f^{(4)}(z) \end{aligned}$$

for some z between x_0 and x_2.

Simpson’s Rule (Graphical Interpretation)

$$\int_{x_0}^{x_2} f(x)\,dx = \frac{h}{3}[f(x_0) + 4f(x_1) + f(x_2)] - \frac{h^5}{90} f^{(4)}(z)$$



Example: Simpson's Rule

$$\int_{x_0}^{x_2} f(x)\,dx = \frac{h}{3}\left[f(x_0) + 4f(x_1) + f(x_2)\right] - \frac{h^5}{90} f^{(4)}(z)$$

where $x_0 < z < x_2$.

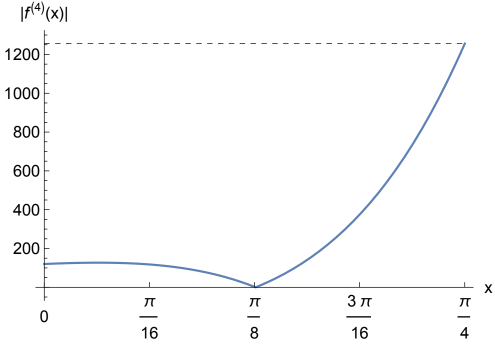
$$\int_0^{\pi/4} e^{3x} \sin 2x\,dx = \frac{2 + 3e^{3\pi/4}}{13} \approx 2.5837$$

$$\text{Absolute error} \approx 0.00493223$$

$$\text{Error bound} \approx 0.130283$$

Determination of Error Bound

$$f(x) = e^{3x}\sin(2x)$$
$$f^{(4)}(x) = e^{3x}(120\cos(2x) - 119\sin(2x))$$



$$\text{error} \leq \max_{0\leq z\leq \pi/4} |f^{(4)}(z)|\,\frac{h^5}{90} = (1255.54)\frac{(\pi/8)^5}{90} \approx 0.130283$$

Precision

Definition

The **degree of accuracy** or **precision** of a quadrature formula is the largest positive integer n such that the formula is exact for x^k for all $k = 0, 1, \ldots, n$.

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Question: what are the precisions of the Trapezoidal Rule (1) and Simpson's Rule (2)?

$$\int_a^b f(x)\,dx = \frac{h}{2}[f(x_0) + f(x_1)] - \frac{h^3}{12} f''(z) \tag{1}$$

$$\int_{x_0}^{x_2} f(x)\,dx = \frac{h}{3}[f(x_0) + 4f(x_1) + f(x_2)] - \frac{h^5}{90} f^{(4)}(z) \tag{2}$$

Precision

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$$\int_{x_0}^{x_2} f(x)\,dx = \frac{h}{3}[f(x_0) + 4f(x_1) + f(x_2)] - \frac{h^5}{90} f^{(4)}(z) \tag{2}$$

Remark: since summation and integration are linear operations, the precision of a quadrature method is n if and only if the method is exact for all polynomials of degree n or less and not exact for some polynomial of degree $n + 1$.

Newton-Cotes Formulas

$$\int_a^b f(x)\,dx$$

- The Trapezoidal Rule (1) and Simpson's Rule (2) are examples of **Newton-Cotes** quadrature formulas.

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- The Trapezoidal Rule (1) and Simpson's Rule (2) are examples of **Newton-Cotes** quadrature formulas.
- Newton-Cotes formulas can be divided into two types:

closed:	methods that use nodes in the closed interval $[a, b]$ including a and b themselves.
open:	methods that use nodes in the open interval (a, b).

Closed Newton-Cotes Formulas (1 of 2)

$$\int_a^b f(x)\,dx$$

The $(n+1)$-point closed Newton-Cotes formula uses as nodes

$$\begin{aligned}
x_0 &= a \\
x_1 &= a + \frac{(b-a)}{n} \\
&\vdots \\
x_i &= a + \frac{(b-a)i}{n} \\
&\vdots \\
x_n &= b.
\end{aligned}$$

Closed Newton-Cotes Formulas (2 of 2)

Remarks:

- The Trapezoidal Rule is an $n = 1$ closed Newton-Cotes formula.
- Simpson's Rule is an $n = 2$ closed Newton-Cotes formula.

Closed Newton-Cotes Formulas (2 of 2)

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- The Trapezoidal Rule is an $n = 1$ closed Newton-Cotes formula.
- Simpson's Rule is an $n = 2$ closed Newton-Cotes formula.
- The $n = 3$ closed Newton-Cotes formula is known as **Simpson's Three-Eighths Rule**:

$$\int_{x_0}^{x_3} f(x)\,dx = \frac{3h}{8}[f(x_0) + 3f(x_1) + 3f(x_2) + f(x_3)] - \frac{3h^5}{80} f^{(4)}(z).$$

Simpson's Three-Eighths Rule: $n = 3$

$$\int_{x_0}^{x_3} f(x)\,dx = \frac{3h}{8}\left[f(x_0) + 3f(x_1) + 3f(x_2) + f(x_3)\right] - \frac{3h^5}{80} f^{(4)}(z)$$

where $x_0 < z < x_3$.

Example

$$\int_0^{\pi/4} e^{3x} \sin 2x \, dx = \frac{2 + 3e^{3\pi/4}}{13} \approx 2.58579$$

$$\text{Absolute error} \approx 0.00283958$$

$$\text{Error bound} \approx 0.0579033$$

General Closed Newton-Cotes Formula (1 of 2)

Theorem

Suppose that $\sum_{i=0}^{n} a_i f(x_i)$ *denotes the* $(n+1)$*-point closed Newton-Cotes formula with* $x_0 = a$, $x_n = b$, *and* $h = (b-a)/n$. *If* n *is even and* $f \in \mathcal{C}^{n+2}[a,b]$ *there exists* $z \in (a,b)$ *such that*

$$\int_a^b f(x)\,dx = \sum_{i=0}^{n} a_i f(x_i) + \frac{h^{n+3} f^{(n+2)}(z)}{(n+2)!} \int_0^n t^2(t-1)\cdots(t-n)\,dt.$$

If n *is odd and* $f \in \mathcal{C}^{n+1}[a,b]$ *there exists* $z \in (a,b)$ *such that*

$$\int_a^b f(x)\,dx = \sum_{i=0}^{n} a_i f(x_i) + \frac{h^{n+2} f^{(n+1)}(z)}{(n+1)!} \int_0^n t(t-1)\cdots(t-n)\,dt.$$

General Closed Newton-Cotes Formula (2 of 2)

In these formulas

$$a_i = \int_{x_0}^{x_n} L_i(x)\,dx = \int_{x_0}^{x_n} \prod_{\substack{j=0\\ j\neq i}}^{n} \frac{(x - x_j)}{(x_i - x_j)}\,dx.$$

Example: $n = 4$

$$a_0 = \int_{x_0}^{x_0+4h} \prod_{\substack{j=0 \\ j\neq 0}}^{4} \frac{(x - x_0 - jh)}{(x_0 - x_0 - jh)}\, dx = \frac{14h}{45}$$

$$a_1 = \int_{x_0}^{x_0+4h} \prod_{\substack{j=0 \\ j\neq 1}}^{4} \frac{(x - x_0 - jh)}{(x_1 - x_0 - jh)}\, dx = \frac{64h}{45}$$

$$a_2 = \int_{x_0}^{x_0+4h} \prod_{\substack{j=0 \\ j\neq 2}}^{4} \frac{(x - x_0 - jh)}{(x_2 - x_0 - jh)}\, dx = \frac{24h}{45}$$

$$a_3 = \int_{x_0}^{x_0+4h} \prod_{\substack{j=0 \\ j\neq 3}}^{4} \frac{(x - x_0 - jh)}{(x_3 - x_0 - jh)}\, dx = \frac{64h}{45}$$

$$a_4 = \int_{x_0}^{x_0+4h} \prod_{\substack{j=0 \\ j\neq 4}}^{4} \frac{(x - x_0 - jh)}{(x_4 - x_0 - jh)}\, dx = \frac{14h}{45}$$

Closed Newton-Cotes: $n = 4$

$$\int_{x_0}^{x_4} f(x)\,dx = \frac{2h}{45}\left[7f(x_0) + 32f(x_1) + 12f(x_2) + 32f(x_3) + 7f(x_4)\right] - \frac{8h^7}{945} f^{(6)}(z)$$

where $x_0 < z < x_4$.

$$\int_0^{\pi/4} e^{3x} \sin 2x\,dx = \frac{2 + 2e^{3\pi/4}}{13} \approx 2.58797$$

Absolute error ≈ 0.000660176

Error bound ≈ 0.0020451

Open Newton-Cotes Formulas

$$\int_a^b f(x)\,dx$$

The $(n+1)$-point open Newton-Cotes formula uses as nodes

$$\begin{aligned} x_0 &= a + \frac{(b-a)}{n+2} \\ &\vdots \\ x_i &= a + \frac{(b-a)(i+1)}{n+2} \\ &\vdots \\ x_n &= a + \frac{(b-a)(n+1)}{n+2}. \end{aligned}$$

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Remarks: For notational simplicity we denote $x_{-1} = a$ and $x_{n+1} = b$.

General Open Newton-Cotes Formula (1 of 2)

Theorem

Suppose that $\sum_{i=0}^{n} a_i f(x_i)$ *denotes the* $(n+1)$*-point open Newton-Cotes formula with* $x_{-1} = a$, $x_{n+1} = b$, *and* $h = (b-a)/(n+2)$. *If* n *is even and* $f \in \mathcal{C}^{n+2}[a,b]$ *there exists* $z \in (a,b)$ *such that*

$$\int_a^b f(x)\,dx = \sum_{i=0}^{n} a_i f(x_i) + \frac{h^{n+3} f^{(n+2)}(z)}{(n+2)!} \int_{-1}^{n+1} t^2 (t-1) \cdots (t-n)\,dt.$$

If n *is odd and* $f \in \mathcal{C}^{n+1}[a,b]$ *there exists* $z \in (a,b)$ *such that*

$$\int_a^b f(x)\,dx = \sum_{i=0}^{n} a_i f(x_i) + \frac{h^{n+2} f^{(n+1)}(z)}{(n+1)!} \int_{-1}^{n+1} t (t-1) \cdots (t-n)\,dt.$$

General Open Newton-Cotes Formula (2 of 2)

In these formulas

$$a_i = \int_{x_{-1}}^{x_{n+1}} L_i(x)\,dx = \int_{x_{-1}}^{x_{n+1}} \prod_{\substack{j=0 \\ j\neq i}}^{n} \frac{(x - x_j)}{(x_i - x_j)}\,dx.$$

Example: $n = 2$

$$a_0 = \int_{x_0-h}^{x_0+2h+h} \prod_{\substack{j=0 \\ j\neq 0}}^{2} \frac{(x - x_0 - jh)}{(x_0 - x_0 - jh)}\, dx = \frac{8h}{3}$$

$$a_1 = \int_{x_0-h}^{x_0+2h+h} \prod_{\substack{j=0 \\ j\neq 1}}^{2} \frac{(x - x_0 - jh)}{(x_1 - x_0 - jh)}\, dx = -\frac{4h}{3}$$

$$a_2 = \int_{x_0-h}^{x_0+2h+h} \prod_{\substack{j=0 \\ j\neq 2}}^{2} \frac{(x - x_0 - jh)}{(x_2 - x_0 - jh)}\, dx = \frac{8h}{3}$$

Midpoint Rule: $n = 0$

$$\int_{x_{-1}}^{x_1} f(x)\,dx = 2hf(x_0) + \frac{h^3}{3} f''(z)$$

where $x_{-1} < z < x_1$.

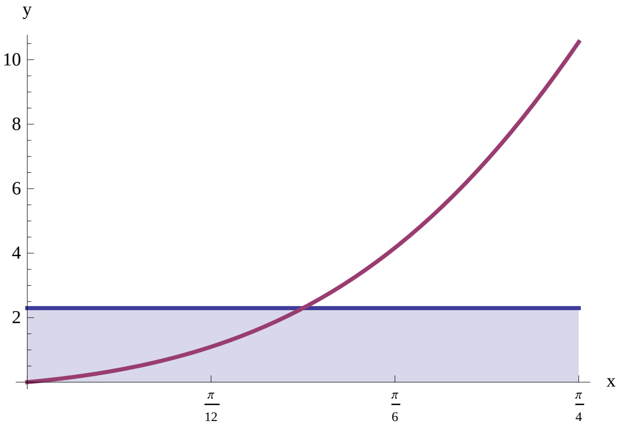
Example

$$\int_0^{\pi/4} e^{3x} \sin 2x\,dx = \frac{2 + 3e^{3\pi/4}}{13} \approx 1.80391$$

$$\text{Absolute error} \approx 0.784714$$

$$\text{Error bound} \approx 1.14941$$

Midpoint Rule (Graphical Interpretation)



Open Newton-Cotes: $n = 1$

$$\int_{x_{-1}}^{x_2} f(x)\,dx = \frac{3h}{2}\left[f(x_0) + f(x_1)\right] + \frac{3h^3}{4} f''(z)$$

where $x_{-1} < z < x_2$.

Example

$$\int_0^{\pi/4} e^{3x} \sin 2x \, dx = \frac{2 + 3e^{3\pi/4}}{13} \approx 2.06663$$

$$\text{Absolute error} \approx 0.521996$$

$$\text{Error bound} \approx 0.766276$$

Open Newton-Cotes: $n = 2$

$$\int_{x_{-1}}^{x_3} f(x)\,dx = \frac{4h}{3}\left[2f(x_0) - f(x_1) + 2f(x_2)\right] + \frac{14h^5}{45} f^{(4)}(z)$$

where $x_{-1} < z < x_3$.

Example

$$\int_0^{\pi/4} e^{3x} \sin 2x\,dx = \frac{2 + 3e^{3\pi/4}}{13} \approx 2.591996$$

$$\text{Absolute error} \approx 0.00307787$$

$$\text{Error bound} \approx 0.113997$$

Open Newton-Cotes: $n = 3$

$$\int_{x_{-1}}^{x_4} f(x)\,dx = \frac{5h}{24}\left[11f(x_0) + f(x_1) + f(x_2) + 11f(x_3)\right] + \frac{95h^5}{144}f^{(4)}(z)$$

where $x_{-1} < z < x_4$.

Example

$$\int_0^{\pi/4} e^{3x}\sin 2x\,dx = \frac{2 + 3e^{3\pi/4}}{13} \approx 2.59119$$

$$\text{Absolute error} \approx 0.00255695$$

$$\text{Error bound} \approx 0.0792118$$

Relationship between n and h

$$\int_a^b f(x)\,dx$$

- For the Closed Newton-Cotes Formulas: $h = \dfrac{b-a}{n}$ and $x_i = a + i\,h$ for $i = 0, 1, \ldots, n$.
- For the Open Newton-Cotes Formulas: $h = \dfrac{b-a}{n+2}$ and $x_i = a + (i+1)h$ for $i = 0, 1, \ldots, n$.

Homework

- Read Section 4.3
- Exercises: 1cf, 3cf, 5cf, 7cf, 25, 26