Matrix Inversion

MATH 375 *Numerical Analysis*

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Review of Linear Algebra

Definition

Two matrices A and B are **equal** if they have the same dimensions and if $a_{ij} = b_{ij}$ for all i and j.

Definition

If A and B are two $n \times m$ matrices then the **sum** of A and B, denoted $A + B$ is the $n \times m$ matrix whose entries are $a_{ij} + b_{ij}$ for all $i = 1, 2, \ldots, n$ and $j = 1, 2, \ldots, m$.

Definition

If A is an $n \times m$ matrix and if $\lambda \in \mathbb{R}$ then the **scalar product** λA is the $n \times m$ matrix whose entries are λa_{ij} for all $i = 1, 2, \ldots, n$ and $j = 1, 2, \ldots, m$.

Matrix Properties

Theorem

If $A, B, C \in \mathbb{R}^{n \times m}$ and if $\lambda, \mu \in \mathbb{R}$ then the following properties hold.

1. $A + B = B + A$ *(commutativity)*
2. $(A + B) + C = A + (B + C)$ *(associativity)*
3. $A + 0 = A$ *(additive identity)*
4. $A + (-A) = 0$ *(additive inverse)*
5. $\lambda(A + B) = \lambda A + \lambda B$ *(distributive property)*
6. $(\lambda + \mu)A = \lambda A + \mu A$ *(distributive property)*
7. $\lambda(\mu A) = (\lambda\mu)A$ *(associativity)*
8. $(1)A = A$ *(scalar multiplicative identity)*

Matrix Multiplication

Definition

Let $A \in \mathbb{R}^{n \times p}$ and $B \in \mathbb{R}^{p \times m}$. The matrix product of A and B, denoted AB is the $n \times m$ matrix C whose entries are

$$c_{ij} = \sum_{k=1}^{p} a_{ik} b_{kj}$$

for all $i = 1, 2, \ldots, n$ and $j = 1, 2, \ldots, m$.

Remark: c_{ij} is the dot product of the ith row of A with the jth column of B.

Special Matrices

Definition

- A **square matrix** has the same number of rows as columns.
- A **diagonal matrix** is a square matrix with entries d_{ij} and $d_{ij} = 0$ if $i \neq j$.
- The **identity matrix of order** n, denoted I_n is a diagonal matrix with 1's on the diagonal.
- An $n \times n$ **upper triangular matrix** U has entries u_{ij} and for each $j = 1, 2, \ldots, n$, $u_{ij} = 0$ for $i = j+1, j+2, \ldots, n$.
- An $n \times n$ **lower triangular matrix** L has entries l_{ij} and for each $j = 1, 2, \ldots, n$, $l_{ij} = 0$ for $i = 1, \ldots, j-1$.

Remarks

- The set of all $n \times m$ matrices whose entries are real numbers together with the operations of matrix addition and scalar multiplication form a **vector space**.
- Matrix multiplication is in general not commutative.
- If $A \in \mathbb{R}^{n \times n}$ then $I_n A = A I_n$.

Properties of Matrix Multiplication

Theorem

Let $A \in \mathbb{R}^{n\times m}$, $B \in \mathbb{R}^{m\times k}$, $C \in \mathbb{R}^{k\times p}$, $D \in \mathbb{R}^{m\times k}$ *and* $\lambda \in \mathbb{R}$ *then*

1. $A(BC) = (AB)C$
2. $A(B + D) = AB + AD$
3. $I_m B = B$ *and* $BI_k = B$
4. $\lambda(AB) = (\lambda A)B = A(\lambda B)$.

Linear Systems

We can re-write the linear system of equations

$$\begin{aligned}
a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n &= b_1 \\
a_{21}x_1 + a_{22}x_2 + \cdots + a_{2n}x_n &= b_2 \\
&\vdots \\
a_{n1}x_1 + a_{n2}x_2 + \cdots + a_{nn}x_n &= b_n
\end{aligned}$$

as the matrix equation $A\mathbf{x} = \mathbf{b}$ where

$$A = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{bmatrix}, \quad \mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix}, \quad \text{and} \quad \mathbf{b} = \begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_n \end{bmatrix}.$$

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Remark: if we can find the **inverse** of the matrix A we can solve for $\mathbf{x}$.

Matrix Inverse

Definition

The matrix $A \in \mathbb{R}^{n \times n}$ is **non-singular** if there exists a matrix $A^{-1} \in \mathbb{R}^{n \times n}$ such that

$$AA^{-1} = A^{-1}A = I_n.$$

The matrix A^{-1} is called the **inverse** of A. A matrix lacking an inverse is called **singular**.

Example

If $A = \begin{bmatrix} 1 & 2 & 0 \\ 2 & 1 & -1 \\ 3 & 1 & 1 \end{bmatrix}$ then $A^{-1} = \dfrac{1}{8}\begin{bmatrix} -2 & 2 & 2 \\ 5 & -1 & -1 \\ 1 & -5 & 3 \end{bmatrix}$ and

$$A^{-1}A = AA^{-1} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}.$$

Example: Linear System

Re-write the linear system as a matrix equation $A\mathbf{x} = \mathbf{b}$ and use A^{-1} to solve for $\mathbf{x}$.

$$\begin{aligned} x_1 + 2x_2 &= 1 \\ 2x_1 + x_2 - x_3 &= 1 \\ 3x_1 + x_2 + x_3 &= 1 \end{aligned}$$

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$$\begin{aligned} \begin{bmatrix} 1 & 2 & 0 \\ 2 & 1 & -1 \\ 3 & 1 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} &= \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} \\ \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} &= \frac{1}{8} \begin{bmatrix} -2 & 2 & 2 \\ 5 & -1 & -1 \\ 1 & -5 & 3 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} \\ \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} &= \begin{bmatrix} \frac{1}{4} \\ \frac{3}{8} \\ -\frac{1}{8} \end{bmatrix} \end{aligned}$$

Finding a Matrix Inverse

We can use Gaussian elimination and back substitution to find the inverse of a matrix.

Given matrix $A = [a_{ij}]$ let $A^{-1} = [b_{ij}]$ be a matrix of n^2 unknowns.

We must solve a system of n^2 linear equations.

$$\sum_{k=1}^{n} a_{ik} b_{kj} = \delta_{ij} = \begin{cases} 1 & \text{if } i = j, \\ 0 & \text{if } i \neq j \end{cases}$$

for $i = 1, 2, \ldots, n$ and $j = 1, 2, \ldots, n$.

Augmented Form

Using the augmented matrix structure:

$$\left[\begin{array}{c|c} A & I_n \end{array}\right] = \left[\begin{array}{cccc|cccc} a_{11} & a_{12} & \cdots & a_{1n} & 1 & 0 & \cdots & 0 \\ a_{21} & a_{22} & \cdots & a_{2n} & 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} & 0 & 0 & \cdots & 1 \end{array}\right]$$

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Reduce A to upper triangular form:

$$\left[\begin{array}{cccc|cccc} u_{11} & u_{12} & \cdots & u_{1n} & 1 & 0 & \cdots & 0 \\ 0 & u_{22} & \cdots & u_{2n} & l_{21} & 1 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & u_{nn} & l_{n1} & l_{n2} & \cdots & 1 \end{array}\right]$$

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Use back substitution on each column in the lower triangular matrix to find A^{-1}.

Example

Find the inverse of $A = \begin{bmatrix} 1 & 2 & 0 \\ 2 & 1 & -1 \\ 3 & 1 & 1 \end{bmatrix}$.

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$$\left[\begin{array}{ccc|ccc} 1 & 2 & 0 & 1 & 0 & 0 \\ 2 & 1 & -1 & 0 & 1 & 0 \\ 3 & 1 & 1 & 0 & 0 & 1 \end{array}\right]$$

$$\mapsto \left[\begin{array}{ccc|ccc} 1 & 2 & 0 & 1 & 0 & 0 \\ 0 & -3 & -1 & -2 & 1 & 0 \\ 0 & -5 & 1 & -3 & 0 & 1 \end{array}\right]$$

$$\mapsto \left[\begin{array}{ccc|ccc} 1 & 2 & 0 & 1 & 0 & 0 \\ 0 & -3 & -1 & -2 & 1 & 0 \\ 0 & 0 & \frac{8}{3} & \frac{1}{3} & -\frac{5}{3} & 1 \end{array}\right]$$

Back Substitution

$$\left[\begin{array}{ccc|c} 1 & 2 & 0 & 1 \\ 0 & -3 & -1 & -2 \\ 0 & 0 & \frac{8}{3} & \frac{1}{3} \end{array}\right] \Longrightarrow \begin{bmatrix} b_{11} \\ b_{21} \\ b_{31} \end{bmatrix} = \begin{bmatrix} -\frac{1}{4} \\ \frac{5}{8} \\ \frac{1}{8} \end{bmatrix}$$

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$$A^{-1} = \frac{1}{8} \begin{bmatrix} -2 & 2 & 2 \\ 5 & -1 & -1 \\ 1 & -5 & 3 \end{bmatrix}$$

Row Reduction Operation Counts

Multiplications/divisions: at the ith stage of the reduction we perform $(n-i)(n+1)+(n-i)=(n-i)(n+2)$ multiplications/divisions.

$$\sum_{i=1}^{n-1}(n-i)(n+2)=\frac{n^3+n^2-2n}{2}$$

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Additions/subtractions: at the *i*th stage of the reduction we perform $(n-i)(n+1)$ additions/subtractions.

$$\sum_{i=1}^{n-1}(n-i)(n+1)=\frac{n^3-n}{2}$$

Back Substitution Operation Counts

Multiplications/divisions: for each column we perform $(n^2+n)/2$ multiplications/divisions.

$$\sum_{i=1}^{n} \frac{n^2+n}{2} = \frac{n^3+n^2}{2}$$

Back Substitution Operation Counts

Multiplications/divisions: for each column we perform $(n^2 + n)/2$ multiplications/divisions.

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Additions/subtractions: for each column we perform $(n^2 - n)/2$ additions/subtractions.

$$\sum_{i=1}^{n} \frac{n^2 - n}{2} = \frac{n^3 - n^2}{2}$$

Matrix Inversion Operation Counts

Total operation counts needed for inversion of an $n \times n$ matrix.

$$\text{Multiplications/divisions: } n^3 + n^2 - n$$

$$\text{Additions/subtractions: } \frac{2n^3 - n^2 - n}{2}$$

Symmetric Matrices

Definition

If matrix $A = [a_{ij}] \in \mathbb{R}^{n \times m}$ then the **transpose** of A, denoted A^t is $A^t = [A_{ji}] \in \mathbb{R}^{m \times n}$.

Definition

If matrix $A = A^t$ then A is said to be **symmetric**.

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If matrix $A = A^t$ then A is said to be **symmetric**.

Theorem
1. $\left(A^t\right)^t = A$.
2. $(A + B)^t = A^t + B^t$.
3. $(A\,B)^t = B^t\,A^t$.
4. *If A is non-singular then* $\left(A^{-1}\right)^t = \left(A^t\right)^{-1}$.

Homework

- Read Section 6.3.
- Exercises: 3, 5, 11, 13