Error Analysis for Iterative Methods

MATH 375 *Numerical Analysis*

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Objectives

We wish to investigate and measure the order of convergence of the iterative root-finding schemes, such as Newton's Method.

Order of Convergence

Definition

Suppose the sequence $\{p_n\}_{n=0}^{\infty}$ converges to p with $p_n \neq p$ for all n. If there exist positive constants α and λ for which

$$\lim_{n\to\infty} \frac{|p_{n+1} - p|}{|p_n - p|^{\alpha}} = \lambda$$

then $\{p_n\}_{n=0}^{\infty}$ is said to **converge to p of order α with asymptotic error constant λ**.

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Remarks:

- If $\alpha = 1$ and $\lambda < 1$, the convergence is **linear**.
- If $\alpha = 2$, the convergence is **quadratic**.
- Larger α generally means "faster" convergence.

Example (1 of 3)

Find the order of convergence and asymptotic error constant for the sequence $\left\{\frac{1}{n+1}\right\}_{n=0}^{\infty}$.

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Let $p_n = \frac{1}{n+1}$, then $p_n \to 0 = p$.

$$\lim_{n \to \infty} \frac{\left| \frac{1}{n+2} - 0 \right|}{\left| \frac{1}{n+1} - 0 \right|^1} = \lim_{n \to \infty} \frac{n+1}{n+2} = 1$$

Thus $\alpha = 1$ and $\lambda = 1$.

Example (2 of 3)

Find the order of convergence and asymptotic error constant for the sequence $\left\{ \frac{1}{2^n} \right\}_{n=0}^{\infty}$.

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$$\lim_{n\to\infty} \frac{\left| \frac{1}{2^{n+1}} - 0 \right|}{\left| \frac{1}{2^n} - 0 \right|^{\alpha}} = \lim_{n\to\infty} \frac{1}{2} \frac{1}{2^{n(1-\alpha)}} = \frac{1}{2}$$

if $\alpha = 1$. The asymptotic error constant is $\lambda = 1/2$ which implies this sequence is linearly convergent.

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Note: If $\alpha < 1$ the limit is 0 (not positive). If $\alpha > 1$ the sequence diverges.

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$$\lim_{n\to\infty} \frac{\left|2^{-(3/2)^{n+1}} - 0\right|}{\left|2^{-(3/2)^n} - 0\right|^{\alpha}} = \lim_{n\to\infty} 2^{-(3/2)^n[3/2-\alpha]} = 1$$

if $\alpha = 3/2$. The asymptotic error constant is $\lambda = 1$.

Comparison of Convergence

Let $p_n = \dfrac{1}{2^n}$ and $q_n = 2^{-(3/2)^n}$, then

n	p_n	q_n
0	1.00000	0.50000000
1	0.50000	0.35355300
2	0.25000	0.21022400
3	0.12500	0.09638820
4	0.06250	0.02992510
5	0.03125	0.00517671

Note: $p_n \to 0$ and $q_n \to 0$ but the q_n sequence is converging "faster".

Convergence of Iterative Techniques

Remark: most of the root-finding techniques we have considered converge only linearly.

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Theorem

Suppose $g \in \mathcal{C}[a, b]$ *and* $g(x) \in [a, b]$ *for all* $x \in [a, b]$ *and* $g' \in \mathcal{C}(a, b)$ *with*

$$|g'(x)| \le k < 1 \quad \text{for all } x \in (a, b).$$

If $g'(p) \neq 0$ *then for any* $p_0 \neq p$ *in* $[a, b]$*, the sequence* $p_n = g(p_{n-1})$*,* $n \ge 1$ *converges linearly to the unique fixed point* p *in* $[a, b]$*.*

Proof (1 of 2)

- The Fixed-Point Theorem asserts the sequence converges to the unique fixed point p.
- According to the MVT

$$\begin{aligned} \frac{p_{n+1}-p}{p_n-p} &= \frac{g(p_n)-g(p)}{p_n-p} \\ &= g'(z_n) \\ p_{n+1}-p &= g'(z_n)(p_n-p) \end{aligned}$$

  where z_n lies between p and p_n.
- Since $p_n \to p$ as $n \to \infty$, then $z_n \to p$ by the Squeeze Theorem.

Proof (2 of 2)

Since g' is continuous

$$\lim_{n\to\infty} g'(z_n) = g'\left(\lim_{n\to\infty} z_n\right) = g'(p).$$

Therefore

$$\begin{aligned}
\lim_{n\to\infty} \frac{|p_{n+1} - p|}{|p_n - p|^1} &= \lim_{n\to\infty} \frac{|g(p_n) - g(p)|}{|p_n - p|} \\
&= \lim_{n\to\infty} |g'(z_n)| \\
&= |g'(p)| > 0.
\end{aligned}$$

Improving Convergence

Remark: we can only have faster than linear convergence when $g'(p) = 0$.

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Strategy: given an equation $f(x) = 0$ with an unknown root p in the interval $[a, b]$, we want to write down an equivalent fixed-point problem $p = g(p)$ such that $g'(p) = 0$.

Useful Result

Theorem

Let p be a solution of $x = g(x)$ and suppose that $g'(p) = 0$. Suppose further that g'' is continuous with $|g''(x)| < M$ on an open interval I containing p. Then there exists a $\delta > 0$ such that for $p_0 \in [p - \delta, p + \delta]$, the sequence defined by $p_n = g(p_{n-1})$ for $n \geq 1$ converges at least quadratically to p. Moreover, for large values of n,

$$|p_{n+1} - p_n| < \frac{M}{2}|p_n - p|^2.$$

Proof (1 of 4)

- Choose $k \in (0, 1)$ and $\delta > 0$ so that
  - $[p - \delta, p + \delta] \subseteq I$,
  - $|g'(x)| \leq k$, and
  - g'' is continuous.

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  - g'' is continuous.
- As in a previous proof, $p_n \in [p - \delta, p + \delta]$ for $n = 0, 1, \ldots$.
- Find the linear Taylor polynomial for $g(x)$ expanded about p.

Proof (2 of 4)

$$g(x) = g(p) + g'(p)(x-p) + \frac{g''(z(x))}{2}(x-p)^2$$

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p_{n+1} - p &= \frac{g''(z(p_n))}{2}(p_n-p)^2
\end{aligned}
$$

where $z(p_n)$ lies between p_n and p.

Proof (3 of 4)

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\begin{aligned}
p_{n+1} - p &= \frac{g''(z(p_n))}{2}(p_n - p)^2 \\
\frac{|p_{n+1} - p|}{|p_n - p|^2} &= \frac{|g''(z(p_n))|}{2}
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\lim_{n\to\infty} \frac{|p_{n+1} - p|}{|p_n - p|^2} &= \frac{|g''(p)|}{2}
\end{aligned}
$$

by the Squeeze Theorem.

Conclusion: sequence $\{p_n\}_{n=0}^{\infty}$ is quadratically convergent if $g''(p) \neq 0$ and has higher order convergence if $g''(p) = 0$.

Proof (4 of 4)

Recall that

$$|p_{n+1} - p| = \frac{|g''(z(p_n))|}{2} |p_n - p|^2.$$

Since $|g''(x)| < M$ for $x \in [p - \delta, p + \delta]$, then

$$|p_{n+1} - p| < \frac{M}{2} |p_n - p|^2.$$

Developing the Method

- Given that $f(x) = 0$ has a solution p, define

$$g(x) = x - \phi(x)\, f(x).$$

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- g has a fixed point at p.
- We want to choose $\phi(x)$ so that $g'(p) = 0$.

$$\begin{aligned} g'(x) &= 1 - \phi'(x)\, f(x) - \phi(x)\, f'(x) \\ g'(p) &= 1 - \phi'(p)\, f(p) - \phi(p)\, f'(p) \\ 0 &= 1 - \phi(p)\, f'(p) \\ \phi(p) &= \frac{1}{f'(p)} \end{aligned}$$

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- Let $\phi(x) = \dfrac{1}{f'(x)}$ and then

$$g(x) = x - \frac{f(x)}{f'(x)} \quad \Longrightarrow \quad p_n = p_{n-1} - \frac{f(p_{n-1})}{f'(p_{n-1})}$$

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$$g(x) = x - \frac{f(x)}{f'(x)} \quad \Longrightarrow \quad p_n = p_{n-1} - \frac{f(p_{n-1})}{f'(p_{n-1})}$$

Newton's Method is quadratically convergent!

Multiple Roots

Remark: Newton's Method will run into trouble if $f'(x) = 0$.

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Definition

A solution p of $f(x) = 0$ is a **root of multiplicity** m if $f(x)$ can be written as

$$f(x) = (x - p)^m q(x)$$

for $x \neq p$ and where $\lim_{x \to p} q(x) \neq 0$.

A root of multiplicity 1 is called a **simple root**.

Determining the Multiplicity

Theorem

Function $f \in \mathcal{C}[a, b]$ *has a simple root at* $p \in (a, b)$ *if and only if* $f(p) = 0$ *and* $f'(p) \neq 0$*.*

Theorem

Function $f \in \mathcal{C}^m[a, b]$ *has a root of multiplicity* m *at* p *if and only if*

$$0 = f(p) = f'(p) = \cdots = f^{(m-1)}(p)$$

but $f^{(m)}(p) \neq 0$*.*

Example (1 of 2)

Each of the following functions has a root at $p = 1$. Determine the multiplicity of this root for each function.

$$
\begin{aligned}
f(x) &= x^3 - 4x^2 + 5x - 2 \\
g(x) &= x^3 - 6x^2 + 11x - 6
\end{aligned}
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n	$f^{(n)}(x)$	$f^{(n)}(1)$	$g^{(n)}(x)$	$g^{(n)}(1)$
1	$3x^2 - 8x + 5$	0	$3x^2 - 12x + 11$	2
2	$6x - 8$	-2	$6x - 12$	-6
3	6	6	6	6

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2	$6x - 8$	-2	$6x - 12$	-6
3	6	6	6	6

Conclusion: $f(x)$ has a root of multiplicity 2 at $x = 1$ while $g(x)$ has a root of multiplicity 1.

Example (2 of 2)

Using an initial approximation of $p_0 = 0.5$ to the root $p = 1$, use Newton's Method to approximate the root with $\epsilon = 10^{-2}$ for each of the following functions.

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n	$f(x)$ Multiplicity 2 p_n	$g(x)$ Multiplicity 1 p_n
0	0.5000	0.5000
1	0.7143	0.8261
2	0.8429	0.9677
3	0.9164	0.9985
4	0.9567	1.0000
5	0.9779	1.0000

Remark: Newton's Method applied to the function with the root of higher multiplicity converges slower.

Improving Convergence at Roots of High Multiplicity

Define $\mu(x) = \dfrac{f(x)}{f'(x)}$, then

$$
\begin{aligned}
\mu(x) &= \frac{(x-p)^m q(x)}{m(x-p)^{m-1}q(x) + (x-p)^m q'(x)} \\
&= \frac{(x-p)q(x)}{mq(x) + (x-p)q'(x)}
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\mu'(x) &= \frac{m(q(x))^2 + (x-p)^2(q'(x))^2 - (x-p)^2 q(x)q''(x)}{(mq(x) + (x-p)q'(x))^2}
\end{aligned}
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Observe: $\mu(p) = 0$ but $\mu'(p) \neq 0$.

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\end{aligned}$$

Observe: $\mu(p) = 0$ but $\mu'(p) \neq 0$.

Now use Newton's Method to approximate a root of $\mu(x)$.

$$g(x) = x - \frac{\mu(x)}{\mu'(x)} = x - \frac{f(x)f'(x)}{[f'(x)]^2 - f(x)f''(x)}$$

Example

Compare the number of iterations required to approximate the root $p = 1$ for the function $f(x) = x^3 - 4x^2 + 5x - 2$ starting with $p_0 = 0.5$ using Newton's Method and the modified Newton's Method.

Example

Compare the number of iterations required to approximate the root $p = 1$ for the function $f(x) = x^3 - 4x^2 + 5x - 2$ starting with $p_0 = 0.5$ using Newton's Method and the modified Newton's Method.

n	Original p_n	Modified p_n
0	0.5000	0.5000
1	0.7143	1.0526
2	0.8429	1.0015
3	0.9164	1.0000
4	0.9567	1.0000
5	0.9779	1.0000

Homework

- Read Section 2.4.
- Exercises: 1, 5, 7, 9