Jacobi and Gauss-Seidel Iterative Techniques

MATH 375 *Numerical Analysis*

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Objectives

In this lesson we will learn to

- solve linear systems using Jacobi's method,
- solve linear systems using the Gauss-Seidel method, and
- solve linear systems using general iterative methods.

Background

- For small linear systems direct methods are often as efficient (or even more efficient) than the iterative methods to be discussed today.
- For large linear systems particularly those with sparse matrix representations (matrices with many zero entries), the iterative methods can be more efficient that the direct methods.
- Sparse linear systems are often found in applications such as ordinary and partial differential equarions and circuit analysis.

Initial Approximation

Consider the linear system $A\mathbf{x} = \mathbf{b}$ where A is an $n \times n$ matrix and $\mathbf{b} \in \mathbb{R}^n$.

Given an initial approximation $\mathbf{x}^{(0}$ to the solution of the linear system $\mathbf{x}$, iterative techniques generate a sequence of vectors $\{\mathbf{x}^{(k)}\}_{k=0}^{\infty}$ which converge to the solution $\mathbf{x}$.

Jacobi's Method

Given the linear system $A\mathbf{x} = \mathbf{b}$, if $a_{ii} \neq 0$ solve the ith equation of the system for x_i.

$$\begin{aligned} b_i &= a_{i1}x_1 + \cdots + a_{ii}x_i + \cdots + a_{in}x_n \\ x_i &= \frac{b_i}{a_{ii}} - \sum_{j=1, j\neq i}^{n} \frac{a_{ij}x_j}{a_{ii}} \end{aligned}$$

We will have n equations of this form ($1 \leq i \leq n$).

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We will have n equations of this form $(1 \leq i \leq n)$.

Given $\mathbf{x}^{(k)}$ then

$$x_i^{(k+1)} = \frac{b_i}{a_{ii}} - \sum_{j=1, j\neq i}^{n} \frac{a_{ij}x_j^{(k)}}{a_{ii}}$$

for $1 \leq i \leq n$. The process can be repeated until $\frac{\|\mathbf{x}^{(k)} - \mathbf{x}^{(k-1)}\|_\infty}{\|\mathbf{x}^{(k)}\|_\infty} < \epsilon$.

Example

Use Jacobi's method to approximate the solution to the following linear system. Use $\mathbf{x}^{(0)} = \mathbf{0}$ and let $\epsilon = 10^{-3}$.

$$\begin{aligned} -2x_1 + x_2 + \frac{1}{2}x_3 &= 4 \\ x_1 - 2x_2 - \frac{1}{2}x_3 &= -4 \\ x_2 + 2x_3 &= 0 \end{aligned}$$

For purposes of comparison, the exact solution is

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} -16/11 \\ 16/11 \\ -8/11 \end{bmatrix} \approx \begin{bmatrix} -1.454545 \\ 1.454545 \\ -0.727273 \end{bmatrix}.$$

Solution

$$\begin{aligned}
x_1 &= -2 + \frac{1}{2}x_2 + \frac{1}{4}x_3 \\
x_2 &= 2 + \frac{1}{2}x_1 - \frac{1}{4}x_3 \\
x_3 &= -\frac{1}{2}x_2
\end{aligned}$$

Solution

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x_2 &= 2 + \frac{1}{2}x_1 - \frac{1}{4}x_3 \\
x_3 &= -\frac{1}{2}x_2
\end{aligned}
$$

k	$x_1^{(k)}$	$x_2^{(k)}$	$x_3^{(k)}$
0	0.0000	0.0000	0.0000
1	−2.0000	2.0000	0.0000
2	−1.0000	1.0000	−1.0000
3	−1.2500	1.2500	−0.8750
4	−1.5938	1.5938	−0.6250
⋮	⋮	⋮	⋮
19	−1.4552	1.4552	−0.7268
20	−1.4541	1.4541	−0.7276

Matrix Notation for Jacobi's Method (1 of 2)

Matrix A can be decomposed as $A = D - L - U$ where

$$A = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{bmatrix} \qquad D = \begin{bmatrix} a_{11} & 0 & \cdots & 0 \\ 0 & a_{22} & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & a_{nn} \end{bmatrix}$$

$$L = \begin{bmatrix} 0 & 0 & \cdots & 0 \\ -a_{21} & 0 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ -a_{n1} & -a_{n2} & \cdots & 0 \end{bmatrix} \qquad U = \begin{bmatrix} 0 & -a_{12} & \cdots & -a_{1n} \\ 0 & 0 & \cdots & -a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 0 \end{bmatrix}.$$

Matrix Notation for Jacobi's Method (2 of 2)

$$\begin{aligned} A\mathbf{x} &= \mathbf{b} \\ (D - L - U)\mathbf{x} &= \mathbf{b} \\ D\mathbf{x} &= (L + U)\mathbf{x} + \mathbf{b} \\ \mathbf{x} &= D^{-1}(L + U)\mathbf{x} + D^{-1}\mathbf{b} \end{aligned}$$

assuming $a_{ii} \neq 0$ for $1 \leq i \leq n$.

Matrix Notation for Jacobi's Method (2 of 2)

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\begin{aligned}
A\mathbf{x} &= \mathbf{b} \\
(D - L - U)\mathbf{x} &= \mathbf{b} \\
D\mathbf{x} &= (L + U)\mathbf{x} + \mathbf{b} \\
\mathbf{x} &= D^{-1}(L + U)\mathbf{x} + D^{-1}\mathbf{b}
\end{aligned}
$$

assuming $a_{ii} \neq 0$ for $1 \leq i \leq n$.

- Define $T_j = D^{-1}(L + U)$ and $\mathbf{c}_j = D^{-1}\mathbf{b}$.
- The Jacobi method can be expressed in matrix notation as

$$\mathbf{x}^{(k)} = T_j\mathbf{x}^{(k-1)} + \mathbf{c}_j.$$

Example

Express the following linear system in the Jacobi matrix notation.

$$\begin{aligned} -2x_1 + x_2 + \frac{1}{2}x_3 &= 4 \\ x_1 - 2x_2 - \frac{1}{2}x_3 &= -4 \\ x_2 + 2x_3 &= 0 \end{aligned}$$

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Solution

Let $A = \begin{bmatrix} -2 & 1 & 1/2 \\ 1 & -2 & -1/2 \\ 0 & 1 & 2 \end{bmatrix}$ *and* $\mathbf{b} = \begin{bmatrix} 4 \\ -4 \\ 0 \end{bmatrix}$.

Solution

$$D = \begin{bmatrix} -2 & 0 & 0 \\ 0 & -2 & 0 \\ 0 & 0 & 2 \end{bmatrix}$$

$$L + U = \begin{bmatrix} 0 & -1 & -1/2 \\ -1 & 0 & 1/2 \\ 0 & -1 & 0 \end{bmatrix}$$

$$T_j = D^{-1}(L+U) = \begin{bmatrix} 0 & 1/2 & 1/4 \\ 1/2 & 0 & -1/4 \\ 0 & -1/2 & 0 \end{bmatrix}$$

$$\mathbf{c}_j = D^{-1}\mathbf{b} = \begin{bmatrix} -2 \\ 2 \\ 0 \end{bmatrix}$$

Solution

$$
\begin{aligned}
D &= \begin{bmatrix} -2 & 0 & 0 \\ 0 & -2 & 0 \\ 0 & 0 & 2 \end{bmatrix} \\
L + U &= \begin{bmatrix} 0 & -1 & -1/2 \\ -1 & 0 & 1/2 \\ 0 & -1 & 0 \end{bmatrix} \\
T_j &= D^{-1}(L+U) = \begin{bmatrix} 0 & 1/2 & 1/4 \\ 1/2 & 0 & -1/4 \\ 0 & -1/2 & 0 \end{bmatrix} \\
\mathbf{c}_j &= D^{-1}\mathbf{b} = \begin{bmatrix} -2 \\ 2 \\ 0 \end{bmatrix} \\
\begin{bmatrix} x_1^{(k)} \\ x_2^{(k)} \\ x_3^{(k)} \end{bmatrix} &= \begin{bmatrix} 0 & 1/2 & 1/4 \\ 1/2 & 0 & -1/4 \\ 0 & -1/2 & 0 \end{bmatrix} \begin{bmatrix} x_1^{(k-1)} \\ x_2^{(k-1)} \\ x_3^{(k-1)} \end{bmatrix} + \begin{bmatrix} -2 \\ 2 \\ 0 \end{bmatrix}
\end{aligned}
$$

Improving the Jacobi Method

Recall that in the Jacobi method,

$$x_i^{(k)} = \frac{1}{a_{ii}} \left(b_i - \sum_{j=1, j \neq i}^{n} a_{ij} x_j^{(k-1)} \right).$$

- As designed all the components of $\mathbf{x}^{(k-1)}$ are used to calculate $x_i^{(k)}$.
- When $i > 1$ the components $x_j^{(k)}$ for $1 \leq j < i$ have already been calculated and should be more accurate than the components $x_j^{(k-1)}$ for $1 \leq j < i$.
- We can modify the Jacobi method to use $x_j^{(k)}$ for $1 \leq j < i$ in place of $x_j^{(k-1)}$ to improve the convergence of the algorithm. This modification is known as the **Gauss-Seidel iterative technique**.

Gauss-Seidel Method

$$x_i^{(k)} = \frac{1}{a_{ii}} \left(b_i - \sum_{j=1}^{i-1} a_{ij} x_j^{(k)} - \sum_{j=i+1}^{n} a_{ij} x_j^{(k-1)} \right).$$

Example

Use the Gauss-Seidel method to approximate the solution to the following linear system. Use $\mathbf{x}^{(0)} = \mathbf{0}$ and let $\epsilon = 10^{-3}$.

$$\begin{aligned}
-2x_1 + x_2 + \frac{1}{2}x_3 &= 4 \\
x_1 - 2x_2 - \frac{1}{2}x_3 &= -4 \\
x_2 + 2x_3 &= 0
\end{aligned}$$

Solution

$$
\begin{aligned}
x_1^{(k)} &= -2 + \frac{1}{2}x_2^{(k-1)} + \frac{1}{4}x_3^{(k-1)} \\
x_2^{(k)} &= 2 + \frac{1}{2}x_1^{(k)} - \frac{1}{4}x_3^{(k-1)} \\
x_3^{(k)} &= -\frac{1}{2}x_2^{(k)}
\end{aligned}
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Solution

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\begin{aligned}
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\end{aligned}
$$

k	$x_1^{(k)}$	$x_2^{(k)}$	$x_3^{(k)}$
0	0.0000	0.0000	0.0000
1	−2.0000	1.0000	−0.5000
2	−1.6250	1.3125	−0.6523
3	−1.5078	1.4102	−0.7051
4	−1.4712	1.4407	−0.7203
5	−1.4598	1.4502	−0.7251
6	−1.4562	1.4532	−0.7266
7	−1.4551	1.4541	−0.7271

Gauss-Seidel Method in Matrix Form (1 of 2)

$$
\begin{aligned}
x_i^{(k)} &= \frac{1}{a_{ii}}\left(b_i - \sum_{j=1}^{i-1} a_{ij}x_j^{(k)} - \sum_{j=i+1}^{n} a_{ij}x_j^{(k-1)}\right) \\
a_{ii}x_i^{(k)} &= b_i - \sum_{j=1}^{i-1} a_{ij}x_j^{(k)} - \sum_{j=i+1}^{n} a_{ij}x_j^{(k-1)} \\
a_{ii}x_i^{(k)} + \sum_{j=1}^{i-1} a_{ij}x_j^{(k)} &= b_i - \sum_{j=i+1}^{n} a_{ij}x_j^{(k-1)}
\end{aligned}
$$

Gauss-Seidel Method in Matrix Form (2 of 2)

Since for $i = 1, 2, \ldots, n$,

$$a_{ii}x_i^{(k)} + \sum_{j=1}^{i-1} a_{ij}x_j^{(k)} = b_i - \sum_{j=i+1}^{n} a_{ij}x_j^{(k-1)},$$

we can express the linear system as follows:

$$\begin{aligned}
a_{11}x_1^{(k)} &= b_1 - a_{12}x_2^{(k-1)} - a_{13}x_3^{(k-1)} - \cdots - a_{1n}x_n^{(k-1)} \\
a_{21}x_1^{(k)} + a_{22}x_2^{(k)} &= b_2 - a_{23}x_2^{(k-1)} - a_{24}x_3^{(k-1)} - \cdots - a_{2n}x_n^{(k-1)} \\
&\vdots \\
a_{n1}x_1^{(k)} + a_{n2}x_2^{(k)} + \cdots + a_{nn}x_n^{(k)} &= b_n
\end{aligned}$$

This is equivalent to the matrix form

$$\begin{aligned}
(D - L)\mathbf{x}^{(k)} &= \mathbf{b} + U\mathbf{x}^{(k-1)} \\
\mathbf{x}^{(k)} &= (D - L)^{-1}\mathbf{b} + (D - L)^{-1}U\mathbf{x}^{(k-1)} \\
\mathbf{x}^{(k)} &= \mathbf{c}_g + T_g\mathbf{x}^{(k-1)}.
\end{aligned}$$

Example

Express the following linear system in the Gauss-Seidel matrix notation.

$$\begin{aligned} -2x_1 + x_2 + \frac{1}{2}x_3 &= 4 \\ x_1 - 2x_2 - \frac{1}{2}x_3 &= -4 \\ x_2 + 2x_3 &= 0 \end{aligned}$$

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Express the following linear system in the Gauss-Seidel matrix notation.

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Solution

Let $A = \begin{bmatrix} -2 & 1 & 1/2 \\ 1 & -2 & -1/2 \\ 0 & 1 & 2 \end{bmatrix}$ *and* $\mathbf{b} = \begin{bmatrix} 4 \\ -4 \\ 0 \end{bmatrix}$.

Solution

$$
\begin{aligned}
D - L &= \begin{bmatrix} -2 & 0 & 0 \\ 1 & -2 & 0 \\ 0 & 1 & 2 \end{bmatrix} \\
(D - L)^{-1} &= \begin{bmatrix} -1/2 & 0 & 0 \\ -1/4 & -1/2 & 0 \\ 1/8 & 1/4 & 1/2 \end{bmatrix} \\
T_g &= (D - L)^{-1} U = \begin{bmatrix} 0 & 1/2 & 1/4 \\ 0 & 1/4 & -1/8 \\ 0 & -1/8 & 1/16 \end{bmatrix} \\
\mathbf{c}_g &= (D - L)^{-1} \mathbf{b} = \begin{bmatrix} -2 \\ 1 \\ -1/2 \end{bmatrix}
\end{aligned}
$$

Solution

$$D - L = \begin{bmatrix} -2 & 0 & 0 \\ 1 & -2 & 0 \\ 0 & 1 & 2 \end{bmatrix}$$

$$(D - L)^{-1} = \begin{bmatrix} -1/2 & 0 & 0 \\ -1/4 & -1/2 & 0 \\ 1/8 & 1/4 & 1/2 \end{bmatrix}$$

$$T_g = (D - L)^{-1} U = \begin{bmatrix} 0 & 1/2 & 1/4 \\ 0 & 1/4 & -1/8 \\ 0 & -1/8 & 1/16 \end{bmatrix}$$

$$\mathbf{c}_g = (D - L)^{-1} \mathbf{b} = \begin{bmatrix} -2 \\ 1 \\ -1/2 \end{bmatrix}$$

$$\begin{bmatrix} x_1^{(k)} \\ x_2^{(k)} \\ x_3^{(k)} \end{bmatrix} = \begin{bmatrix} 0 & 1/2 & 1/4 \\ 0 & 1/4 & -1/8 \\ 0 & -1/8 & 1/16 \end{bmatrix} \begin{bmatrix} x_1^{(k-1)} \\ x_2^{(k-1)} \\ x_3^{(k-1)} \end{bmatrix} + \begin{bmatrix} -2 \\ 1 \\ -1/2 \end{bmatrix}$$

General Iteration Methods

We have seen that we can express an iterative method for the solution of a linear system in the form:

$$\mathbf{x}^{(k)} = T\,\mathbf{x}^{(k-1)} + \mathbf{c}$$

for $k = 1, 2, \ldots$ where $\mathbf{x}^{(0)}$ is arbitrary.

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We have seen that we can express an iterative method for the solution of a linear system in the form:

$$\mathbf{x}^{(k)} = T\,\mathbf{x}^{(k-1)} + \mathbf{c}$$

for $k = 1, 2, \ldots$ where $\mathbf{x}^{(0)}$ is arbitrary.

We must now establish conditions under which this iterative method will converge to the unique solution of the system $A\mathbf{x} = \mathbf{b}$.

Important Lemma

Lemma

If $\rho(T) < 1$ *then* $(I - T)^{-1}$ *exists and*

$$(I - T)^{-1} = I + T + T^2 + \cdots = \sum_{j=0}^{\infty} T^j.$$

Proof (1 of 2)

$$\begin{aligned} T\mathbf{x} &= \lambda\mathbf{x} \\ \mathbf{x} - T\mathbf{x} &= \mathbf{x} - \lambda\mathbf{x} \\ (I - T)\mathbf{x} &= (1 - \lambda)\mathbf{x} \end{aligned}$$

- Thus λ is an eigenvalue of T if and only if $1 - \lambda$ is an eigenvalue of $I - T$.
- If $\rho(T) < 1$ then for any eigenvalue λ of T, $|\lambda| < 1$, therefore $\lambda \neq 1$.
- Hence $1 - 1 = 0$ cannot be an eigenvalue of $I - T$ which implies $I - T$ is nonsingular.

Proof (2 of 2)

Define $S_m = I + T + T^2 + \cdots + T^m$ for $m = 1, 2, \ldots$.

$$\begin{aligned}
(I - T)S_m &= (I + T + T^2 + \cdots + T^m) - (T + T^2 + \cdots + T^{m+1}) \\
&= I - T^{m+1} \\
\lim_{m\to\infty}(I - T)S_m &= \lim_{m\to\infty}(I - T^{m+1}) \\
(I - T)\lim_{m\to\infty} S_m &= I \text{ (since } T \text{ is convergent)}
\end{aligned}$$

Consequently $(I - T)^{-1} = \sum_{j=0}^{\infty} T^j$.

Convergence of Iterative Methods

Theorem

For any $\mathbf{x}^{(0)} \in \mathbb{R}^n$ *the sequence* $\{\mathbf{x}^{(k)}\}_{k=0}^{\infty}$ *defined by*

$$\mathbf{x}^{(k)} = T\mathbf{x}^{(k-1)} + \mathbf{c} \text{ for } k = 1, 2, \ldots$$

converges to the unique solution of $\mathbf{x} = T\mathbf{x} + \mathbf{c}$ *if and only if* $\rho(T) < 1$.

Proof (1 of 4)

Suppose $\rho(T) < 1$, then by assumption

$$
\begin{aligned}
\mathbf{x}^{(k)} &= T\,\mathbf{x}^{(k-1)} + \mathbf{c} \\
&= T(T\,\mathbf{x}^{(k-2)} + \mathbf{c}) + \mathbf{c} \\
&= T^2\mathbf{x}^{(k-2)} + (I + T)\mathbf{c} \\
&\vdots \\
\mathbf{x}^{(k)} &= T^k\mathbf{x}^{(0)} + (I + T + \cdots T^{k-1})\mathbf{c} \\
\lim_{k\to\infty} \mathbf{x}^{(k)} &= \lim_{k\to\infty} \left[T^k\mathbf{x}^{(0)} + (I + T + \cdots T^{k-1})\mathbf{c}\right] \\
&= \left(\lim_{k\to\infty} T^k\right)\mathbf{x}^{(0)} + \left(\sum_{j=0}^{\infty} T^j\right)\mathbf{c} \\
&= \mathbf{0} + (I - T)^{-1}\mathbf{c} \text{ (since } T \text{ is convergent)} \\
\mathbf{x} &= (I - T)^{-1}\mathbf{c}.
\end{aligned}
$$

Proof (2 of 4)

So far we know the sequence $\{\mathbf{x}^{(k)}\}_{k=0}^{\infty}$ converges to

$$\mathbf{x} = (I - T)^{-1}\mathbf{c}$$

Proof (2 of 4)

So far we know the sequence $\{\mathbf{x}^{(k)}\}_{k=0}^{\infty}$ converges to

$$\begin{aligned} \mathbf{x} &= (I - T)^{-1}\mathbf{c} \\ (I - T)\mathbf{x} &= \mathbf{c} \\ \mathbf{x} &= T\,\mathbf{x} + \mathbf{c}. \end{aligned}$$

Hence **x** is a solution to the linear system.

Proof (3 of 4)

- To prove the converse, let $\mathbf{z}$ be any vector in $\mathbb{R}^n$.
- Let $\mathbf{x}$ be the unique solution to the linear system $\mathbf{x} = T\mathbf{x} + \mathbf{c}$.

Proof (3 of 4)

- To prove the converse, let $\mathbf{z}$ be any vector in $\mathbb{R}^n$.
- Let $\mathbf{x}$ be the unique solution to the linear system $\mathbf{x} = T\mathbf{x} + \mathbf{c}$.
- Define $\mathbf{x}^{(0} = \mathbf{x} - \mathbf{z}$ and for $k \geq 1$ define $\mathbf{x}^{(k)} = T\mathbf{x}^{(k-1)} + \mathbf{c}$.

Proof (3 of 4)

- To prove the converse, let $\mathbf{z}$ be any vector in $\mathbb{R}^n$.
- Let $\mathbf{x}$ be the unique solution to the linear system $\mathbf{x} = T\mathbf{x} + \mathbf{c}$.
- Define $\mathbf{x}^{(0} = \mathbf{x} - \mathbf{z}$ and for $k \geq 1$ define $\mathbf{x}^{(k)} = T\mathbf{x}^{(k-1)} + \mathbf{c}$.
- By assumption $\lim_{k\to\infty} \mathbf{x}^{(k)} = \mathbf{x}$.

Proof (4 of 4)

$$\begin{aligned}
\mathbf{x} - \mathbf{x}^{(k)} &= (T\mathbf{x} + \mathbf{c}) - (T\mathbf{x}^{(k-1)} + \mathbf{x}) \\
&= T(\mathbf{x} - \mathbf{x}^{(k-1)}) \\
&= T\left(T(\mathbf{x} - \mathbf{x}^{(k-2)})\right) \\
&\vdots \\
\mathbf{x} - \mathbf{x}^{(k)} &= T^k(\mathbf{x} - \mathbf{x}^{(0)}) = T^k\mathbf{z}
\end{aligned}$$

Since $\mathbf{z}$ is arbitrary then T is a convergent matrix and hence $\rho(T) < 1$.

Proof (4 of 4)

$$
\begin{aligned}
\mathbf{x} - \mathbf{x}^{(k)} &= (T\mathbf{x} + \mathbf{c}) - (T\mathbf{x}^{(k-1)} + \mathbf{x}) \\
&= T(\mathbf{x} - \mathbf{x}^{(k-1)}) \\
&= T\left(T(\mathbf{x} - \mathbf{x}^{(k-2)})\right) \\
&\vdots \\
\mathbf{x} - \mathbf{x}^{(k)} &= T^k(\mathbf{x} - \mathbf{x}^{(0)}) = T^k\mathbf{z} \\
\lim_{k\to\infty} (\mathbf{x} - \mathbf{x}^{(k)}) &= \lim_{k\to\infty} T^k\mathbf{z} \\
\mathbf{0} &= \lim_{k\to\infty} T^k\mathbf{z}
\end{aligned}
$$

Since $\mathbf{z}$ is arbitrary then T is a convergent matrix and hence $\rho(T) < 1$.

Error Bounds

Corollary

If $\|T\| < 1$ *for any natural matrix norm and* $\mathbf{c}$ *is a fixed vector, then the sequence* $\{\mathbf{x}^{(k)}\}_{k=0}^{\infty}$ *defined by* $\mathbf{x}^{(k)} = T\mathbf{x}^{(k-1)} + \mathbf{c}$ *converges for any* $\mathbf{x}^{(0)}$ *to a vector* $\mathbf{x} \in \mathbb{R}^n$ *with* $\mathbf{x} = T\mathbf{x} + \mathbf{c}$ *with the following error bounds.*

1. $\|\mathbf{x} - \mathbf{x}^{(k)}\| \le \|T\|^k \|\mathbf{x}^{(0)} - \mathbf{x}\|$
2. $\|\mathbf{x} - \mathbf{x}^{(k)}\| \le \dfrac{\|T\|^k}{1 - \|T\|} \|\mathbf{x}^{(1)} - \mathbf{x}^{(0)}\|$

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1. $\|\mathbf{x} - \mathbf{x}^{(k)}\| \leq \|T\|^k \|\mathbf{x}^{(0)} - \mathbf{x}\|$
2. $\|\mathbf{x} - \mathbf{x}^{(k)}\| \leq \dfrac{\|T\|^k}{1 - \|T\|} \|\mathbf{x}^{(1)} - \mathbf{x}^{(0)}\|$

Remark: if we can show that $\rho(T_j) < 1$ and $\rho(T_g) < 1$ then the Jacobi and Gauss-Seidel methods will always converge to the unique solution of the linear system.

Diagonal Dominance

Theorem

If matrix A is strictly diagonally dominant, then for any choice of $\mathbf{x}^{(0)}$, both the Jacobi and Gauss-Seidel methods produce sequences $\{\mathbf{x}^{(k)}\}_{k=0}^{\infty}$ that converge to the unique solution of $A\mathbf{x} = \mathbf{b}$.

Stein-Rosenberg Theorem

Theorem (Stein-Rosenberg)

If $a_{ij} \leq 0$ for each $i \neq j$ and $a_{ii} > 0$ for $i = 1, 2, \ldots, n$ then exactly one of the following statements is true.

1. $0 \leq \rho(T_g) < \rho(T_j) < 1$
2. $0 = \rho(T_g) = \rho(T_j)$
3. $1 < \rho(T_j) < \rho(T_g)$
4. $1 = \rho(T_g) < \rho(T_j)$

Homework

- Read Section 7.3.
- Exercises: 1ac, 3ac, 5ac, 7ac