Interpolation and the Lagrange Polynomial

MATH 375

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Introduction

We often choose **polynomials** to approximate other classes of functions.

Theorem (Weierstrass Approximation Theorem)

If $f \in \mathcal{C}[a, b]$ and $\epsilon > 0$ then there exists a polynomial P such that

$$|f(x) - P(x)| < \epsilon, \text{ for all } x \in [a, b].$$

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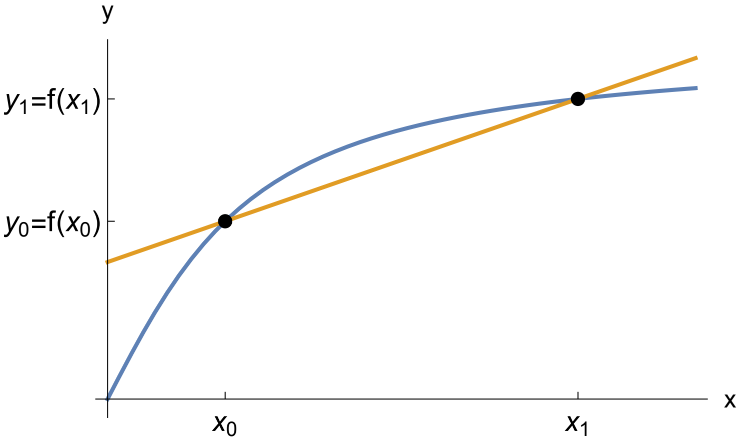
$$|f(x) - P(x)| < \epsilon, \text{ for all } x \in [a, b].$$

- We have used **Taylor polynomials** to approximate functions.

$$P_n(x) = \sum_{k=0}^{n} \frac{f^{(k)}(x_0)}{k!}(x - x_0)^k$$

- Away from x_0 the approximation may be very poor.

Linear Approximation (1 of 2)



Using the two-point formula for a line we determine the equation of the line to be

$$y = f(x_0) + \left(\frac{f(x_1) - f(x_0)}{x_1 - x_0} \right) (x - x_0).$$

Linear Approximation (2 of 2)

We can take another approach to finding the linear approximation. Define

- $L_0(x) = \dfrac{x - x_1}{x_0 - x_1}$,
- $L_1(x) = \dfrac{x - x_0}{x_1 - x_0}$,

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If $f(x)$ is any function then

$$P_1(x) = f(x_0)\,L_0(x) + f(x_1)\,L_1(x)$$

is a linear function which matches $f(x)$ at $x = x_0$ and $x = x_1$.

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We want to extend this idea to higher order polynomials.

Construction of Polynomials

Challenge: given a function $f(x)$, construct a polynomial $P_n(x)$ of degree at most n which matches $f(x)$ at $n+1$ distinct points.

$$\{(x_0, f(x_0)),\ (x_1, f(x_1)),\ \ldots,\ (x_n, f(x_n))\}$$

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Strategy: construct $n+1$ polynomials $L_{n,k}(x)$ of degree n with the property that

$$L_{n,k}(x_i) = \begin{cases} 0 & \text{if } k \neq i, \\ 1 & \text{if } k = i \end{cases}$$

for $k = 0, 1, 2, \ldots, n$.

Lagrange Basis Polynomials

Definition

Suppose $\{x_0, x_1, \ldots, x_n\}$ is a set of $n+1$ distinct points. A **Lagrange Basis Polynomial** is a polynomial of degree n having the form

$$L_{n,k}(x) = \prod_{\substack{i=0 \\ i \neq k}}^{n} \frac{x - x_i}{x_k - x_i}.$$

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Remarks:

- We will let $k = 0, 1, 2, \ldots, n$.
- $L_{n,k}(x_i) = 1$ for $i = k$.
- $L_{n,k}(x_i) = 0$ for $i \neq k$.

Example

If $x_i = i$ for $i = 0, 1, 2$ then we can create three different Lagrange Basis Polynomials.

$$L_{2,0}(x) =$$

$$L_{2,1}(x) =$$

$$L_{2,2}(x) =$$

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$$L_{2,0}(x) = \frac{x-1}{0-1} \cdot \frac{x-2}{0-2} = \frac{1}{2}(x-1)(x-2)$$

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Graph

$$L_{2,0}(x) = \frac{1}{2}(x-1)(x-2)$$
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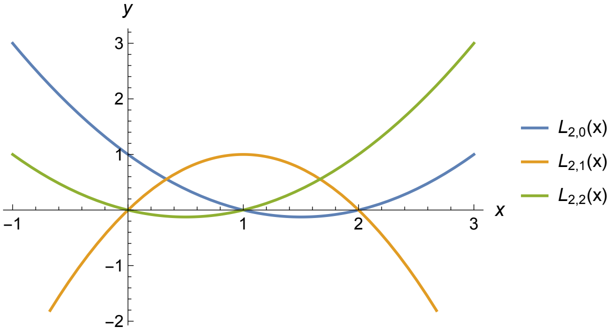
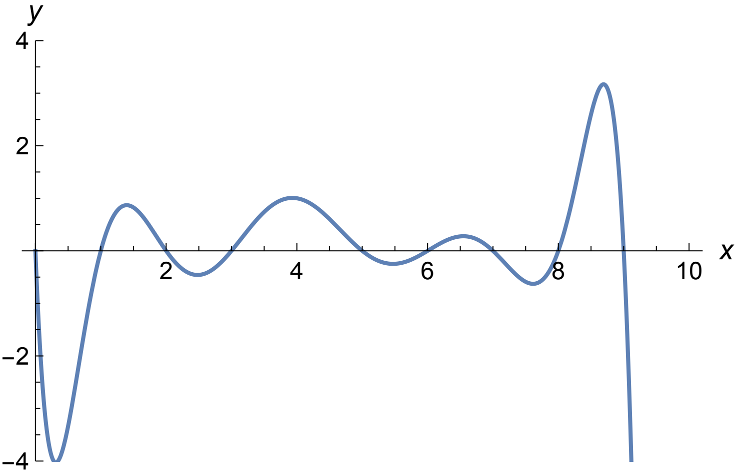


Example

If $n = 9$ and the nodes are $\{0, 1, \ldots, 9\}$ then $L_{9,4}(x)$ has a graph resembling the following.



Lagrange Interpolating Polynomials

Definition

Let $\{x_0, x_1, \ldots, x_n\}$ be a set of $n+1$ distinct points at which the function $f(x)$ is defined. The (unique) **Lagrange Interpolating Polynomial** of $f(x)$ of degree n is the polynomial having the form

$$P_n(x) = \sum_{k=0}^{n} f(x_k) L_{n,k}(x).$$

Properties:

- $L_{n,k}(x_i) = 0$ for all $i \neq k$.
- $L_{n,k}(x_k) = 1$.
- $P(x_k) = f(x_k)$ for $k = 0, 1, \ldots, n$.

Example (1 of 4)

Let $f(x) = \sin x$, $n = 3$, and $x_k = k\pi/3$ for $k = 0, 1, 2, 3$. The four Lagrange Basis Polynomials are listed below.

$$L_{3,0}(x) = \frac{x - \frac{\pi}{3}}{0 - \frac{\pi}{3}} \cdot \frac{x - \frac{2\pi}{3}}{0 - \frac{2\pi}{3}} \cdot \frac{x - \pi}{0 - \pi}$$

$$= \frac{1}{2\pi^3}(\pi - 3x)(2\pi - 3x)(\pi - x)$$

$$L_{3,1}(x) = \frac{9}{2\pi^3}x(2\pi - 3x)(\pi - x)$$

$$L_{3,2}(x) = -\frac{9}{2\pi^3}x(\pi - 3x)(\pi - x)$$

$$L_{3,3}(x) = \frac{1}{2\pi^3}x(\pi - 3x)(2\pi - 3x)$$

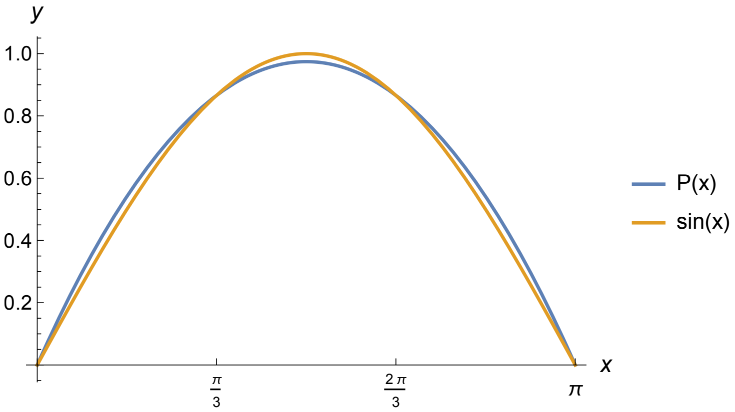
Example (2 of 4)

The Lagrange Interpolating Polynomial of degree 3 for $f(x) = \sin x$ and $x_k = k\pi/3$ for $k = 0, 1, 2, 3$ is

$$\begin{aligned} P_3(x) = &\ (\sin 0)\, L_{3,0}(x) + \left(\sin \frac{\pi}{3}\right) L_{3,1}(x) + \left(\sin \frac{2\pi}{3}\right) L_{3,2}(x) \\ &+ (\sin \pi)\, L_{3,3}(x) \\ &= \frac{\sqrt{3}}{2} \cdot \frac{9}{2\pi^3} x(2\pi - 3x)(\pi - x) - \frac{\sqrt{3}}{2} \cdot \frac{9}{2\pi^3} x(\pi - 3x)(\pi - x) \\ &= \frac{9\sqrt{3}}{4\pi^2} x(\pi - x) \end{aligned}$$

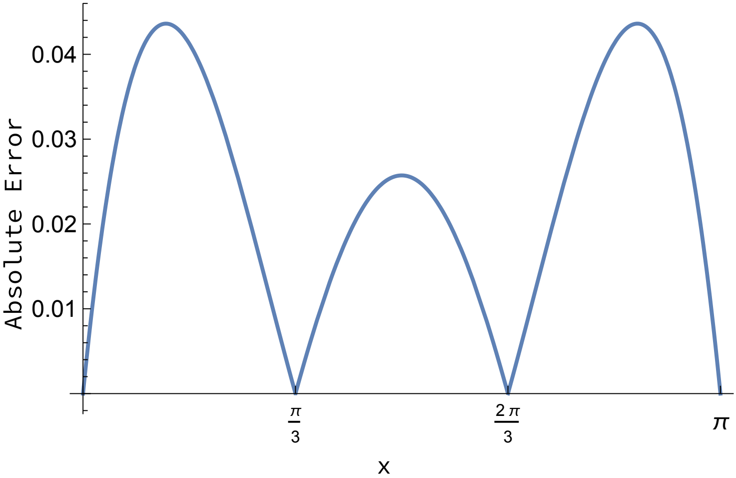
Example (3 of 4)

The graphs of $P_3(x)$ and $\sin x$.



Example (4 of 4)

The graph of $|P_3(x) - \sin x|$.



Lagrange Polynomial with Error Term

Theorem

Suppose $x_0, x_1, \ldots, x_n$ are distinct numbers in the interval $[a, b]$ and suppose $f \in \mathcal{C}^{n+1}[a, b]$. Then for each $x \in [a, b]$ there exists a number $z(x) \in (a, b)$ for which

$$f(x) = P_n(x) + \frac{f^{(n+1)}(z(x))}{(n+1)!}(x - x_0)(x - x_1)\cdots(x - x_n),$$

where $P_n(x)$ is the Lagrange Interpolating Polynomial.

Proof (1 of 3)

Define the function $\Lambda(t)$ as follows:

$$\Lambda(t) = (P_n(x) - f(x)) \frac{(t - x_0)(t - x_1) \cdots (t - x_n)}{(x - x_0)(x - x_1) \cdots (x - x_n)}.$$

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Similarly define function $g(t)$ as

$$\begin{aligned} g(t) &= P(t) - f(t) - \Lambda(t) \\ &= (P(t) - f(t)) - (P_n(x) - f(x)) \frac{(t - x_0)(t - x_1) \cdots (t - x_n)}{(x - x_0)(x - x_1) \cdots (x - x_n)}. \end{aligned}$$

Note:

- $g(x_i) = 0$ for $i = 0, 1, 2, \ldots, n$
- $g(t), g'(t), \ldots, g^{(n)}(t)$ are continuous on $[a, b]$
- $g^{(n+1)}(t)$ exists for interval (a, b).

Proof (2 of 3)

Theorem (Generalized Rolle's)

Suppose $f \in \mathcal{C}[a, b]$ is n times differentiable on (a, b). If f has $n + 1$ distinct roots

$$a \leq x_0 < x_1 < \cdots < x_n \leq b,$$

then there exists $c \in (x_0, x_n) \subset (a, b)$ such that $f^{(n)}(c) = 0$.

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Applying Generalized Rolle's Theorem to function $g(t)$, there is a $z(x) \in (a,b)$ such that $g^{(n+1)}(z(x)) = 0$.

$$\begin{aligned} g^{(n+1)}(z(x)) &= P^{(n+1)}(z(x)) - f^{(n+1)}(z(x)) - \Lambda^{(n+1)}(z(x)) \\ 0 &= 0 - f^{(n+1)}(z(x)) - \Lambda^{(n+1)}(z(x)) \\ f^{(n+1)}(z(x)) &= -\Lambda^{(n+1)}(z(x)) \end{aligned}$$

Proof (3 of 3)

Function $\Lambda(t)$ is a polynomial of degree $n+1$ in variable t.

$$\Lambda(t) = \frac{P_n(x) - f(x)}{(x - x_0)(x - x_1)\cdots(x - x_n)}\left[(t - x_0)(t - x_1)\cdots(t - x_n)\right]$$

$$\Lambda^{(n+1)}(t) = \frac{P_n(x) - f(x)}{(x - x_0)(x - x_1)\cdots(x - x_n)}(n+1)!$$

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$$\Lambda^{(n+1)}(t) = \frac{P_n(x) - f(x)}{(x-x_0)(x-x_1)\cdots(x-x_n)}(n+1)!$$

$$f^{(n+1)}(z(x)) = \frac{f(x) - P_n(x)}{(x-x_0)(x-x_1)\cdots(x-x_n)}(n+1)!$$

$$f(x) - P_n(x) = \frac{f^{(n+1)}(z(x))}{(n+1)!}(x-x_0)(x-x_1)\cdots(x-x_n)$$

Comments

- Compare the error terms of the Lagrange polynomial and the Taylor polynomial.

  Taylor: $$\frac{f^{(n+1)}(z(x))}{(n+1)!}(x-x_0)^{n+1}$$

  Lagrange: $$\frac{f^{(n+1)}(z(x))}{(n+1)!}(x-x_0)(x-x_1)\cdots(x-x_n)$$

- Lagrange polynomials form the basis of many numerical approximations to derivatives and integrals, and thus the error term is important to understanding the errors present in those approximations.

Error in $\sin x$

Earlier we found the Lagrange interpolating polynomial of degree 3 for $f(x) = \sin x$ using nodes $x_k = k\pi/3$ for $k = 0, 1, 2, 3$ was

$$P_3(x) = \frac{9\sqrt{3}}{4\pi^2} x(\pi - x).$$

What is an error bound for this approximation?

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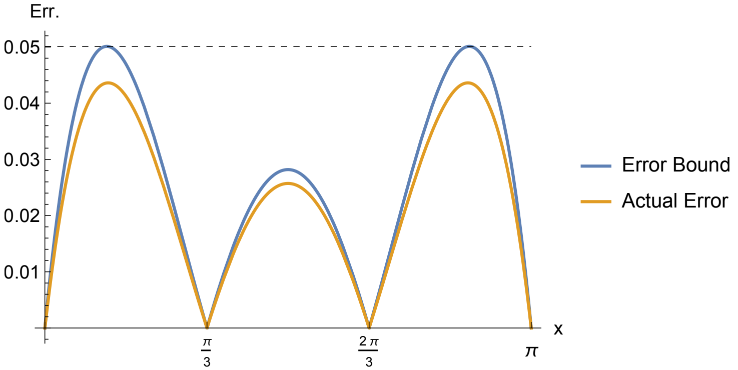
What is an error bound for this approximation?

$$\begin{aligned}
|R(x)| &= \left| \frac{\sin z}{4!}(x - 0)\left(x - \frac{\pi}{3}\right)\left(x - \frac{2\pi}{3}\right)(x - \pi) \right| \\
&\leq \frac{1}{24} \max_{0 \leq x \leq \pi} \left| x\left(x - \frac{\pi}{3}\right)\left(x - \frac{2\pi}{3}\right)(x - \pi) \right| \\
&= \frac{1.20258}{24} = 0.050108
\end{aligned}$$

Error Bound *vs.* Actual Error



Example

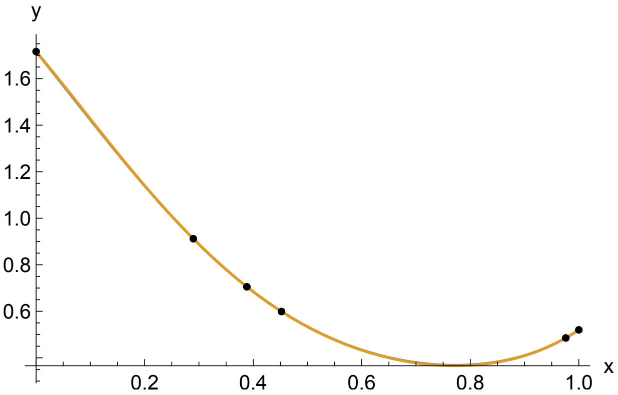
Let $f(x) = e^{\cos(x+1)^2}$ and use the nodes located at

$$\{0, 0.289613, 0.388303, 0.452164, 0.975935, 1\}$$

to find the Lagrange interpolating polynomial of degree at most 5 which interpolates $f(x)$ on the interval $[0, 1]$. Find an error bound for the interpolation.

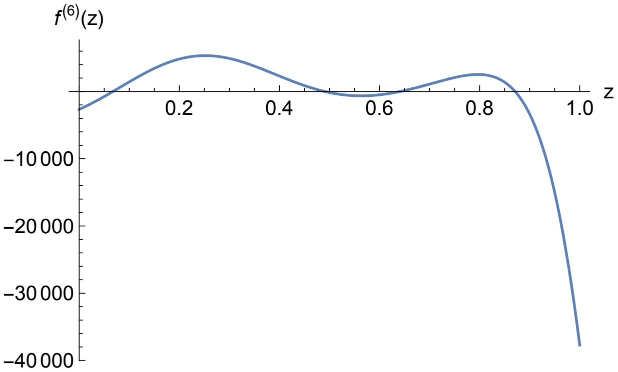
Solution

$$P_5(x) = f(0)L_{5,0}(x) + f(0.289613)L_{5,1}(x) + f(0.388303)L_{5,2}(x) \\ + f(0.452164)L_{5,3}(x) + f(0.975935)L_{5,4}(x) + f(1)L_{5,5}(x)$$



Error Bound

Consider a plot of $f^{(6)}(z)$ on $[0, 1]$.



$$\left| \frac{f^{(6)}(z)}{6!} \prod_{k=0}^{5} (x - x_k) \right| \leq \frac{|f^{(6)}(1)|}{720} \max_{0 \leq x \leq 1} \left| \prod_{k=0}^{5} (x - x_k) \right|$$

$$\leq \frac{(37708.9)(0.002126)}{720} = 0.111322$$

Example

Let $g(x) = \cos e^{(x+1)^2}$ and use the nodes located at

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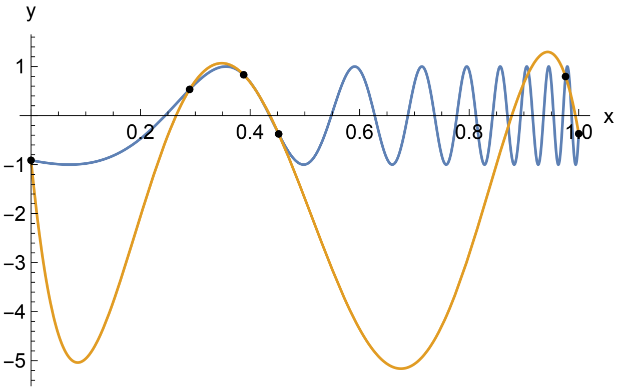

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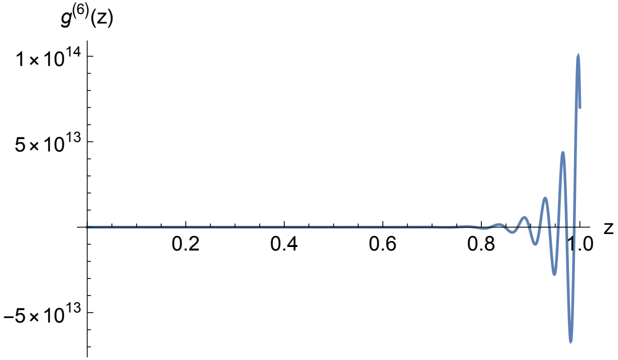

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$$\leq \frac{10^{14}(0.002126)}{720} = 2.95 \times 10^{8}$$

Application

Example

Suppose we are preparing a table of values for $\cos x$ on $[0, \pi]$. The entries in the table will have eight accurate decimal places and we will linearly interpolate between adjacent entries to determine intermediate values. What should the spacing between adjacent x-values be to preserve the eight-decimal-place accuracy in the interpolation?

Solution

$$\begin{aligned}
|\cos x - P_1(x)| &= \left|\frac{-\cos z}{2}\right| |x - x_j||x - x_{j+1}| \quad \text{for some } 0 \le z \le \pi \\
&\le \frac{1}{2} \max_{0 \le z \le \pi} |\cos z| \cdot \max_{x_j \le x \le x_{j+1}} |x - x_j||x - x_{j+1}| \\
&= \frac{1}{2} \max_{jh \le x \le (j+1)h} |(x - jh)(x - (j+1)h| \\
&= \frac{h^2}{8}
\end{aligned}$$

Thus if $h^2/8 < 10^{-8}$ then $h < 2\sqrt{2} \times 10^{-4} \approx 0.000282$.

Connections with Root Finding

Recall the Secant method: given x_0 and x_1 then

$$x_{n+1} = x_n - \frac{f(x_n)(x_n - x_{n-1})}{f(x_n) - f(x_{n-1})}.$$

- The expression $\dfrac{f(x_n) - f(x_{n-1})}{x_n - x_{n-1}}$ is the slope of the secant line through $(x_{n-1}, f(x_{n-1}))$ and $(x_n, f(x_n))$.
- The expression $\dfrac{f(x_n) - f(x_{n-1})}{x_n - x_{n-1}}$ can also be thought of as the derivative of the linear Lagrange interpolating function for $f(x)$ at points $\{(x_{n-1}, f(x_{n-1})), (x_n, f(x_n))\}$.

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A generalization of the Secant method would be **Sidi's method**,

$$x_{n+1} = x_n - \frac{f(x_n)}{P'_{n,k}(x_n)}$$

where $P_{n,k}(x)$ is the Lagrange interpolating polynomial passing through

$$\{(x_{n-k}, f(x_{n-k})), (x_{n-k+1}, f(x_{n-k+1})), \ldots, (x_n, f(x_n))\}.$$

Sidi's Method of degree 3

Find a root of $J_{7/2}(x)$ in the interval $[6, 8]$ using Sidi's method of degree 3 and the four initial approximations at $x \in \{6, 8, 6.5, 7.5\}$. What is the rate of convergence?

Results

n	Newton x_n	Secant x_n	Sidi x_n
0	6.00000	6.00000	6.00000
1	7.16766	8.00000	8.00000
2	6.98412	7.06918	6.50000
3	6.98793	6.96811	7.50000
4	6.98793	6.98806	6.93156
5	6.98793	6.98793	6.98785
6	6.98793	6.98793	6.98793

Rate of Convergence

Consider the sequence of values of $\dfrac{|x_{n+1} - x|}{|x_n - x|^{\alpha}}$ where $x = 6.98793$ for various values of α.

n	x_n	$\alpha = 1$	$\alpha = \frac{1+\sqrt{5}}{2}$	$\alpha = 2$
0	6.00000	1.02443	1.03215	1.03694
1	8.00000	0.48211	0.47855	0.47637
2	6.50000	1.04947	1.63521	2.15084
3	7.50000	0.11009	0.16649	0.21499
4	6.93156	0.00150	0.00887	0.02660
5	6.98785	0.00003	0.01150	0.413517
6	6.98793			

It appears the rate of convergence of Sidi's Method with $k = 3$ is $\dfrac{1+\sqrt{5}}{2} < \alpha < 2$.

Homework

- Read Section 3.1
- Exercises: 1, 3, 5a, 7a, 13, 17