Norms of Vectors and Matrices

MATH 375 *Numerical Analysis*

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Spring 2022

Objectives

- We discussed direct techniques for solving linear systems in Chapter 6.
- In Chapter 7 we will discuss iterative techniques for solving linear systems.
- We have seen iterative methods before when we discussed fixed point methods (notably Newton's method) for solving scalar equations.
- Today we will explore ways of measuring the distance between vectors and also matrices.

Vector Norm

Definition

A **vector norm** on $\mathbb{R}^n$ is a function denoted $\|\cdot\|$ from $\mathbb{R}^n \to \mathbb{R}$ with the following properties:

1. $\|\mathbf{x}\| \geq 0$ for all $\mathbf{x} \in \mathbb{R}^n$.
2. $\|\mathbf{x}\| = 0$ if and only if $\mathbf{x} = \mathbf{0}$.
3. $\|\alpha\,\mathbf{x}\| = |\alpha|\,\|\mathbf{x}\|$ for all $\alpha \in \mathbb{R}$ and for all $\mathbf{x} \in \mathbb{R}^n$.
4. $\|\mathbf{x} + \mathbf{y}\| \leq \|\mathbf{x}\| + \|\mathbf{y}\|$ for all $\mathbf{x}, \mathbf{y} \in \mathbb{R}^n$.

Remark: the last property is called the **triangle inequality**.

Euclidean Norm

There are several functions which possess the four properties of a vector norm.

Definition

The l_2**-norm** of $\mathbf{x} \in \mathbb{R}^n$ is defined as

$$\|\mathbf{x}\|_2 = \left(\sum_{k=1}^{n} x_k^2 \right)^{1/2} .$$

This is also called the **Euclidean norm**.

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The l_∞**-norm** of $\mathbf{x} \in \mathbb{R}^n$ is defined as

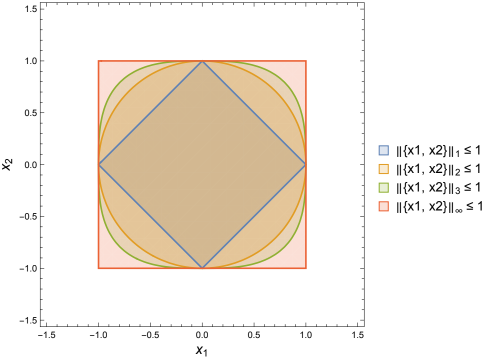
$$\|\mathbf{x}\|_\infty = \max_{1 \le k \le n} |x_k|.$$

l_p-Norm

Definition

For $p \geq 1$ the l_p**-norm** of $\mathbf{x} \in \mathbb{R}^n$ is defined as

$$\|\mathbf{x}\|_p = \left(\sum_{k=1}^{n} |x_k|^p \right)^{1/p} .$$



Example

Let $\mathbf{x} = \langle 1, -2, 1/2 \rangle$ and find

- $\|\mathbf{x}\|_2$
- $\|\mathbf{x}\|_\infty$

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- $\|\mathbf{x}\|_\infty$

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- $\|\mathbf{x}\|_\infty = \max_{1 \le k \le 3}\{|1|, |-2|, |1/2|\} = 2$

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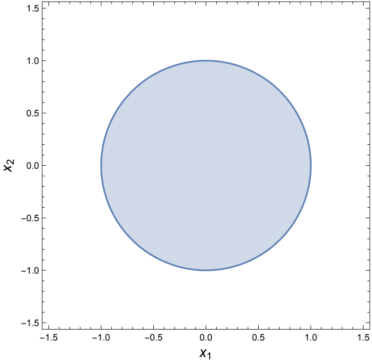
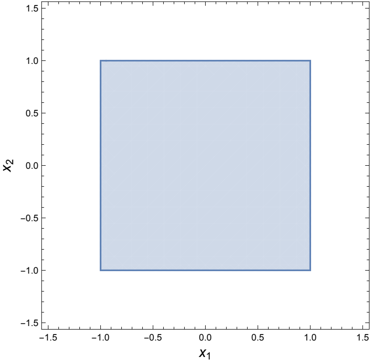
Sketch the vectors in $\mathbb{R}^2$ for which

- $\|\mathbf{x}\|_2 \leq 1$,
- $\|\mathbf{x}\|_\infty \leq 1$.

Solution



$$\|\mathbf{x}\|_2 \leq 1$$



$$\|\mathbf{x}\|_\infty \leq 1$$

Cauchy-Schwarz Inequality

Theorem

For all $\mathbf{x}, \mathbf{y} \in \mathbb{R}^n$*,*

$$\mathbf{x}^t\mathbf{y} = \sum_{k=1}^{n} x_k y_k = \mathbf{x} \cdot \mathbf{y} \leq \left(\sum_{k=1}^{n} x_i^2 \right)^{1/2} \left(\sum_{k=1}^{n} y_i^2 \right)^{1/2} = \|\mathbf{x}\|_2 \|\mathbf{y}\|_2.$$

Proof (1 of 2)

- If $\mathbf{x} = \mathbf{0}$ or $\mathbf{y} = \mathbf{0}$ the result is trivially true, so suppose $\mathbf{x} \neq \mathbf{0}$ and $\mathbf{y} \neq \mathbf{0}$.

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- Let $\lambda \in \mathbb{R}$ then

$$
\begin{aligned}
0 &\leq \|\mathbf{x} - \lambda \mathbf{y}\|_2^2 \\
&= \sum_{k=1}^{n} (x_k - \lambda y_k)^2 \\
&= \sum_{k=1}^{n} (x_k^2 - 2\lambda x_k y_k + \lambda^2 y_k^2)
\end{aligned}
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0 &\leq \sum_{k=1}^{n} x_k^2 - 2\lambda \sum_{k=1}^{n} x_k y_k + \lambda^2 \sum_{k=1}^{n} y_k^2 \\
2\lambda \sum_{k=1}^{n} x_k y_k &\leq \sum_{k=1}^{n} x_k^2 + \lambda^2 \sum_{k=1}^{n} y_k^2 .
\end{aligned}$$

Proof (2 of 2)

Suppose $\lambda = \dfrac{\|\mathbf{x}\|_2}{\|\mathbf{y}\|_2}$ then

$$2\lambda \sum_{k=1}^{n} x_k y_k \quad \leq \quad \sum_{k=1}^{n} x_k^2 + \lambda^2 \sum_{k=1}^{n} y_k^2$$

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$$\begin{aligned} 2\lambda \sum_{k=1}^{n} x_k y_k &\leq \sum_{k=1}^{n} x_k^2 + \lambda^2 \sum_{k=1}^{n} y_k^2 \\ 2\frac{\|\mathbf{x}\|_2}{\|\mathbf{y}\|_2} \sum_{k=1}^{n} x_k y_k &\leq \|\mathbf{x}\|_2^2 + \left(\frac{\|\mathbf{x}\|_2}{\|\mathbf{y}\|_2}\right)^2 \|\mathbf{y}\|_2^2 \end{aligned}$$

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&= 2\|\mathbf{x}\|_2^2 \\
\frac{1}{\|\mathbf{y}\|_2} \sum_{k=1}^{n} x_k y_k &\leq \|\mathbf{x}\|_2 \\
\sum_{k=1}^{n} x_k y_k &\leq \|\mathbf{x}\|_2 \|\mathbf{y}\|_2.
\end{aligned}
$$

Distance

Definition

If $\mathbf{x}, \mathbf{y} \in \mathbb{R}^n$ the l_2**-distance** between $\mathbf{x}$ and $\mathbf{y}$ is

$$\|\mathbf{x} - \mathbf{y}\|_2 = \left(\sum_{k=1}^{n} (x_k - y_k)^2 \right)^{1/2}.$$

The l_∞**-distance** between $\mathbf{x}$ and $\mathbf{y}$ is

$$\|\mathbf{x} - \mathbf{y}\|_\infty = \max_{1 \le k \le n} |x_k - y_k|.$$

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Definition

A sequence of vectors $\{\mathbf{x}^{(k)}\}_{k=1}^{\infty}$ in $\mathbb{R}^n$ is said to **converge** to vector $\mathbf{x}$ with respect to norm $\|\cdot\|$ if given $\epsilon > 0$, there exists $N \in \mathbb{N}$ such that $\|\mathbf{x}^{(k)} - \mathbf{x}\| < \epsilon$ for all $k \ge N$.

Example

Suppose the solution to a linear system is $\mathbf{x} = \langle 1, -2, 3 \rangle$ and by Gaussian elimination and back-substitution we approximate the solution by $\hat{\mathbf{x}} = \langle 0.9, -1.99, 2.95 \rangle$.

Find:

- $\|\mathbf{x} - \hat{\mathbf{x}}\|_2$
- $\|\mathbf{x} - \hat{\mathbf{x}}\|_\infty$

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Find:

- $\|\mathbf{x} - \hat{\mathbf{x}}\|_2 = \sqrt{(1 - 0.9)^2 + (-2 - (-1.99))^2 + (3 - 2.95)^2} = 0.11225$
- $\|\mathbf{x} - \hat{\mathbf{x}}\|_\infty$

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- $\|\mathbf{x} - \hat{\mathbf{x}}\|_\infty = \max_{1 \leq k \leq 3}\{|1-0.9|, |-2-(-1.99)|, |3-2.95|\} = 0.1$

Convergence Result

Theorem

The sequence of vectors $\left\{\mathbf{x}^{(k)}\right\}_{k=1}^{\infty}$ *converges to* $\mathbf{x} \in \mathbb{R}^n$ *with respect to* $\|\cdot\|_\infty$ *if and only if*

$$\lim_{k\to\infty} x_i^{(k)} = x_i \quad \textit{for } i = 1, 2, \ldots, n.$$

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Proof.

- Suppose $\{\mathbf{x}^{(k)}\}_{k=1}^{\infty}$ converges to $\mathbf{x}$ with respect to $\|\cdot\|_\infty$.
- Given $\epsilon > 0$ there exists $N \in \mathbb{N}$ such that for all $k \geq N$

$$\max_{1\leq i\leq n} \left| x_i^{(k)} - x_i \right| = \|\mathbf{x}^{(k)} - \mathbf{x}\|_\infty < \epsilon.$$

- This implies $\left| x_i^{(k)} - x_i \right| < \epsilon$ for $i = 1, 2, \ldots, n$ which implies $\lim_{k\to\infty} x_i^{(k)} = x_i$ for $i = 1, 2, \ldots, n$.

□

Proof

- Suppose $\lim_{k\to\infty} x_i^{(k)} = x_i$ for $i = 1, 2, \ldots, n$.
- Given $\epsilon > 0$ let N_i be a positive integer with the property that $\left|x_i^{(k)} - x_i\right| < \epsilon$ when $k \geq N_i$.
- Define $N = \max_{1\leq i\leq n}\{N_i\}$, then if $k \geq N$ we have

$$\max_{1\leq i\leq n}\left|x_i^{(k)} - x_i\right| = \|\mathbf{x}^{(k)} - \mathbf{x}\|_\infty < \epsilon.$$

- This implies $\mathbf{x}^{(k)} \to \mathbf{x}$ with respect to norm $\|\cdot\|_\infty$.

Example

Let $\mathbf{x}^{(k)} = \left\langle \frac{1}{k}, \frac{\sin k}{k}, 1 + \frac{\cos k}{k} \right\rangle$ and find

$$\lim_{k \to \infty} \mathbf{x}^{(k)}$$

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$$\lim_{k \to \infty} \mathbf{x}^{(k)} = \langle 0, 0, 1 \rangle.$$

Equivalence of Norms

Theorem

For all $\mathbf{x} \in \mathbb{R}^n$ *we have* $\|\mathbf{x}\|_\infty \leq \|\mathbf{x}\|_2 \leq \sqrt{n}\|\mathbf{x}\|_\infty$.

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- Let x_j be the component of $\mathbf{x}$ such that $\|\mathbf{x}\|_\infty = |x_j|$.

$$\|\mathbf{x}\|_\infty^2 = |x_j|^2 = x_j^2 \leq \sum_{k=1}^{n} x_k^2 = \|\mathbf{x}\|_2^2$$

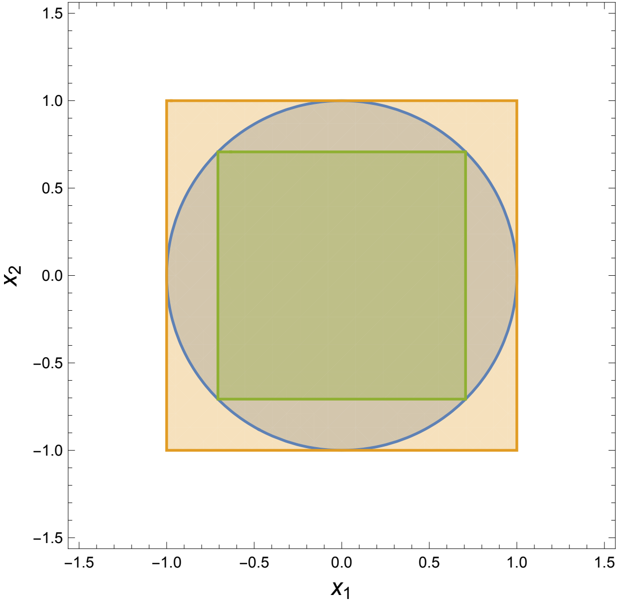
  Hence $\|\mathbf{x}\|_\infty \leq \|\mathbf{x}\|_2$.

- Note that

$$\|\mathbf{x}\|_2^2 = \sum_{k=1}^{n} x_k^2 \leq \sum_{k=1}^{n} x_j^2 = n x_j^2 = n\|\mathbf{x}\|_\infty^2.$$

  Hence $\|\mathbf{x}\|_2 \leq \sqrt{n}\|\mathbf{x}\|_\infty$.

Illustration



Matrix Norm

Definition

A **matrix norm** on $\mathbb{R}^{n\times n}$ is a real-valued function $\|\cdot\|$ satisfying for all matrices A, $B \in \mathbb{R}^{n\times n}$ and for all $\alpha \in \mathbb{R}$:

1. $\|A\| \geq 0$
2. $\|A\| = 0$ if and only if $A = 0^{n\times n}$
3. $\|\alpha A\| = |\alpha| \, \|A\|$
4. $\|A + B\| \leq \|A\| + \|B\|$
5. $\|A B\| \leq \|A\| \, \|B\|$

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Remark: the distance between A and B with respect to the matrix norm is $\|A - B\|$.

Induced Norms

Theorem

If $\|\cdot\|$ *is a vector norm on* $\mathbb{R}^n$ *then*

$$\|A\| = \max_{\|\mathbf{x}\|=1} \|A\mathbf{x}\|$$

is a matrix norm.

Remark: this is a matrix norm **induced** by the vector norm.

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Corollary

For any vector $\mathbf{z} \neq \mathbf{0}$, matrix A and norm $\|\cdot\|$ we have $\|A\mathbf{z}\| \leq \|A\| \, \|\mathbf{z}\|$.

Proof

- If $\mathbf{z} \neq \mathbf{0}$ then $\mathbf{z}/\|\mathbf{z}\|$ is a unit vector.
- Using the induced norm on matrix A we have

$$\max_{\|\mathbf{x}\|=1} \|A\mathbf{x}\| = \max_{\|\mathbf{z}\|\neq 0} \left\| A \frac{\mathbf{z}}{\|\mathbf{z}\|} \right\| = \max_{\|\mathbf{z}\|\neq 0} \frac{\|A\mathbf{z}\|}{\|\mathbf{z}\|} = \|A\|.$$

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- Therefore,

$$\|A\| \, \|\mathbf{z}\| \geq \frac{\|A\mathbf{z}\|}{\|\mathbf{z}\|} \|\mathbf{z}\| = \|A\mathbf{z}\|.$$

Comments

Using the l_2- and l_∞-norms (for vectors) we have induced norms (for matrices) of

$$\|A\|_2 = \max_{\|\mathbf{x}\|_2=1} \|A\mathbf{x}\|_2$$
$$\|A\|_\infty = \max_{\|\mathbf{x}\|_\infty=1} \|A\mathbf{x}\|_\infty.$$

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$$\|A\|_\infty = \max_{\|\mathbf{x}\|_\infty=1} \|A\mathbf{x}\|_\infty .$$

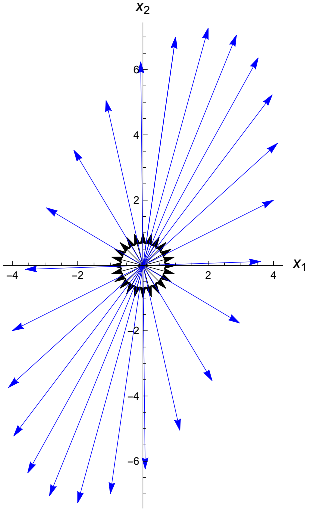
The norm of a matrix is the maximum "stretch" it gives to a unit vector.

Example: l_2-norm

Consider the matrix

$$A = \begin{bmatrix} 1 & 4 \\ 7 & 2 \end{bmatrix}$$

and the l_2-norm on $\mathbb{R}^2$.



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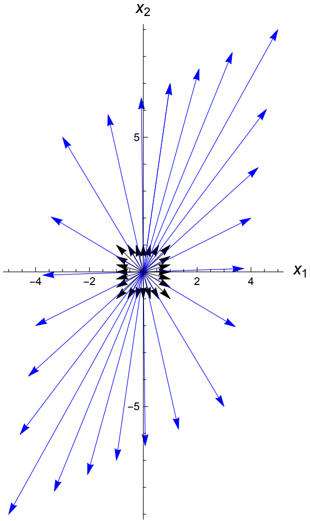

Infinity Norm of Matrices

Theorem

If $A \in \mathbb{R}^{n\times n}$ then

$$\|A\|_\infty = \max_{1\le i\le n} \sum_{j=1}^{n} |a_{ij}|,$$

in other words, the ∞-norm of A is the row with the largest summed magnitudes.

Proof (1 of 4)

- Let $\mathbf{x} \in \mathbb{R}^n$ with $\|\mathbf{x}\|_\infty = 1$.

$$\begin{aligned} \|A\mathbf{x}\|_\infty &= \max_{1 \le i \le n} |(A\mathbf{x})_i| \\ &= \max_{1 \le i \le n} \left| \sum_{j=1}^{n} a_{ij} x_j \right| \end{aligned}$$

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&= \max_{1 \le i \le n} \left| \sum_{j=1}^{n} a_{ij} x_j \right| \\
&\le \max_{1 \le i \le n} \sum_{j=1}^{n} |a_{ij}| \, |x_j| \\
&\le \left(\max_{1 \le i \le n} \sum_{j=1}^{n} |a_{ij}| \right) \left(\max_{1 \le j \le n} |x_j| \right).
\end{aligned}$$

Proof (2 of 4)

- Since $\|\mathbf{x}\|_\infty = 1$ then $\max_{1 \le j \le n} |x_j| = 1$ and

$$\|A\mathbf{x}\|_\infty \le \max_{1 \le i \le n} \sum_{j=1}^{n} |a_{ij}|.$$

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- Thus

$$\|A\|_\infty = \max_{\|\mathbf{x}\|_\infty = 1} \|A\mathbf{x}\|_\infty \le \max_{1 \le i \le n} \sum_{j=1}^{n} |a_{ij}|.$$

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$$\|A\|_\infty = \max_{\|\mathbf{x}\|_\infty = 1} \|A\mathbf{x}\|_\infty \le \max_{1 \le i \le n} \sum_{j=1}^{n} |a_{ij}|.$$

- Let $p \in \{1, 2, \ldots, n\}$ be such that

$$\sum_{j=1}^{n} |a_{pj}| = \max_{1 \le i \le n} \sum_{j=1}^{n} |a_{ij}|.$$

Proof (3 of 4)

- Define $\mathbf{x} \in \mathbb{R}^n$ as

$$x_j = \begin{cases} 1 & \text{if } a_{pj} \geq 0, \\ -1 & \text{if } a_{pj} < 0. \end{cases}$$

  Then $\|\mathbf{x}\|_\infty = 1$.

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  Then $\|\mathbf{x}\|_\infty = 1$.

- Therefore $a_{pj} x_j = |a_{pj}|$ for $j = 1, 2, \ldots, n$ and

$$\|A\mathbf{x}\|_\infty = \max_{1 \leq i \leq n} \left| \sum_{j=1}^{n} a_{ij} x_j \right|$$

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- Therefore $a_{pj}x_j = |a_{pj}|$ for $j = 1, 2, \ldots, n$ and

$$\begin{aligned} \|A\mathbf{x}\|_\infty &= \max_{1 \leq i \leq n} \left| \sum_{j=1}^{n} a_{ij}x_j \right| \\ &\geq \left| \sum_{j=1}^{n} a_{pj}x_j \right| = \left| \sum_{j=1}^{n} |a_{pj}| \right| \\ &= \max_{1 \leq i \leq n} \sum_{j=1}^{n} |a_{ij}|. \end{aligned}$$

Proof (4 of 4)

Thus we have shown that

$$\max_{1\le i\le n}\sum_{j=1}^{n}|a_{ij}| \le \|A\|_\infty \le \max_{1\le i\le n}\sum_{j=1}^{n}|a_{ij}|$$

which implies

$$\|A\|_\infty = \max_{1\le i\le n}\sum_{j=1}^{n}|a_{ij}|.$$

Proof (4 of 4)

Thus we have shown that

$$\max_{1 \le i \le n} \sum_{j=1}^{n} |a_{ij}| \le \|A\|_\infty \le \max_{1 \le i \le n} \sum_{j=1}^{n} |a_{ij}|$$

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Example

Let $A = \begin{bmatrix} 1 & 3 & 2 \\ -4 & 0 & 1 \\ 5 & 0 & 2 \end{bmatrix}$ and find

$$\|A\|_\infty$$

Proof (4 of 4)

Thus we have shown that

$$\max_{1 \le i \le n} \sum_{j=1}^{n} |a_{ij}| \le \|A\|_\infty \le \max_{1 \le i \le n} \sum_{j=1}^{n} |a_{ij}|$$

which implies

$$\|A\|_\infty = \max_{1 \le i \le n} \sum_{j=1}^{n} |a_{ij}|.$$

Example

Let $A = \begin{bmatrix} 1 & 3 & 2 \\ -4 & 0 & 1 \\ 5 & 0 & 2 \end{bmatrix}$ and find

$$\|A\|_\infty = 7.$$

Homework

- Read Section 7.1.
- Exercises: 1, 3, 4, 5, 13