

Error Analysis of ODE Solvers

MATH 375 Numerical Analysis

J Robert Buchanan

Department of Mathematics

Fall 2022

Objectives

In this lesson we will learn:

- ▶ the multivariable form of Taylor's theorem,
- ▶ to determine the order of convergence of various ODE solvers.

Taylor's Theorem

Theorem

Suppose $f(t, y)$ and all of its partial derivatives of order $n + 1$ and lower are continuous on the rectangle $D = \{(t, y) \mid a \leq t \leq b, c \leq y \leq d\}$ and let $(t_0, y_0) \in D$. Then for every $(t, y) \in D$ there exist $\xi \in (a, b)$ and $\mu \in (c, d)$ such that

$$\begin{aligned} f(t, y) &= f(t_0, y_0) + (t - t_0)f^{(1,0)}(t_0, y_0) + (y - y_0)f^{(0,1)}(t_0, y_0) \\ &\quad + \frac{(t - t_0)^2}{2}f^{(2,0)}(t_0, y_0) + (t - t_0)(y - y_0)f^{(1,1)}(t_0, y_0) + \frac{(y - y_0)^2}{2}f^{(0,2)}(t_0, y_0) \\ &\quad + \cdots + \frac{1}{n!} \sum_{j=0}^n \binom{n}{j} (t - t_0)^{n-j} (y - y_0)^j f^{(n-j,j)}(t_0, y_0) \\ &\quad + \frac{1}{(n+1)!} \sum_{j=0}^{n+1} \binom{n+1}{j} (t - t_0)^{n+1-j} (y - y_0)^j f^{(n+1-j,j)}(\xi, \mu). \end{aligned}$$

Remarks:

- ▶ The last term is the Taylor remainder.

- ▶ $f^{(j,k)}(t, y) = \frac{\partial^{j+k} f}{\partial t^j \partial y^k}$

Example: $f(t, y) = y - t^2 + 1$

Expand $f(t, y)$ about the point $(0, 1/2)$.

$$f(t, y) = y - t^2 + 1$$

$$f^{(1,0)}(t, y) = -2t$$

$$f^{(0,1)}(t, y) = 1$$

$$f^{(1,1)}(t, y) = 0$$

$$f^{(2,0)}(t, y) = -2$$

$$f^{(0,2)}(t, y) = 0$$

$$f(0, 1/2) = \frac{3}{2}$$

$$f^{(1,0)}(0, 1/2) = 0$$

$$f^{(0,1)}(0, 1/2) = 1$$

$$f^{(1,1)}(0, 1/2) = 0$$

$$f^{(2,0)}(0, 1/2) = -2$$

$$f^{(0,2)}(0, 1/2) = 0$$

All higher order derivatives vanish.

All higher order terms vanish.

Example: $f(t, y) = y - t^2 + 1$

The Taylor expansion can be written as:

$$\begin{aligned} f(t, y) &= f(0, 1/2) + (t - 0)f^{(1,0)}(0, 1/2) + (y - 1/2)f^{(0,1)}(0, 1/2) \\ &\quad + \frac{(t - 0)^2}{2}f^{(2,0)}(0, 1/2) + (t - 0)(y - 1/2)f^{(1,1)}(0, 1/2) \\ &\quad + \frac{(y - 1/2)^2}{2}f^{(0,2)}(0, 1/2) + R(\xi, \mu) \\ &= \frac{3}{2} + t(0) + (y - \frac{1}{2})(1) + \frac{t^2}{2}(-2) + t(y - \frac{1}{2})(0) + \frac{(y - 1/2)^2}{2}(0) \\ &= \frac{3}{2} + y - \frac{1}{2} - t^2. \end{aligned}$$

Note: for this example the Taylor remainder is identically zero.

Solving an IVP

The solution to the IVP:

$$y'(t) = f(t, y)$$

$$y(t_0) = y_0$$

can be written as a Taylor polynomial expanded about t_0 .

$$y(t_0 + h) = y(t_0) + y'(t_0)h + y''(t_0)\frac{h^2}{2} + y'''(t_0)\frac{h^3}{6} + O(h^4)$$

Each derivative of $y(t)$ can be replaced by an expression involving partial derivatives of $f(t, y)$.

Partial Derivatives

$$y'(t) = f(t, y)$$

$$\begin{aligned}y''(t) &= \frac{d}{dt} [f(t, y)] = f^{(1,0)}(t, y) + f^{(0,1)}(t, y)y'(t) \\ &= f^{(1,0)}(t, y) + f^{(0,1)}(t, y)f(t, y)\end{aligned}$$

$$\begin{aligned}y'''(t) &= f^{(2,0)}(t, y) + f^{(1,1)}(t, y)f(t, y) \\ &\quad + f^{(0,1)}(t, y) \left[f^{(1,0)}(t, y) + f^{(0,1)}(t, y)f(t, y) \right] \\ &\quad + \left[f^{(1,1)}(t, y) + f^{(0,2)}(t, y)f(t, y) \right] f(t, y) \\ &= f^{(2,0)}(t, y) + 2f^{(1,1)}(t, y)f(t, y) + f^{(0,2)}(t, y)(f(t, y))^2 \\ &\quad + f^{(0,1)}(t, y)f^{(1,0)}(t, y) + \left(f^{(0,1)}(t, y) \right)^2 f(t, y)\end{aligned}$$

Re-write Taylor Expansion

$$y(t_0 + h) = y(t_0) + f h + \left(f^{(1,0)} + f^{(0,1)} f \right) \frac{h^2}{2} \\ + \left(f^{(2,0)} + 2f^{(1,1)} f + f^{(0,2)} (f)^2 + f^{(0,1)} f^{(1,0)} + \left(f^{(0,1)} \right)^2 f \right) \frac{h^3}{6} + O(h^4)$$

Note: each instance of f or one of its partial derivatives is evaluated at (t_0, y_0) .

ODE Solvers

All of the methods for approximating solutions to an IVP have taken the form:

$$k_1 = f(t_0, y_0)$$

$$k_2 = f(t_0 + \beta_2 h, y_0 + \beta_2 h k_1)$$

$$k_3 = f(t_0 + \beta_3 h, y_0 + \beta_3 h k_2)$$

$$\vdots$$

$$k_s = f(t_0 + \beta_s h, y_0 + \beta_s h k_{s-1})$$

$$y(t_0 + h) = y(t_0) + (\alpha_1 k_1 + \alpha_2 k_2 + \cdots + \alpha_s k_s) h.$$

Use Taylor's theorem to expand each expression for k_i about (t_0, y_0) .

Taylor Expansion of $f(t_0 + \beta_j h, y_0 + \beta_j h k_{j-1})$

$$\begin{aligned}k_j &= f(t_0 + \beta_j h, y_0 + \beta_j h k_{j-1}) \\&= f(t_0, y_0) + \beta_j h f^{(1,0)}(t_0, y_0) + \beta_j h k_{j-1} f^{(0,1)}(t_0, y_0) \\&\quad + \frac{h^2 \beta_j^2}{2} f^{(2,0)}(t_0, y_0) + h^2 \beta_j^2 k_{j-1} f^{(1,1)}(t_0, y_0) \\&\quad + \frac{h^2 \beta_j^2 k_{j-1}^2}{2} f^{(0,2)}(t_0, y_0) + O(h^3)\end{aligned}$$

Taylor Expansion of k_1

When $j = 1$ then $\beta_1 = 0$ and

$$k_1 = f(t_0 + \beta_1 h, y_0 + \beta_1 h k_0) = f(t_0, y_0).$$

Taylor Expansion of k_2

When $j = 2$,

$$\begin{aligned}k_2 &= f(t_0 + \beta_2 h, y_0 + \beta_2 h k_1) \\&= f + \beta_2 h f^{(1,0)} + \beta_2 h k_1 f^{(0,1)} + \frac{h^2 \beta_2^2}{2} f^{(2,0)} + h^2 \beta_2^2 k_1 f^{(1,1)} + \frac{h^2 \beta_2^2 k_1^2}{2} f^{(0,2)} + O(h^3) \\&= f + \beta_2 h (f^{(1,0)} + f f^{(0,1)}) + \frac{h^2 \beta_2^2}{2} (f^{(2,0)} + 2f f^{(1,1)} + (f)^2 f^{(0,2)}) + O(h^3)\end{aligned}$$

Taylor Expansion of k_3 (1 of 2)

When $j = 3$,

$$\begin{aligned}k_3 &= f(t_0 + \beta_3 h, y_0 + \beta_3 h k_2) \\&= f + \beta_3 h f^{(1,0)} + \beta_3 h k_2 f^{(0,1)} + \frac{h^2 \beta_3^2}{2} f^{(2,0)} + h^2 \beta_3^2 k_2 f^{(1,1)} + \frac{h^2 \beta_3^2 k_2^2}{2} f^{(0,2)} + O(h^3) \\&= f + \beta_3 h (f^{(1,0)} + k_2 f^{(0,1)}) + \frac{h^2 \beta_3^2}{2} (f^{(2,0)} + 2k_2 f^{(1,1)} + k_2^2 f^{(0,2)}) + O(h^3)\end{aligned}$$

We still must substitute k_2 , but we need only the terms which will contribute powers of h less than 3.

Taylor Expansion of k_3 (2 of 2)

$$k_2 = f + \beta_2 h(f^{(1,0)} + f f^{(0,1)}) + \frac{h^2 \beta_2^2}{2} (f^{(2,0)} + 2f f^{(1,1)} + (f)^2 f^{(0,2)}) + O(h^3)$$

$$= f + \beta_2 h(f^{(1,0)} + f f^{(0,1)}) + O(h^2)$$

$$= f + O(h)$$

$$k_2^2 = \left(f + \beta_2 h(f^{(1,0)} + f f^{(0,1)}) + \frac{h^2 \beta_2^2}{2} (f^{(2,0)} + 2f f^{(1,1)} + (f)^2 f^{(0,2)}) + O(h^3) \right)^2$$

$$= (f)^2 + O(h)$$

Therefore,

$$k_3 = f + \beta_3 h(f^{(1,0)} + k_2 f^{(0,1)}) + \frac{h^2 \beta_3^2}{2} (f^{(2,0)} + 2k_2 f^{(1,1)} + k_2^2 f^{(0,2)}) + O(h^3)$$

$$= f + \beta_3 h(f^{(1,0)} + [f + \beta_2 h(f^{(1,0)} + f f^{(0,1)})] f^{(0,1)}) + \frac{h^2 \beta_3^2}{2} (f^{(2,0)} + 2(f) f^{(1,1)} + (f)^2 f^{(0,2)}) + O(h^3)$$

$$= f + \beta_3 h(f^{(1,0)} + f f^{(0,1)}) + \beta_3 h^2 \left[\beta_2 (f^{(1,0)} + f f^{(0,1)}) f^{(0,1)} + \frac{\beta_3}{2} (f^{(2,0)} + 2(f) f^{(1,1)} + (f)^2 f^{(0,2)}) \right]$$

$$+ O(h^3)$$

Summary

We have found the Taylor expansion of the first-order ODE:

$$y(t_0 + h) = y(t_0) + f h + \left(f^{(1,0)} + f^{(0,1)} f \right) \frac{h^2}{2} \\ + \left(f^{(2,0)} + 2f^{(1,1)} f + f^{(0,2)} (f)^2 + f^{(0,1)} f^{(1,0)} + \left(f^{(0,1)} \right)^2 f \right) \frac{h^3}{6} + O(h^4)$$

and the Taylor expansion of the ODE solver:

$$y(t_0 + h) = y(t_0) + (\alpha_1 k_1 + \alpha_2 k_2 + \alpha_3 k_3) h \\ = y(t_0) + \alpha_1 f h \\ + \alpha_2 \left(f + \beta_2 h (f^{(1,0)} + f f^{(0,1)}) + \frac{h^2 \beta_2^2}{2} (f^{(2,0)} + 2f f^{(1,1)} + (f)^2 f^{(0,2)}) + O(h^3) \right) h \\ + \alpha_3 \left(f + \beta_3 h (f^{(1,0)} + f f^{(0,1)}) + \beta_3 h^2 \left[\beta_2 (f^{(1,0)} + f f^{(0,1)}) f^{(0,1)} \right. \right. \\ \left. \left. + \frac{\beta_3}{2} (f^{(2,0)} + 2(f) f^{(1,1)} + (f)^2 f^{(0,2)}) \right] + O(h^3) \right) h.$$

Subtract the first equation from the second to determine the truncation error.

Subtraction (1 of 3)

It is easier to organize the subtraction by grouping like powers of h .

$$h^0: y(t_0) - y(t_0) = 0$$

$$h^1: \alpha_1 f + \alpha_2 f + \alpha_3 f - f$$

$$h^2: (\alpha_2 \beta_2 + \alpha_3 \beta_3)(f^{(1,0)} + f f^{(0,1)}) - \frac{1}{2}(f^{(1,0)} + f f^{(0,1)})$$

$$h^3:$$

$$\begin{aligned} & \frac{\alpha_2 \beta_2^2}{2} (f^{(2,0)} + 2f f^{(1,1)} + (f)^2 f^{(0,2)}) \\ & + \alpha_3 \beta_3 \left[\beta_2 (f^{(1,0)} + f f^{(0,1)}) f^{(0,1)} + \frac{\beta_3}{2} (f^{(2,0)} + 2(f) f^{(1,1)} + (f)^2 f^{(0,2)}) \right] \\ & - \frac{1}{6} \left(f^{(2,0)} + 2f^{(1,1)} f + f^{(0,2)} (f)^2 + f^{(0,1)} f^{(1,0)} + (f^{(0,1)})^2 f \right) \end{aligned}$$

Subtraction (2 of 3)

If the difference contains no h^1 term, then

$$0 = \alpha_1 f + \alpha_2 f + \alpha_3 f - f = (\alpha_1 + \alpha_2 + \alpha_3 - 1)f$$

which implies $\alpha_1 + \alpha_2 + \alpha_3 = 1$.

Subtraction (2 of 3)

If the difference contains no h^1 term, then

$$0 = \alpha_1 f + \alpha_2 f + \alpha_3 f - f = (\alpha_1 + \alpha_2 + \alpha_3 - 1)f$$

which implies $\alpha_1 + \alpha_2 + \alpha_3 = 1$.

If the difference contains no h^2 term, then

$$\begin{aligned} 0 &= (\alpha_2 \beta_2 + \alpha_3 \beta_3)(f^{(1,0)} + f f^{(0,1)}) - \frac{1}{2}(f^{(1,0)} + f f^{(0,1)}) \\ &= \left(\alpha_2 \beta_2 + \alpha_3 \beta_3 - \frac{1}{2} \right) (f^{(1,0)} + f f^{(0,1)}) \end{aligned}$$

which implies $\alpha_2 \beta_2 + \alpha_3 \beta_3 = \frac{1}{2}$.

Subtraction (3 of 3)

We will collect the terms in the coefficient of h^3 .

$$\begin{aligned} & \frac{\alpha_2 \beta_2^2}{2} (f^{(2,0)} + 2f f^{(1,1)} + (f)^2 f^{(0,2)}) \\ & + \alpha_3 \beta_3 \left[\beta_2 (f^{(1,0)} + f f^{(0,1)}) f^{(0,1)} + \frac{\beta_3}{2} (f^{(2,0)} + 2(f) f^{(1,1)} + (f)^2 f^{(0,2)}) \right] \\ & - \frac{1}{6} \left(f^{(2,0)} + 2f^{(1,1)} f + f^{(0,2)} (f)^2 + f^{(0,1)} f^{(1,0)} + (f^{(0,1)})^2 f \right) \\ = & \frac{1}{2} \left[(\alpha_2 \beta_2^2 + \alpha_3 \beta_3^2) - \frac{1}{3} \right] (f^{(2,0)} + 2(f) f^{(1,1)} + (f)^2 f^{(0,2)}) \\ & + \left(\alpha_3 \beta_2 \beta_3 - \frac{1}{6} \right) \left(f^{(1,0)} f^{(0,1)} + f (f^{(0,1)})^2 \right) \end{aligned}$$

If the difference contains no h^3 term, then

$$\begin{aligned} \alpha_2 \beta_2^2 + \alpha_3 \beta_3^2 &= \frac{1}{3} \\ \alpha_3 \beta_2 \beta_3 &= \frac{1}{6}. \end{aligned}$$

Summary

The truncation error will be $O(h^4)$ provided the following equations are satisfied.

$$\begin{aligned}\alpha_1 + \alpha_2 + \alpha_3 &= 1 \\ \alpha_2\beta_2 + \alpha_3\beta_3 &= \frac{1}{2} \\ \alpha_2\beta_2^2 + \alpha_3\beta_3^2 &= \frac{1}{3} \\ \alpha_3\beta_2\beta_3 &= \frac{1}{6}.\end{aligned}$$

Let's see if any of the ODE solvers we have tried satisfy these conditions.

Half-Open Newton Cotes Solver (1 of 2)

$$y(t_{i+1}) = y(t_i) + h \left(\frac{1}{4}k_1 + \frac{3}{4}k_3 \right)$$

where

$$k_1 = f(t_i, y(t_i))$$

$$k_2 = f\left(t_i + \frac{h}{3}, y(t_i) + \frac{h}{3}k_1\right)$$

$$k_3 = f\left(t_i + \frac{2h}{3}, y(t_i) + \frac{2h}{3}k_2\right),$$

Thus $\alpha_1 = \frac{1}{4}$, $\alpha_2 = 0$, $\alpha_3 = \frac{3}{4}$, $\beta_2 = \frac{1}{3}$, and $\beta_3 = \frac{2}{3}$.

Half-Open Newton Cotes Solver (2 of 2)

$$\alpha_1 + \alpha_2 + \alpha_3 = \frac{1}{4} + 0 + \frac{3}{4} = 1$$

$$\alpha_2\beta_2 + \alpha_3\beta_3 = 0 \cdot \frac{1}{3} + \frac{3}{4} \cdot \frac{2}{3} = \frac{1}{2}$$

$$\alpha_2\beta_2^2 + \alpha_3\beta_3^2 = 0 \left(\frac{1}{3}\right)^2 + \frac{3}{4} \left(\frac{2}{3}\right)^2 = \frac{1}{3}$$

$$\alpha_3\beta_2\beta_3 = \frac{3}{4} \cdot \frac{1}{3} \cdot \frac{2}{3} = \frac{1}{6}.$$

Thus the half-open Newton-Cotes ODE solver is (at least) $O(h^4)$.

Open Newton-Cotes: $n = 1$ Solver (1 of 2)

The solution to the IVP is approximated by

$$y(t_{i+1}) = y(t_i) + h \left(\frac{1}{2}k_2 + \frac{1}{2}k_3 \right)$$

where

$$k_1 = f(t_i, y(t_i))$$

$$k_2 = f\left(t_i + \frac{h}{3}, y(t_i) + \frac{h}{3}k_1\right)$$

$$k_3 = f\left(t_i + \frac{2h}{3}, y(t_i) + \frac{2h}{3}k_2\right).$$

Thus $\alpha_1 = 0$, $\alpha_2 = \frac{1}{2}$, $\alpha_3 = \frac{1}{2}$, $\beta_2 = \frac{1}{3}$, and $\beta_3 = \frac{2}{3}$.

Open Newton-Cotes: $n = 1$ Solver (2 of 2)

$$\alpha_1 + \alpha_2 + \alpha_3 = 0 + \frac{1}{2} + \frac{1}{2} = 1$$

$$\alpha_2\beta_2 + \alpha_3\beta_3 = \frac{1}{2} \cdot \frac{1}{3} + \frac{1}{2} \cdot \frac{2}{3} = \frac{1}{2}$$

$$\alpha_2\beta_2^2 + \alpha_3\beta_3^2 = \frac{1}{2} \left(\frac{1}{3}\right)^2 + \frac{1}{2} \left(\frac{2}{3}\right)^2 \neq \frac{1}{3}$$

$$\alpha_3\beta_2\beta_3 = \frac{3}{4} \cdot \frac{1}{3} \cdot \frac{2}{3} = \frac{1}{6}.$$

Thus the open Newton-Cotes $n = 1$ ODE solver is $O(h^3)$.

Trapezoidal ODE Solver (1 of 2)

The solution to the IVP is approximated by

$$y(t_{i+1}) = y(t_i) + h \left(\frac{1}{2}k_1 + \frac{1}{2}k_2 \right)$$

where

$$k_1 = f(t_i, y(t_i))$$

$$k_2 = f(t_i + h, y(t_i) + h k_1).$$

Thus $\alpha_1 = \frac{1}{2}$, $\alpha_2 = \frac{1}{2}$, $\alpha_3 = 0$, $\beta_2 = 1$, and β_3 is undefined.

Trapezoidal ODE Solver (2 of 2)

$$\alpha_1 + \alpha_2 + \alpha_3 = \frac{1}{2} + \frac{1}{2} + 0 = 1$$

$$\alpha_2\beta_2 + \alpha_3\beta_3 = \frac{1}{2} \cdot 1 + 0 \cdot \beta_3 = \frac{1}{2}$$

$$\alpha_2\beta_2^2 + \alpha_3\beta_3^2 = \frac{1}{2} (1)^2 + 0\beta_3^2 \neq \frac{1}{3}$$

$$\alpha_3\beta_2\beta_3 = 0 \cdot 1 \cdot \beta_2 \neq \frac{1}{6}.$$

Thus the Trapezoidal ODE solver is $O(h^3)$.

Euler's ODE Solver (1 of 2)

The solution to the IVP is approximated by

$$y(t_{i+1}) = y(t_i) + h k_1$$

where

$$k_1 = f(t_i, y(t_i))$$

Thus $\alpha_1 = 1$, $\alpha_2 = \alpha_3 = 0$, β_2 is undefined, and β_3 is undefined.

Euler's ODE Solver (2 of 2)

$$\alpha_1 + \alpha_2 + \alpha_3 = 1 + 0 + 0 = 1$$

$$\alpha_2\beta_2 + \alpha_3\beta_3 = 0 \cdot \beta_2 + 0 \cdot \beta_3 \neq \frac{1}{2}$$

$$\alpha_2\beta_2^2 + \alpha_3\beta_3^2 = 0 \cdot \beta_2^2 + 0 \cdot \beta_3^2 \neq \frac{1}{3}$$

$$\alpha_3\beta_2\beta_3 = 0 \cdot \beta_2\beta_3 \neq \frac{1}{6}.$$

Thus Euler's ODE solver is $O(h^2)$.

Simpson's ODE Solver (1 of 2)

The solution to the IVP:

$$\begin{aligned}y' &= f(t, y) \\ y(t_0) &= y_0\end{aligned}$$

is approximated by

$$y(t_{i+1}) = y(t_i) + h \left(\frac{1}{6}k_1 + \frac{2}{3}k_2 + \frac{1}{6}k_3 \right)$$

where

$$\begin{aligned}k_1 &= f(t_i, y(t_i)) \\ k_2 &= f\left(t_i + \frac{h}{2}, y(t_i) + \frac{h}{2}k_1\right) \\ k_3 &= f(t_i + h, y(t_i) + hk_2).\end{aligned}$$

Thus $\alpha_1 = \frac{1}{6}$, $\alpha_2 = \frac{2}{3}$, $\alpha_3 = \frac{1}{6}$, $\beta_2 = \frac{1}{2}$, and $\beta_3 = 1$.

Simpson's ODE Solver (2 of 2)

$$\alpha_1 + \alpha_2 + \alpha_3 = \frac{1}{6} + \frac{2}{3} + \frac{1}{6} = 1$$

$$\alpha_2\beta_2 + \alpha_3\beta_3 = \frac{2}{3} \cdot \frac{1}{2} + \frac{1}{6} \cdot 1 = \frac{1}{2}$$

$$\alpha_2\beta_2^2 + \alpha_3\beta_3^2 = \frac{2}{3} \left(\frac{1}{2}\right)^2 + \frac{1}{6} (1)^2 = \frac{1}{3}$$

$$\alpha_3\beta_2\beta_3 = \frac{1}{6} \cdot \frac{1}{2} \cdot 1 \neq \frac{1}{6}.$$

Thus Simpson's ODE solver is $O(h^3)$.

Remarks

We have so far explored **five** ODE solvers.

Euler: needs only one function evaluation, $O(h^2)$ error.

Trapezoidal: (also known as **improved Euler**) requires two function evaluations, $O(h^3)$ error.

Open $n = 1$ NC: requires three function evaluations, $O(h^3)$ error, never used in favor of improved Euler method.

Half-open NC: (also known as **Heun's 3rd order method**) requires three function evaluations, $O(h^4)$ error.

Simpson's: requires three function evaluation, $O(h^3)$ error, never used in favor of Heun's method.

These are examples of **Runge-Kutta** methods named for Carl Runge (1856-1927) and Martin Kutta (1867-1944).

Classic Runge-Kutta $O(h^5)$

The solution to the IVP:

$$y' = f(t, y)$$

$$y(t_0) = y_0$$

is approximated by

$$y(t_0 + h) = y(t_0) + h \left(\frac{1}{6}k_1 + \frac{1}{3}k_2 + \frac{1}{3}k_3 + \frac{1}{6}k_4 \right)$$

$$k_1 = f(t_0, y_0)$$

$$k_2 = f \left(t_0 + \frac{h}{2}, y_0 + \frac{h}{2}k_1 \right)$$

$$k_3 = f \left(t_0 + \frac{h}{2}, y_0 + \frac{h}{2}k_2 \right)$$

$$k_4 = f(t_0 + h, y_0 + hk_3)$$

Derivation (1 of 9)

Expand $y(t)$ about t_0 as a 4th-order Taylor polynomial.

$$y(t_0 + h) = y(t_0) + hy'(t_0) + \frac{h^2}{2}y''(t_0) + \frac{h^3}{6}y'''(t_0) + \frac{h^4}{24}y^{(4)}(t_0) + O(h^5)$$

We have already shown that

$$y'(t_0) = f(t_0, y_0) = k_1$$

$$y''(t_0) = f^{(1,0)}(t_0, y_0) + f^{(0,1)}(t_0, y_0)f(t_0, y_0) = f^{(1,0)} + f^{(0,1)}f$$

$$y'''(t_0) = f^{(2,0)} + 2f^{(1,1)}f + f^{(0,2)}(f)^2 + f^{(1,0)}f^{(0,1)} + (f^{(0,1)})^2f.$$

Derivation (2 of 9)

$$\begin{aligned}y^{(4)}(t_0) &= \frac{d}{dt} \left[f^{(2,0)} + 2f^{(1,1)}f + f^{(0,2)}(f)^2 + f^{(1,0)}f^{(0,1)} + (f^{(0,1)})^2 f \right] \\&= f^{(3,0)} + f^{(2,1)}f + 2(f^{(2,1)}f + f^{(1,2)}(f)^2 + f^{(1,1)}f^{(1,0)} + f^{(1,1)}f^{(0,1)}f) \\&\quad + (f^{(1,2)} + f^{(0,3)}f)(f)^2 + 2f^{(0,2)}f(f^{(1,0)} + f^{(0,1)}f) \\&\quad + (f^{(2,0)} + f^{(1,1)}f)f^{(0,1)} + f^{(1,0)}(f^{(1,1)} + f^{(0,2)}f) \\&\quad + 2f^{(0,1)}(f^{(1,1)} + f^{(0,2)}f)f + (f^{(0,1)})^2(f^{(1,0)} + f^{(0,1)}f) \\&= f^{(0,3)}(f)^3 + 3f^{(1,2)}(f)^2 + 4f^{(0,1)}f^{(0,2)}(f)^2 + 3f^{(2,1)}f + 5f^{(2,0)}f^{(0,1)}f \\&\quad + (f^{(0,1)})^3f + f^{(1,0)}f^{(0,2)}f + f^{(1,0)}(f^{(0,1)})^2 + f^{(2,0)}f^{(0,1)} + f^{(3,0)} + 3f^{(1,0)}f^{(1,1)}\end{aligned}$$

Derivation (3 of 9)

$$\begin{aligned}y(t_0 + h) &= y(t_0) + hy'(t_0) + \frac{h^2}{2}y''(t_0) + \frac{h^3}{6}y'''(t_0) + \frac{h^4}{24}y^{(4)}(t_0) + O(h^5) \\&= y(t_0) + hf + \frac{h^2}{2}(f^{(1,0)} + f^{(0,1)}f) \\&\quad + \frac{h^3}{6}(f^{(2,0)} + 2f^{(1,1)}f + f^{(0,2)}(f)^2 + f^{(1,0)}f^{(0,1)} + (f^{(0,1)})^2f) \\&\quad + \frac{h^4}{24}(f^{(0,3)}(f)^3 + 3f^{(1,2)}(f)^2 + 4f^{(0,1)}f^{(0,2)}(f)^2 + 3f^{(2,1)}f + 5f^{(2,0)}f^{(0,1)}f \\&\quad + (f^{(0,1)})^3f + f^{(1,0)}f^{(0,2)}f + f^{(1,0)}(f^{(0,1)})^2 + f^{(2,0)}f^{(0,1)} + f^{(3,0)} + 3f^{(1,0)}f^{(1,1)}) \\&\quad + O(h^5)\end{aligned}$$

Derivation (4 of 9)

Expand $f(t_0 + \beta_j h, y_0 + \beta_j h k_{j-1})$ as a 3rd-order Taylor polynomial about (t_0, y_0) using the multivariable version of Taylor's theorem.

$$\begin{aligned} & f(t_0 + \beta_j h, y_0 + \beta_j h k_{j-1}) \\ = & f + \beta_j h f^{(1,0)} + \beta_j h k_{j-1} f^{(0,1)} \\ & + \frac{1}{2}(\beta_j^2 h^2 f^{(2,0)} + 2h^2 \beta_j^2 k_{j-1} f^{(1,1)} + \beta_j^2 h^2 k_{j-1}^2 f^{(0,2)}) \\ & + \frac{1}{6}(\beta_j^3 h^3 f^{(3,0)} + 3\beta_j^3 h^3 k_{j-1} f^{(2,1)} + 3\beta_j^3 h^3 k_{j-1}^2 f^{(1,2)} + \beta_j^3 h^3 k_{j-1}^3 f^{(0,3)}) + O(h^4) \\ = & f + \beta_j h (f^{(1,0)} + k_{j-1} f^{(0,1)}) + \frac{\beta_j^2 h^2}{2} (f^{(2,0)} + 2k_{j-1} f^{(1,1)} + k_{j-1}^2 f^{(0,2)}) \\ & + \frac{\beta_j^3 h^3}{6} (f^{(3,0)} + 3k_{j-1} f^{(2,1)} + 3k_{j-1}^2 f^{(1,2)} + k_{j-1}^3 f^{(0,3)}) + O(h^4) \end{aligned}$$

Derivation (5 of 9)

$$k_1 = f(t_0, y_0)$$

$$\begin{aligned} k_2 &= f + \beta_2 h (f^{(1,0)} + k_1 f^{(0,1)}) + \frac{\beta_2^2 h^2}{2} (f^{(2,0)} + 2k_1 f^{(1,1)} + k_1^2 f^{(0,2)}) \\ &\quad + \frac{\beta_2^3 h^3}{6} (f^{(3,0)} + 3k_1 f^{(2,1)} + 3k_1^2 f^{(1,2)} + k_1^3 f^{(0,3)}) + O(h^4) \\ &= f + \beta_2 h (f^{(1,0)} + f f^{(0,1)}) + \frac{\beta_2^2 h^2}{2} (f^{(2,0)} + 2f f^{(1,1)} + f^2 f^{(0,2)}) \\ &\quad + \frac{\beta_2^3 h^3}{6} (f^{(3,0)} + 3f f^{(2,1)} + 3f^2 f^{(1,2)} + f^3 f^{(0,3)}) + O(h^4) \end{aligned}$$

Derivation (5 of 9)

$$k_1 = f(t_0, y_0)$$

$$\begin{aligned}k_2 &= f + \beta_2 h (f^{(1,0)} + k_1 f^{(0,1)}) + \frac{\beta_2^2 h^2}{2} (f^{(2,0)} + 2k_1 f^{(1,1)} + k_1^2 f^{(0,2)}) \\ &\quad + \frac{\beta_2^3 h^3}{6} (f^{(3,0)} + 3k_1 f^{(2,1)} + 3k_1^2 f^{(1,2)} + k_1^3 f^{(0,3)}) + O(h^4) \\ &= f + \beta_2 h (f^{(1,0)} + f f^{(0,1)}) + \frac{\beta_2^2 h^2}{2} (f^{(2,0)} + 2f f^{(1,1)} + f^2 f^{(0,2)}) \\ &\quad + \frac{\beta_2^3 h^3}{6} (f^{(3,0)} + 3f f^{(2,1)} + 3f^2 f^{(1,2)} + f^3 f^{(0,3)}) + O(h^4)\end{aligned}$$

For future use note that

$$k_2^2 = f^2 + 2h \beta_2 (f^{(1,0)} + f f^{(0,1)}) f + O(h^2)$$

$$k_2^3 = f^3 + O(h).$$

Derivation (6 of 9)

$$\begin{aligned}k_3 &= f + \beta_3 h(f^{(1,0)} + k_2 f^{(0,1)}) + \frac{\beta_3^2 h^2}{2}(f^{(2,0)} + 2k_2 f^{(1,1)} + k_2^2 f^{(0,2)}) \\&\quad + \frac{\beta_3^3 h^3}{6}(f^{(3,0)} + 3k_2 f^{(2,1)} + 3k_2^2 f^{(1,2)} + k_2^3 f^{(0,3)}) + O(h^4) \\&= f + \beta_3 h \left(f^{(1,0)} + \left[f + \beta_2 h(f^{(1,0)} + f f^{(0,1)}) + \frac{\beta_2^2 h^2}{2}(f^{(2,0)} + 2f f^{(1,1)} + f^2 f^{(0,2)}) \right] f^{(0,1)} \right) \\&\quad + \frac{\beta_3^2 h^2}{2} \left(f^{(2,0)} + 2 \left[f + \beta_2 h(f^{(1,0)} + f f^{(0,1)}) \right] f^{(1,1)} + \left[f^2 + 2h \beta_2 (f^{(1,0)} + f f^{(0,1)}) f \right] f^{(0,2)} \right) \\&\quad + \frac{\beta_3^3 h^3}{6} (f^{(3,0)} + 3f f^{(2,1)} + 3f^2 f^{(1,2)} + f^3 f^{(0,3)}) + O(h^4) \\&= f + \beta_3 h(f^{(1,0)} + f f^{(0,1)}) + \beta_2 \beta_3 h^2 (f^{(1,0)} + f f^{(0,1)}) f^{(0,1)} \\&\quad + \frac{\beta_3^2 h^2}{2} (f^{(2,0)} + 2f f^{(1,1)} + f^2 f^{(0,2)}) + \frac{\beta_2^2 \beta_3 h^3}{2} (f^{(2,0)} + 2f f^{(1,1)} + f^2 f^{(0,2)}) f^{(0,1)} \\&\quad + \beta_2 \beta_3^2 h^3 (f^{(1,0)} + f f^{(0,1)}) (f^{(1,1)} + f f^{(0,2)}) + \frac{\beta_3^3 h^3}{6} (f^{(3,0)} + 3f f^{(2,1)} + 3f^2 f^{(1,2)} + f^3 f^{(0,3)}) \\&\quad + O(h^4)\end{aligned}$$

Derivation (6 of 9)

$$\begin{aligned}k_3 &= f + \beta_3 h(f^{(1,0)} + k_2 f^{(0,1)}) + \frac{\beta_3^2 h^2}{2}(f^{(2,0)} + 2k_2 f^{(1,1)} + k_2^2 f^{(0,2)}) \\&\quad + \frac{\beta_3^3 h^3}{6}(f^{(3,0)} + 3k_2 f^{(2,1)} + 3k_2^2 f^{(1,2)} + k_2^3 f^{(0,3)}) + O(h^4) \\&= f + \beta_3 h \left(f^{(1,0)} + \left[f + \beta_2 h(f^{(1,0)} + f f^{(0,1)}) + \frac{\beta_2^2 h^2}{2}(f^{(2,0)} + 2f f^{(1,1)} + f^2 f^{(0,2)}) \right] f^{(0,1)} \right) \\&\quad + \frac{\beta_3^2 h^2}{2} \left(f^{(2,0)} + 2 \left[f + \beta_2 h(f^{(1,0)} + f f^{(0,1)}) \right] f^{(1,1)} + \left[f^2 + 2h \beta_2 (f^{(1,0)} + f f^{(0,1)}) f \right] f^{(0,2)} \right) \\&\quad + \frac{\beta_3^3 h^3}{6} (f^{(3,0)} + 3f f^{(2,1)} + 3f^2 f^{(1,2)} + f^3 f^{(0,3)}) + O(h^4) \\&= f + \beta_3 h(f^{(1,0)} + f f^{(0,1)}) + \beta_2 \beta_3 h^2 (f^{(1,0)} + f f^{(0,1)}) f^{(0,1)} \\&\quad + \frac{\beta_3^2 h^2}{2} (f^{(2,0)} + 2f f^{(1,1)} + f^2 f^{(0,2)}) + \frac{\beta_2^2 \beta_3 h^3}{2} (f^{(2,0)} + 2f f^{(1,1)} + f^2 f^{(0,2)}) f^{(0,1)} \\&\quad + \beta_2 \beta_3^2 h^3 (f^{(1,0)} + f f^{(0,1)}) (f^{(1,1)} + f f^{(0,2)}) + \frac{\beta_3^3 h^3}{6} (f^{(3,0)} + 3f f^{(2,1)} + 3f^2 f^{(1,2)} + f^3 f^{(0,3)}) \\&\quad + O(h^4)\end{aligned}$$

For future use note that

$$k_3^2 = f^2 + 2h \beta_3 (f^{(1,0)} + f f^{(0,1)}) f + O(h^2)$$

$$k_3^3 = f^3 + O(h).$$

Derivation (7 of 9)

$$\begin{aligned}k_4 &= f + \beta_4 h(f^{(1,0)} + k_3 f^{(0,1)}) + \frac{\beta_4^2 h^2}{2}(f^{(2,0)} + 2k_3 f^{(1,1)} + k_3^2 f^{(0,2)}) \\&\quad + \frac{\beta_4^3 h^3}{6}(f^{(3,0)} + 3k_3 f^{(2,1)} + 3k_2^2 f^{(1,2)} + k_3^3 f^{(0,3)}) + O(h^4) \\&= f + \beta_4 h(f^{(1,0)} + (f + \beta_3 h(f^{(1,0)} + f f^{(0,1)})) + \beta_2 \beta_3 h^2(f^{(1,0)} + f f^{(0,1)})f^{(0,1)}) \\&\quad + \frac{\beta_3^2 h^2}{2}(f^{(2,0)} + 2f f^{(1,1)} + f^2 f^{(0,2)})f^{(0,1)} \\&\quad + \frac{\beta_4^2 h^2}{2}(f^{(2,0)} + 2(f + \beta_3 h(f^{(1,0)} + f f^{(0,1)}))f^{(1,1)} + (f^2 + 2\beta_3 h(f^{(1,0)} + f f^{(0,1)}))f f^{(0,2)}) \\&\quad + \frac{\beta_4^3 h^3}{6}(f^{(3,0)} + 3f f^{(2,1)} + 3f^2 f^{(1,2)} + f^3 f^{(0,3)}) + O(h^4) \\&= f + \beta_4 h(f^{(1,0)} + f f^{(0,1)}) + \beta_3 \beta_4 h^2(f^{(1,0)} + f f^{(0,1)})f^{(0,1)} + \frac{\beta_4^2 h^2}{2}(f^{(2,0)} + 2f f^{(1,1)} + f^2 f^{(0,2)}) \\&\quad + \beta_2 \beta_3 \beta_4 h^3(f^{(1,0)} + f f^{(0,1)})(f^{(0,1)})^2 + \frac{\beta_3^2 \beta_4 h^3}{2}(f^{(2,0)} + 2f f^{(1,1)} + f^2 f^{(0,2)})f^{(0,1)} \\&\quad + \beta_3 \beta_4^2 h^3(f^{(1,0)} + f f^{(0,1)})(f^{(1,1)} + f f^{(0,2)}) + \frac{\beta_4^3 h^3}{6}(f^{(3,0)} + 3f f^{(2,1)} + 3f^2 f^{(1,2)} + f^3 f^{(0,3)}) \\&\quad + O(h^4)\end{aligned}$$

Derivation (8 of 9)

Thus far we have two expressions for $y(t_0 + h)$:

$$\begin{aligned}y(t_0 + h) &= y(t_0) + hf + \frac{h^2}{2}(f^{(1,0)} + f^{(0,1)}f) \\ &\quad + \frac{h^3}{6}(f^{(2,0)} + 2f^{(1,1)}f + f^{(0,2)}(f)^2 + f^{(1,0)}f^{(0,1)} + (f^{(0,1)})^2f) \\ &\quad + \frac{h^4}{24}(f^{(0,3)}(f)^3 + 3f^{(1,2)}(f)^2 + 4f^{(0,1)}f^{(0,2)}(f)^2 + 3f^{(2,1)}f + 5f^{(2,0)}f^{(0,1)}f \\ &\quad + (f^{(0,1)})^3f + f^{(1,0)}f^{(0,2)}f + f^{(1,0)}(f^{(0,1)})^2 + f^{(2,0)}f^{(0,1)} + f^{(3,0)} + 3f^{(1,0)}f^{(1,1)}) \\ &\quad + O(h^5) \\ &= y(t_0) + h \left(\frac{1}{6}k_1 + \frac{1}{3}k_2 + \frac{1}{3}k_3 + \frac{1}{6}k_4 \right) + O(h^5)\end{aligned}$$

Matching coefficients of h^n for $n = 0, 1, 2, 3, 4$ produces 10 equations with 7 unknowns.

Derivation (9 of 9)

$$\alpha_1 + \alpha_2 + \alpha_3 + \alpha_4 = 1$$

$$\alpha_2\beta_2 + \alpha_3\beta_3 + \alpha_4\beta_4 = \frac{1}{2}$$

$$\alpha_2\beta_2^2 + \alpha_3\beta_3^2 + \alpha_4\beta_4^2 = \frac{1}{3}$$

$$\alpha_3\beta_2\beta_3 + \alpha_4\beta_3\beta_4 = \frac{1}{6}$$

$$\alpha_2\beta_2^3 + \alpha_3\beta_3^3 + \alpha_4\beta_4^3 = \frac{1}{4}$$

$$\alpha_3\beta_2\beta_3^2 + \alpha_4\beta_3\beta_4^2 = \frac{1}{8}$$

$$2\alpha_3\beta_2\beta_3^2 + 2\alpha_4\beta_3\beta_4^2 + \alpha_3\beta_2^2\beta_3 + \alpha_4\beta_3^2\beta_4 = \frac{1}{3}$$

$$\alpha_3\beta_2\beta_3^2 + \alpha_4\beta_3\beta_4^2 + \alpha_3\beta_2^2\beta_3 + \alpha_4\beta_3^2\beta_4 = \frac{5}{24}$$

$$\alpha_3\beta_2^2\beta_3 + \alpha_4\beta_3^2\beta_4 = \frac{1}{12}$$

$$\alpha_4\beta_2\beta_3\beta_4 = \frac{1}{24}$$

From the last equation

$$\alpha_4 = \frac{1}{24\beta_2\beta_3\beta_4}.$$

Eliminate α_4 from the first 9 equations.

Derivation (9 of 9)

$$\alpha_1 + \alpha_2 + \alpha_3 + \frac{1}{24\beta_2\beta_3\beta_4} = 1$$

$$\alpha_2\beta_2 + \alpha_3\beta_3 + \frac{1}{24\beta_2\beta_3} = \frac{1}{2}$$

$$\alpha_2\beta_2^2 + \alpha_3\beta_3^2 + \frac{\beta_4}{24\beta_2\beta_3} = \frac{1}{3}$$

$$\alpha_3\beta_2\beta_3 + \frac{1}{24\beta_2} = \frac{1}{6}$$

$$\alpha_2\beta_2^3 + \alpha_3\beta_3^3 + \frac{\beta_4^2}{24\beta_2\beta_3} = \frac{1}{4}$$

$$\alpha_3\beta_2\beta_3^2 + \frac{\beta_4}{24\beta_2} = \frac{1}{8}$$

$$2\alpha_3\beta_2\beta_3^2 + \frac{\beta_4}{12\beta_2} + \alpha_3\beta_2^2\beta_3 + \frac{\beta_3}{24\beta_2} = \frac{1}{3}$$

$$\alpha_3\beta_2\beta_3^2 + \frac{\beta_4}{24\beta_2} + \alpha_3\beta_2^2\beta_3 + \frac{\beta_3}{24\beta_2} = \frac{5}{24}$$

$$\alpha_3\beta_2^2\beta_3 + \frac{\beta_3}{24\beta_2} = \frac{1}{12}$$

From the 4th equation

$$\alpha_3 = \frac{4\beta_2 - 1}{24\beta_2^2\beta_3}$$

Eliminate α_3 from the remaining equations.

Derivation (9 of 9)

$$\alpha_1 + \alpha_2 + \frac{4\beta_2 - 1}{24\beta_2^2\beta_3} + \frac{1}{24\beta_2\beta_3\beta_4} = 1$$

$$\alpha_2\beta_2 + \frac{4\beta_2 - 1}{24\beta_2^2} + \frac{1}{24\beta_2\beta_3} = \frac{1}{2}$$

$$\alpha_2\beta_2^2 + \frac{(4\beta_2 - 1)\beta_3}{24\beta_2^2} + \frac{\beta_4}{24\beta_2\beta_3} = \frac{1}{3}$$

$$\alpha_2\beta_2^3 + \frac{(4\beta_2 - 1)\beta_3^2}{24\beta_2^2} + \frac{\beta_4^2}{24\beta_2\beta_3} = \frac{1}{4}$$

$$\frac{(4\beta_2 - 1)\beta_3}{24\beta_2} + \frac{\beta_4}{24\beta_2} = \frac{1}{8}$$

$$\frac{(4\beta_2 - 1)\beta_3}{12\beta_2} + \frac{\beta_4}{12\beta_2} + \frac{4\beta_2 - 1}{24} + \frac{\beta_3}{24\beta_2} = \frac{1}{3}$$

$$\frac{(4\beta_2 - 1)\beta_3}{24\beta_2} + \frac{\beta_4}{24\beta_2} + \frac{4\beta_2 - 1}{24} + \frac{\beta_3}{24\beta_2} = \frac{5}{24}$$

$$\frac{4\beta_2 - 1}{24} + \frac{\beta_3}{24\beta_2} = \frac{1}{12}$$

Solve the last equation for β_3 ,

$$\beta_3 = 3\beta_2 - 4\beta_2^2$$

and eliminate β_3 from the remaining equations.

Derivation (9 of 9)

$$\alpha_1 + \alpha_2 + \frac{4\beta_2 - 1}{24(3\beta_2^3 - 4\beta_2^4)} + \frac{1}{24(3\beta_2^2 - 4\beta_2^3)\beta_4} = 1$$

$$\alpha_2\beta_2 + \frac{4\beta_2 - 1}{24\beta_2^2} + \frac{1}{24(3\beta_2^2 - 4\beta_2^3)} = \frac{1}{2}$$

$$\alpha_2\beta_2^2 + \frac{(4\beta_2 - 1)(3 - 4\beta_2)}{24\beta_2} + \frac{\beta_4}{24(3\beta_2^2 - 4\beta_2^3)} = \frac{1}{3}$$

$$\alpha_2\beta_2^3 + \frac{(4\beta_2 - 1)(3 - 4\beta_2)^2}{24} + \frac{\beta_4^2}{24(3\beta_2^2 - 4\beta_2^3)} = \frac{1}{4}$$

$$\frac{(4\beta_2 - 1)(3 - 4\beta_2)}{24} + \frac{\beta_4}{24\beta_2} = \frac{1}{8}$$

$$\frac{(4\beta_2 - 1)(3 - 4\beta_2)}{12} + \frac{\beta_4}{12\beta_2} + \frac{4\beta_2 - 1}{24} + \frac{3 - 4\beta_2}{24} = \frac{1}{3}$$

$$\frac{(4\beta_2 - 1)(3 - 4\beta_2)}{24} + \frac{\beta_4}{24\beta_2} + \frac{4\beta_2 - 1}{24} + \frac{3 - 4\beta_2}{24} = \frac{5}{24}$$

Solve the last equation for β_4 ,

$$\beta_4 = \beta_2(16\beta_2^2 - 16\beta_2 + 6)$$

and eliminate β_4 from the remaining equations.

Derivation (9 of 9)

$$\alpha_1 + \alpha_2 + \frac{4\beta_2 - 1}{24(3\beta_2^3 - 4\beta_2^4)} + \frac{1}{24(3\beta_2^3 - 4\beta_2^3)(16\beta_2^2 - 16\beta_2 + 6)} = 1$$

$$\alpha_2\beta_2 + \frac{4\beta_2 - 1}{24\beta_2^2} + \frac{1}{24(3\beta_2^2 - 4\beta_2^3)} = \frac{1}{2}$$

$$\alpha_2\beta_2^2 + \frac{(4\beta_2 - 1)(3 - 4\beta_2)}{24\beta_2} + \frac{16\beta_2^2 - 16\beta_2 + 6}{24(3\beta_2 - 4\beta_2^2)} = \frac{1}{3}$$

$$\alpha_2\beta_2^3 + \frac{(4\beta_2 - 1)(3 - 4\beta_2)^2}{24} + \frac{(16\beta_2^2 - 16\beta_2 + 6)^2}{24(3 - 4\beta_2)} = \frac{1}{4}$$

$$\frac{(4\beta_2 - 1)(3 - 4\beta_2)}{24} + \frac{16\beta_2^2 - 16\beta_2 + 6}{24} = \frac{1}{8}$$

$$\frac{(4\beta_2 - 1)(3 - 4\beta_2)}{12} + \frac{16\beta_2^2 - 16\beta_2 + 6}{12} + \frac{4\beta_2 - 1}{24} + \frac{3 - 4\beta_2}{24} = \frac{1}{3}$$

Derivation (9 of 9)

$$\alpha_1 + \alpha_2 + \frac{4\beta_2 - 1}{24(3\beta_2^3 - 4\beta_2^4)} + \frac{1}{24(3\beta_2^3 - 4\beta_2^3)(16\beta_2^2 - 16\beta_2 + 6)} = 1$$

$$\alpha_2\beta_2 + \frac{4\beta_2 - 1}{24\beta_2^2} + \frac{1}{24(3\beta_2^2 - 4\beta_2^3)} = \frac{1}{2}$$

$$\alpha_2\beta_2^2 + \frac{(4\beta_2 - 1)(3 - 4\beta_2)}{24\beta_2} + \frac{16\beta_2^2 - 16\beta_2 + 6}{24(3\beta_2 - 4\beta_2^2)} = \frac{1}{3}$$

$$\alpha_2\beta_2^3 + \frac{(4\beta_2 - 1)(3 - 4\beta_2)^2}{24} + \frac{(16\beta_2^2 - 16\beta_2 + 6)^2}{24(3 - 4\beta_2)} = \frac{1}{4}$$

$$\frac{(4\beta_2 - 1)(3 - 4\beta_2)}{24} + \frac{16\beta_2^2 - 16\beta_2 + 6}{24} = \frac{1}{8}$$

$$\frac{(4\beta_2 - 1)(3 - 4\beta_2)}{12} + \frac{16\beta_2^2 - 16\beta_2 + 6}{12} + \frac{4\beta_2 - 1}{24} + \frac{3 - 4\beta_2}{24} = \frac{1}{3}$$

Solve the 2nd equation for α_2 ,

$$\alpha_2 = \frac{2 - 16\beta_2 + 52\beta_2^2 - 48\beta_2^3}{24\beta_2^3(3 - 4\beta_2)}$$

and eliminate α_2 from the remaining equations.

Derivation (9 of 9)

$$\begin{aligned}\alpha_1 + \frac{2 - 16\beta_2 + 52\beta_2^2 - 48\beta_2^3}{24\beta_2^3(3 - 4\beta_2)} + \frac{4\beta_2 - 1}{24(3\beta_2^3 - 4\beta_2^4)} + \frac{1}{24(3\beta_2^3 - 4\beta_2^3)(16\beta_2^2 - 16\beta_2 + 6)} &= 1 \\ \frac{2 - 16\beta_2 + 52\beta_2^2 - 48\beta_2^3}{24\beta_2(3 - 4\beta_2)} + \frac{(4\beta_2 - 1)(3 - 4\beta_2)}{24\beta_2} + \frac{16\beta_2^2 - 16\beta_2 + 6}{24(3\beta_2 - 4\beta_2^2)} &= \frac{1}{3} \\ \frac{2 - 16\beta_2 + 52\beta_2^2 - 48\beta_2^3}{24(3 - 4\beta_2)} + \frac{(4\beta_2 - 1)(3 - 4\beta_2)^2}{24} + \frac{(16\beta_2^2 - 16\beta_2 + 6)^2}{24(3 - 4\beta_2)} &= \frac{1}{4} \\ \frac{(4\beta_2 - 1)(3 - 4\beta_2)}{24} + \frac{16\beta_2^2 - 16\beta_2 + 6}{24} &= \frac{1}{8} \\ \frac{(4\beta_2 - 1)(3 - 4\beta_2)}{12} + \frac{16\beta_2^2 - 16\beta_2 + 6}{12} + \frac{4\beta_2 - 1}{24} + \frac{3 - 4\beta_2}{24} &= \frac{1}{3}\end{aligned}$$

Derivation (9 of 9)

$$\begin{aligned}\alpha_1 + \frac{2 - 16\beta_2 + 52\beta_2^2 - 48\beta_2^3}{24\beta_2^3(3 - 4\beta_2)} + \frac{4\beta_2 - 1}{24(3\beta_2^3 - 4\beta_2^4)} + \frac{1}{24(3\beta_2^3 - 4\beta_2^3)(16\beta_2^2 - 16\beta_2 + 6)} &= 1 \\ \frac{2 - 16\beta_2 + 52\beta_2^2 - 48\beta_2^3}{24\beta_2(3 - 4\beta_2)} + \frac{(4\beta_2 - 1)(3 - 4\beta_2)}{24\beta_2} + \frac{16\beta_2^2 - 16\beta_2 + 6}{24(3\beta_2 - 4\beta_2^2)} &= \frac{1}{3} \\ \frac{2 - 16\beta_2 + 52\beta_2^2 - 48\beta_2^3}{24(3 - 4\beta_2)} + \frac{(4\beta_2 - 1)(3 - 4\beta_2)^2}{24} + \frac{(16\beta_2^2 - 16\beta_2 + 6)^2}{24(3 - 4\beta_2)} &= \frac{1}{4} \\ \frac{(4\beta_2 - 1)(3 - 4\beta_2)}{24} + \frac{16\beta_2^2 - 16\beta_2 + 6}{24} &= \frac{1}{8} \\ \frac{(4\beta_2 - 1)(3 - 4\beta_2)}{12} + \frac{16\beta_2^2 - 16\beta_2 + 6}{12} + \frac{4\beta_2 - 1}{24} + \frac{3 - 4\beta_2}{24} &= \frac{1}{3}\end{aligned}$$

The 2nd equation depends only on β_2 and can be solved.

Derivation (9 of 9)

$$\frac{2 - 16\beta_2 + 52\beta_2^2 - 48\beta_2^3}{24\beta_2(3 - 4\beta_2)} + \frac{(4\beta_2 - 1)(3 - 4\beta_2)}{24\beta_2} + \frac{16\beta_2^2 - 16\beta_2 + 6}{24\beta_2(3 - 4\beta_2)} = \frac{1}{3}$$

$$2 - 16\beta_2 + 52\beta_2^2 - 48\beta_2^3 + (4\beta_2 - 1)(3 - 4\beta_2)^2 + 16\beta_2^2 - 16\beta_2 + 6 = 8\beta_2(3 - 4\beta_2)$$

$$16\beta_2^3 - 12\beta_2^2 + 4\beta_2 - 1 = 0$$

$$(2\beta_2 - 1)(8\beta_2^2 - 2\beta_2 + 1) = 0$$

$$\beta_2 = \frac{1}{2}$$

Derivation (9 of 9)

$$\alpha_1 + \alpha_2 + \alpha_3 + \alpha_4 = 1$$

$$\alpha_2\beta_2 + \alpha_3\beta_3 + \alpha_4\beta_4 = \frac{1}{2}$$

$$\alpha_2\beta_2^2 + \alpha_3\beta_3^2 + \alpha_4\beta_4^2 = \frac{1}{3}$$

$$\alpha_3\beta_2\beta_3 + \alpha_4\beta_3\beta_4 = \frac{1}{6}$$

$$\alpha_2\beta_2^3 + \alpha_3\beta_3^3 + \alpha_4\beta_4^3 = \frac{1}{4}$$

$$\alpha_3\beta_2\beta_3^2 + \alpha_4\beta_3\beta_4^2 = \frac{1}{8}$$

$$2\alpha_3\beta_2\beta_3^2 + 2\alpha_4\beta_3\beta_4^2 + \alpha_3\beta_2^2\beta_3 + \alpha_4\beta_3^2\beta_4 = \frac{1}{3}$$

$$\alpha_3\beta_2\beta_3^2 + \alpha_4\beta_3\beta_4^2 + \alpha_3\beta_2^2\beta_3 + \alpha_4\beta_3^2\beta_4 = \frac{5}{24}$$

$$\alpha_3\beta_2^2\beta_3 + \alpha_4\beta_3^2\beta_4 = \frac{1}{12}$$

$$\alpha_4\beta_2\beta_3\beta_4 = \frac{1}{24}$$

$$\alpha_1 = \frac{1}{6}$$

$$\alpha_2 = \frac{1}{3}$$

$$\alpha_3 = \frac{1}{3}$$

$$\alpha_4 = \frac{1}{6}$$

$$\beta_2 = \frac{1}{2}$$

$$\beta_3 = \frac{1}{2}$$

$$\beta_4 = 1$$

Example

Consider the initial-value problem:

$$\begin{aligned}\frac{dy}{dt} &= y - t^2 + 1 \\ y(0) &= \frac{1}{2}.\end{aligned}$$

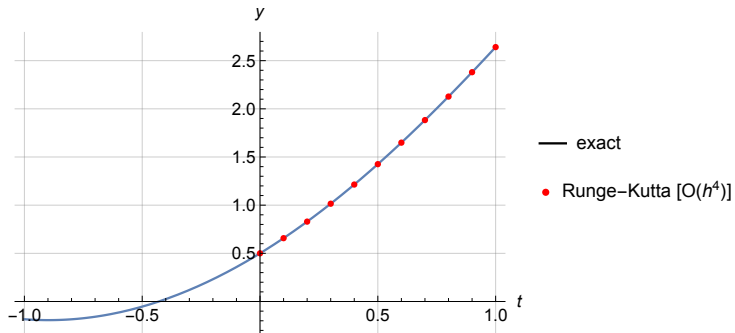
The exact solution is $y(t) = (1 + t)^2 - \frac{1}{2}e^t$.

Approximate the solution to this IVP for $0 \leq t \leq 1$ using the Runge-Kutta method with error $O(h^4)$.

Results

t	RK-5	Exact	Error
0.0	0.50000	0.5	0.0
0.1	0.65741	0.657415	1.660×10^{-7}
0.2	0.82930	0.829299	3.449×10^{-7}
0.3	1.01507	1.01507	5.378×10^{-7}
0.4	1.21409	1.21409	7.455×10^{-7}
0.5	1.42564	1.42564	9.690×10^{-7}
0.6	1.64894	1.64894	1.209×10^{-6}
0.7	1.88312	1.88312	1.468×10^{-6}
0.8	2.12723	2.12723	1.745×10^{-6}
0.9	2.38020	2.3802	2.043×10^{-6}
1.0	2.64086	2.64086	2.362×10^{-6}

Illustration



The ODE-solver based on the classic Runge-Kutta method appears to give accurate results.

Order of Convergence (1 of 2)

Consider the absolute error in the calculation of $y(1)$ for the various ODE solving methods.

h	Euler	Trap	Open	1/2-Open	Simpson	R-K
1/2	3.909×10^{-1}	1.252×10^{-1}	8.272×10^{-3}	4.430×10^{-3}	3.992×10^{-2}	1.256×10^{-3}
1/4	2.219×10^{-1}	3.537×10^{-2}	1.723×10^{-3}	5.876×10^{-4}	1.048×10^{-2}	8.714×10^{-5}
1/8	1.195×10^{-1}	9.367×10^{-3}	3.755×10^{-4}	7.493×10^{-5}	2.668×10^{-3}	5.713×10^{-6}
1/16	6.219×10^{-2}	2.407×10^{-3}	8.617×10^{-5}	9.433×10^{-6}	6.721×10^{-4}	3.653×10^{-7}
1/32	3.176×10^{-2}	6.098×10^{-4}	2.053×10^{-5}	1.182×10^{-6}	1.686×10^{-4}	2.308×10^{-8}
1/64	1.605×10^{-2}	1.534×10^{-4}	5.003×10^{-6}	1.480×10^{-7}	4.221×10^{-5}	1.451×10^{-9}
1/128	8.070×10^{-3}	3.849×10^{-5}	1.234×10^{-6}	1.851×10^{-8}	1.056×10^{-5}	9.092×10^{-11}

Divide the absolute error reported in each row by the absolute error in the row above.

Order of Convergence (2 of 2)

Each entry in the table below is nearly an integer power of $\frac{1}{2}$.

Euler	Trap	Open	1/2-Open	Simpson	R-K
0.567759	0.282401	0.20827	0.132658	0.262451	0.069353
0.538382	0.264851	0.217939	0.12751	0.254687	0.065561
0.520562	0.256969	0.229501	0.125887	0.251879	0.063940
0.510663	0.253352	0.238256	0.125346	0.250812	0.063198
0.505432	0.251641	0.243687	0.125148	0.250372	0.062843
0.502742	0.250811	0.246723	0.125067	0.250178	0.062670

The entries in the R-K column are nearly $(\frac{1}{2})^4$, so R-K is often referred to as a 4th order ODE solver.

Homework

- ▶ Read Section 6.4.