Introduction to Ordinary Differential Equations

MATH 375 *Numerical Analysis*

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Background

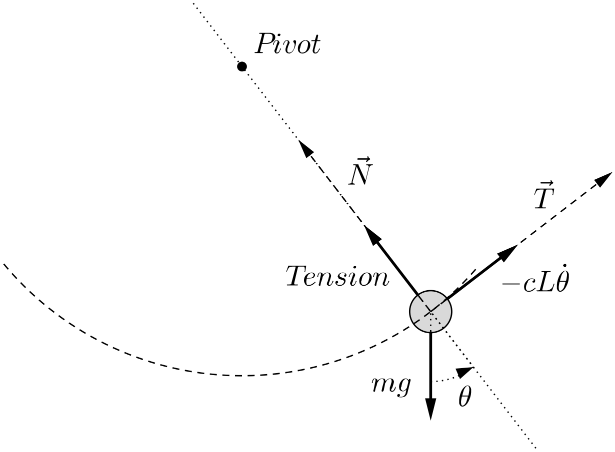
- **Ordinary Differential Equations** (ODEs) are mathematical models of quantities and their rates of change with respect to a single independent variable.
- ODEs have numerous applications in the STEM fields.
- The independent variable is often time (denoted t) or position (denoted x in Cartesian coordinates and θ) in polar coordinates).
- ODEs are often derived from well-known laws of motion (particularly Newton's 2nd Law: $\mathbf{F} = m\mathbf{a}$) and **free body diagrams**).

Free Body Diagrams

A free body diagram is an idealized depiction of an object and the forces acting on it. Common forces include:

- gravity: acting downward with magnitude $m\,g$ where m is the mass of the body.
- tension/compression: transmitted through ropes, wires, chains, rods, dowels, poles, *etc*.
- spring: proportional to the amount of deformation, directed opposite the deformation.
- normal: perpendicular away from a surface to keep the object from sinking into the surface.
- friction: opposing motion with proportionality constant μ (coefficient of friction). Often divided into static and dynamic friction.
- applied: by other objects (*e.g.*, people, motors, *etc*).

Example: Pendulum



Forces on Pendulum

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In the tangential direction, the velocity of the pendulum bob is $L\,\frac{d\theta}{dt}$, the friction is $-\mu\,L\,\frac{d\theta}{dt}$, and the acceleration is $L\,\frac{d^2\theta}{dt^2}$.

Pendulum ODE

Since the pendulum bob is accelerating only in the tangential direction, then according to Newton's 2nd Law of Motion,

$$
\begin{aligned}
m\,\mathbf{a}_T &= -m\,\mathbf{g}\sin\theta - \mu\,L\,\frac{d\theta}{dt} \\
m\,L\,\frac{d^2\theta}{dt^2} &= -m\,g\sin\theta - \mu\,L\,\frac{d\theta}{dt} \\
\frac{d^2\theta}{dt^2} + \frac{\mu}{m}\,\frac{d\theta}{dt} + \frac{g}{L}\sin\theta &= 0.
\end{aligned}
$$

- The unknown is θ.
- The ODE is described as **second-order** since the highest derivative of the unknown present is $\frac{d^2\theta}{dt^2}$.
- The ODE is **nonlinear** because of the term $\sin\theta$.

Initial Value Problem

If two side conditions are specified (for instance the initial position $\theta(0)$ and the initial velocity $\theta'(0)$) the system of equations,

$$\begin{aligned} \frac{d^2\theta}{dt^2} + \frac{\mu}{m}\frac{d\theta}{dt} + \frac{g}{L}\sin\theta &= 0 \\ \theta(0) &= \theta_0 \\ \theta'(0) &= \omega_0, \end{aligned}$$

is called an **initial value problem** (IVP).

Example

Show that $\theta(t) = e^{2t}$ is a solution to the ODE:

$$3\frac{d^2\theta}{dt^2} - 8\frac{d\theta}{dt} + 4\theta = 0.$$

Solution

$$\begin{aligned} 3\frac{d^2}{dt^2}[e^{2t}] - 8\frac{d}{dt}[e^{2t}] + 4e^{2t} &= 0 \\ 3(4e^{2t}) - 8(2e^{2t}) + 4e^{2t} &= 0 \\ 12e^{2t} - 16e^{2t} + 4e^{2t} &= 0 \\ 0 &= 0 \end{aligned}$$

Example

Show that $y(t) = \sqrt{1+t^2}$ is a solution to the IVP:

$$\frac{dy}{dt} = \frac{t}{y}$$
$$y(0) = 1.$$

Solution

We can check $y(0) = \sqrt{1+(0)^2} = \sqrt{1} = 1$, *so the initial condition is satisfied.*

$$\frac{d}{dt}\left[\sqrt{1+t^2}\right] = \frac{t}{\sqrt{1+t^2}}$$
$$\frac{t}{\sqrt{1+t^2}} = \frac{t}{\sqrt{1+t^2}}$$

Finding Solutions to ODEs

Provided the ODE is of a special form:

- first or second order,
- linear,
- separable, or
- exact

we can usually find the solution.

The solution technique, **separation of variables**, uses elementary calculus and integration.

Separation of Variables

Consider the general, first-order ODE:

$$\frac{dy}{dt} = f(t, y).$$

If this ODE can be algebraically manipulated into the form:

$$M(t)\,dt + N(y)\,dy = 0$$

we say that the ODE is **separable**.

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We can separate the independent and dependent variables as

$$N(y)\,dy = -M(t)\,dt,$$

and integrate both sides.

Examples

Verify that the following first-order ODEs are separable.

$$\frac{dy}{dt} = \frac{t^2}{y(1+t^3)}$$

$$\frac{dy}{dt} = \frac{y^2}{t}$$

$$\frac{dy}{dt} = \frac{3t^2}{3y^2-4}$$

Solution: $\frac{dy}{dt} = \frac{t^2}{y(1+t^3)}$

Find the general solution to the ODE:

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$$\int y\,dy = \int \frac{t^2}{1+t^3}\,dt$$

Solution: $\dfrac{dy}{dt} = \dfrac{t^2}{y(1+t^3)}$

Find the general solution to the ODE:

$$\begin{aligned} \frac{dy}{dt} &= \frac{t^2}{y(1+t^3)} \\ y\,dy &= \frac{t^2}{1+t^3}\,dt \\ \int y\,dy &= \int \frac{t^2}{1+t^3}\,dt \\ \frac{1}{2}y^2 &= \frac{1}{3}\ln\left|1+t^3\right| + C \end{aligned}$$

Solution: $\frac{dy}{dt} = \frac{y^2}{t}$

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Solution: $\dfrac{dy}{dt} = \dfrac{y^2}{t}$

Find the general solution to the ODE:

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\frac{dy}{dt} &= \frac{y^2}{t} \\
\frac{1}{y^2}\,dy &= \frac{1}{t}\,dt \\
\int \frac{1}{y^2}\,dy &= \int \frac{1}{t}\,dt \\
-\frac{1}{y} &= \ln|t| + C \\
y &= \frac{-1}{\ln|t| + C}
\end{aligned}$$

Solution: $\frac{dy}{dt} = \frac{3t^2}{3y^2 - 4}$

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(3y^2 - 4)\,dy &= 3t^2\,dt \\
\int (3y^2 - 4)\,dy &= \int 3t^2\,dt \\
y^3 - 4y &= t^3 + C
\end{aligned}$$

Homework

- Read Section 6.1.
- Exercises: