Pivoting Strategies

MATH 375 *Numerical Analysis*

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Gaussian Elimination

Given the augmented matrix $A = [a_{ij}]_{i=1,\dots,n,\, j=1,\dots,n+1}$:

STEP 1 For $i = 1, 2, \dots, n-1$ set

$$p = \min_{i \le j \le n} \{j \mid a_{ji} \neq 0\}$$

- If $p \neq i$ then transpose rows i and p.
- For $j = i+1, i+2, \dots, n$ replace row j by the sum of row j and $-\frac{a_{ji}}{a_{ii}}$ times row i.

STEP 2 Set $x_n = \frac{a_{n,n+1}}{a_{nn}}$.

STEP 3 For $i = n-1, n-2, \dots, 1$ set

$$x_i = \frac{a_{i,n+1} - \sum_{j=i+1}^{n} a_{ij} x_j}{a_{ii}}$$

Comments

- The steps of the Gaussian elimination in red implement the process known as **pivoting**.
- For each $i \in \{1, 2, \ldots, n\}$ the reduced augmented matrix must have $a_{ii} \neq 0$ in order to find a unique solution to the linear system.

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- For each $i \in \{1, 2, \ldots, n\}$ the reduced augmented matrix must have $a_{ii} \neq 0$ in order to find a unique solution to the linear system.
- The element a_{ii} is used to calculate

$$
\begin{aligned}
m_{ji} &= \frac{a_{ji}}{a_{ii}} \quad \text{for } j = i + 1, \ldots, n \\
x_i &= \frac{a_{i,n+1} - \sum_{j=i+1}^{n} a_{ij} x_j}{a_{ii}}
\end{aligned}
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\end{aligned}
$$

- If a_{ii} is small in magnitude, then any error present in its value will be magnified by the division operation.

Example

$$
\begin{aligned}
3.03x_1 - 12.1x_2 + 14.0x_3 &= -119 \\
-3.03x_1 + 12.1x_2 - 7.00x_3 &= 120 \\
6.11x_1 - 14.2x_2 + 21.0x_3 &= -139
\end{aligned}
$$

Exact solution:

$$(x_1, x_2, x_3) = \left(0, 10, \frac{1}{7}\right).$$

Approximate the solution using 3-digit chopping arithmetic.

Gaussian Elimination

$$\left[\begin{array}{rrr|r} 3.03 & -12.1 & 14.0 & -119 \\ -3.03 & 12.1 & -7.00 & 120 \\ 6.11 & -14.2 & 21.0 & -139 \end{array}\right]$$

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$$\mapsto \left[\begin{array}{rrr|r} 3.03 & -12.1 & 14.0 & -119 \\ 0.00 & 0.00 & 7.00 & 1.00 \\ 0.02 & 10.1 & -7.10 & 100 \end{array}\right]$$

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Back Substitution

$$\left[\begin{array}{ccc|c} 3.03 & -12.1 & 14.0 & -119 \\ 0.02 & 10.1 & -7.10 & 100 \\ 0.00 & 0.00 & 7.00 & 1.00 \end{array}\right]$$

$$x_3 = \frac{1.00}{7.00} = 0.142$$

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$$\begin{aligned} x_3 &= \frac{1.00}{7.00} = 0.142 \\ x_2 &= \frac{100 - (-7.10)(0.142)}{10.1} = \frac{100 + 1.00}{10.1} = 10.0 \end{aligned}$$

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$(x_1, x_2, x_3) = (0.330, 10.0, 0.142)$

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$(x_1, x_2, x_3) = (0.330, 10.0, 0.142)$
Recall the "exact" solution was $(0, 10, 1/7)$.

Partial Pivoting

Idea: to avoid the magnification of round-off error when dividing by a small value a_{ii}, find the largest magnitude element in the ith column below the diagonal.

$$|a_{pi}| = \max_{i \le j \le n} |a_{ji}|$$

At the ith stage of row reduction, swap row i and row p.

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At the ith stage of row reduction, swap row i and row p.

Remarks:

- This modification to the algorithm is called **partial pivoting** or **maximal column pivoting**.
- This process will ensure all row multipliers lie in interval $[-1, 1]$.

Algorithm: Partial Pivoting

Given the augmented matrix $A = [a_{ij}]_{i=1,\dots,n,\, j=1,\dots,n+1}$:

STEP 1 For $i = 1, 2, \dots, n-1$ do STEPS 2–4.

STEP 2 Let p be the smallest integer with $i \le p \le n$ such that

$$|a_{pi}| = \max_{i \le j \le n} \{|a_{ji}|\}.$$

STEP 3 If $p \ne i$ then transpose rows i and p.

STEP 4 For $j = i+1, i+2, \dots, n$ replace row j by the sum of row j and $-\dfrac{a_{ji}}{a_{ii}}$ times row i.

STEP 5 Set $x_n = \dfrac{a_{n,n+1}}{a_{nn}}$.

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\end{aligned}
$$

Exact solution:

$$(x_1, x_2, x_3) = \left(0, 10, \frac{1}{7}\right).$$

Approximate the solution using Gaussian elimination with partial pivoting and 3-digit chopping arithmetic.

Solution: Partial Pivoting

$$\left[\begin{array}{rrr|r} 3.03 & -12.1 & 14.0 & -119 \\ -3.03 & 12.1 & -7.00 & 120 \\ 6.11 & -14.2 & 21.0 & -139 \end{array}\right]$$

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$$\mapsto \left[\begin{array}{rrr|r} 6.11 & -14.2 & 21.0 & -139 \\ -0.01 & 5.08 & 3.30 & 51.2 \\ 0.01 & -5.08 & 3.70 & -50.2 \end{array}\right]$$

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$$\begin{aligned} x_3 &= \frac{1.00}{7.00} = 0.142 \\ x_2 &= \frac{51.2 - (3.30)(0.142)}{5.08} = \frac{51.2 - 0.468}{5.08} = 10.0 \end{aligned}$$

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Back Substitution

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$(x_1, x_2, x_3) = (0.163, 10.0, 0.142)$

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- We may wish to keep track of the largest magnitude element of each row of the matrix.

$$s_i = \max_{1 \leq j \leq n} \{|a_{ij}|\}$$

- If we swap rows according to p being the row number with $i \leq p \leq n$ such that

$$\frac{|a_{pi}|}{s_p} = \max_{i \leq j \leq n} \frac{|a_{ji}|}{s_j}$$

we are performing **scaled-column pivoting**.

Gaussian Elimination with Scaled Column Pivoting

Given the augmented matrix $A = [a_{ij}]_{i=1,\ldots,n,\, j=1,\ldots,n+1}$:

STEP 1 For $i = 1, 2, \ldots, n$, set $s_i = \max\limits_{1 \le j \le n} \{|a_{ij}|\}$.

STEP 2 For $i = 1, 2, \ldots, n-1$ do STEPS 3–5.

STEP 3 Let p be the smallest integer with $i \le p \le n$ such that

$$\frac{|a_{pi}|}{s_p} = \max_{i \le j \le n} \{\frac{|a_{ji}|}{s_j}\}.$$

STEP 4 If $p \neq i$ then transpose rows i and p.

STEP 5 For $j = i+1, i+2, \ldots, n$ replace row j by the sum of row j and $-\frac{a_{ji}}{a_{ii}}$ times row i.

STEP 6 Set $x_n = \frac{a_{n,n+1}}{a_{nn}}$.

STEP 7 For $i = n-1, n-2, \ldots, 1$ set

$$x_i = \frac{a_{i,n+1} - \sum_{j=i+1}^{n} a_{ij} x_j}{a_{ii}}$$

Remarks

- The entries in the augmented matrix are not modified by the scaling factors s_i. The scaling is only used to determine the appropriate pivots.
- Scaling factors s_i are computed only once and must be interchanged when swapping rows.
- Implementation of this algorithm calls for additional operations: more multiplications/divisions and **comparisons**.

Operation Count: Comparisons

- To calculate $\{s_i\}_{i=1}^{n}$ requires comparisons.
- At the ith stage of the row reduction we perform comparisons.

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Total comparisons:

$$
\begin{aligned}
n(n-1)+\sum_{i=1}^{n-1}(n-i) &= n(n-1)+\sum_{i=1}^{n-1} n-\sum_{i=1}^{n-1} i \\
&= n(n-1)+n(n-1)-\frac{n(n-1)}{2} \\
&= \frac{3}{2}n(n-1)
\end{aligned}
$$

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At the *i*th stage of the row reduction we perform new multiplications/divisions (beyond ordinary row reduction).

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Total multiplications/divisions:

$$\begin{aligned}
\sum_{i=1}^{n-1}(n-i+1) &= \sum_{i=1}^{n-1}(n+1) - \sum_{i=1}^{n-1} i \\
&= (n+1)(n-1) - \frac{n(n-1)}{2} \\
&= \frac{(n-1)(n+2)}{2}
\end{aligned}$$

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Remark: Gaussian elimination with scaled column pivoting remains an $O(n^3)$ operation.

Example

$$
\begin{aligned}
3.3330x_1 + 15920x_2 - 10.333x_3 &= 7953 \\
2.2220x_1 + 16.710x_2 + 9.6120x_3 &= 0.965 \\
-1.5611x_1 + 5.1792x_2 - 1.6855x_3 &= 2.714
\end{aligned}
$$

Exact solution $(x_1, x_2, x_3) = (1, \frac{1}{2}, -1)$.

Approximate the solution using 3-digit chopping arithmetic, partial pivoting and separately, scaled column pivoting.

Solution: Partial Pivoting

$$\left[\begin{array}{rrr|r} 3.33 & 15900 & -10.3 & 7950 \\ 2.22 & 16.7 & 9.61 & 0.965 \\ -1.56 & 5.17 & -1.68 & 2.71 \end{array}\right]$$

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$$\mapsto \left[\begin{array}{rrr|r} 3.33 & 15900 & -10.3 & 7950 \\ 0.01 & -10400 & 16.4 & -5280 \\ -0.01 & 7440 & 6.50 & 3720 \end{array}\right]$$

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$$\mapsto \left[\begin{array}{ccc|c} 3.33 & 15900 & -10.3 & 7950 \\ 0.01 & -10400 & 16.4 & -5280 \\ -0.01 & 10.0 & 18.2 & -50.0 \end{array}\right]$$

Back Substitution

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$$x_3 = \frac{-50.0}{18.2} = -2.74$$

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$$\begin{aligned} x_3 &= \frac{-50.0}{18.2} = -2.74 \\ x_2 &= \frac{-5280 - (16.4)(-2.74)}{-10400} = \frac{-5280 + 44.9}{-10400} \\ &= \frac{-5240}{-10400} = 0.503 \end{aligned}$$

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x_1 &= \frac{7950 - (-10.3)(-2.74) - (15900)(0.503)}{3.33} \\
&= \frac{7950 - 28.2 - 7990}{3.33} = \frac{7930 - 7990}{3.33} = \frac{-60}{3.33} = -18.0
\end{aligned}$$

Back Substitution

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$(x_1, x_2, x_3) = (-18.0, 0.503, -2.74)$

Solution: Scaled Column Pivoting (1 of 2)

$$\left[\begin{array}{rrr|r} 3.33 & 15900 & -10.3 & 7950 \\ 2.22 & 16.7 & 9.61 & 0.965 \\ -1.56 & 5.17 & -1.68 & 2.71 \end{array}\right]$$

1. Calculate s_i for $i = 1, 2, 3$.

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2. Perform pivoting.

$$\begin{aligned} \frac{|a_{11}|}{s_1} &= \frac{3.33}{15900} = 0.000209 \\ \frac{|a_{21}|}{s_2} &= \frac{2.22}{16.7} = 0.132 \\ \frac{|a_{31}|}{s_3} &= \frac{1.56}{5.17} = 0.301 \end{aligned}$$

Interchange rows 1 and 3.

Solution: Scaled Column Pivoting (2 of 2)

$$\left[\begin{array}{rrr|r} 3.33 & 15900 & -10.3 & 7950 \\ 2.22 & 16.7 & 9.61 & 0.965 \\ -1.56 & 5.17 & -1.68 & 2.71 \end{array}\right] \quad \begin{array}{c} 0.000209 \\ 0.132 \\ 0.301 \end{array}$$

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$$\mapsto \left[\begin{array}{rrr|r} -1.56 & 5.17 & -1.68 & 2.71 \\ 2.22 & 16.7 & 9.61 & 0.965 \\ 3.33 & 15900 & -10.3 & 7950 \end{array}\right] \quad \begin{array}{c} 0.301 \\ 0.132 \\ 0.000209 \end{array}$$

Solution: Scaled Column Pivoting (2 of 2)

$$\left[\begin{array}{rrr|r} 3.33 & 15900 & -10.3 & 7950 \\ 2.22 & 16.7 & 9.61 & 0.965 \\ -1.56 & 5.17 & -1.68 & 2.71 \end{array}\right] \quad \begin{array}{c} 0.000209 \\ 0.132 \\ 0.301 \end{array}$$

$$\mapsto \left[\begin{array}{rrr|r} -1.56 & 5.17 & -1.68 & 2.71 \\ 2.22 & 16.7 & 9.61 & 0.965 \\ 3.33 & 15900 & -10.3 & 7950 \end{array}\right] \quad \begin{array}{c} 0.301 \\ 0.132 \\ 0.000209 \end{array}$$

$$\mapsto \left[\begin{array}{rrr|r} -1.56 & 5.17 & -1.68 & 2.71 \\ 0.01 & 24.0 & 7.23 & 4.80 \\ 0.01 & 15900 & 10.1 & 7950 \end{array}\right] \quad \begin{array}{c} 0.301 \\ 0.132 \\ 0.000209 \end{array}$$

Solution: Scaled Column Pivoting (2 of 2)

$$\left[\begin{array}{rrr|r} 3.33 & 15900 & -10.3 & 7950 \\ 2.22 & 16.7 & 9.61 & 0.965 \\ -1.56 & 5.17 & -1.68 & 2.71 \end{array}\right] \begin{array}{c} 0.000209 \\ 0.132 \\ 0.301 \end{array}$$

$$\mapsto \left[\begin{array}{rrr|r} -1.56 & 5.17 & -1.68 & 2.71 \\ 2.22 & 16.7 & 9.61 & 0.965 \\ 3.33 & 15900 & -10.3 & 7950 \end{array}\right] \begin{array}{c} 0.301 \\ 0.132 \\ 0.000209 \end{array}$$

$$\mapsto \left[\begin{array}{rrr|r} -1.56 & 5.17 & -1.68 & 2.71 \\ 0.01 & 24.0 & 7.23 & 4.80 \\ 0.01 & 15900 & 10.1 & 7950 \end{array}\right] \begin{array}{c} 0.301 \\ 0.132 \\ 0.000209 \end{array}$$

$$\mapsto \left[\begin{array}{rrr|r} -1.56 & 5.17 & -1.68 & 2.71 \\ 0.01 & 15900 & 10.1 & 7950 \\ 0.01 & 24.0 & 7.23 & 4.80 \end{array}\right] \begin{array}{c} 0.301 \\ 0.000209 \\ 0.132 \end{array}$$

Solution: Scaled Column Pivoting (2 of 2)

$$\left[\begin{array}{ccc|c} 3.33 & 15900 & -10.3 & 7950 \\ 2.22 & 16.7 & 9.61 & 0.965 \\ -1.56 & 5.17 & -1.68 & 2.71 \end{array}\right] \begin{array}{c} 0.000209 \\ 0.132 \\ 0.301 \end{array}$$

$$\mapsto \left[\begin{array}{ccc|c} -1.56 & 5.17 & -1.68 & 2.71 \\ 2.22 & 16.7 & 9.61 & 0.965 \\ 3.33 & 15900 & -10.3 & 7950 \end{array}\right] \begin{array}{c} 0.301 \\ 0.132 \\ 0.000209 \end{array}$$

$$\mapsto \left[\begin{array}{ccc|c} -1.56 & 5.17 & -1.68 & 2.71 \\ 0.01 & 24.0 & 7.23 & 4.80 \\ 0.01 & 15900 & 10.1 & 7950 \end{array}\right] \begin{array}{c} 0.301 \\ 0.132 \\ 0.000209 \end{array}$$

$$\mapsto \left[\begin{array}{ccc|c} -1.56 & 5.17 & -1.68 & 2.71 \\ 0.01 & 15900 & 10.1 & 7950 \\ 0.01 & 24.0 & 7.23 & 4.80 \end{array}\right] \begin{array}{c} 0.301 \\ 0.000209 \\ 0.132 \end{array}$$

$$\mapsto \left[\begin{array}{ccc|c} -1.56 & 5.17 & -1.68 & 2.71 \\ 0.01 & 15900 & 10.1 & 7950 \\ 0.01 & 0.200 & 7.21 & -7.10 \end{array}\right] \begin{array}{c} 0.301 \\ 0.000209 \\ 0.132 \end{array}$$

Back Substitution

$$\left[\begin{array}{ccc|c} -1.56 & 5.17 & -1.68 & 2.71 \\ 0.01 & 15900 & 10.1 & 7950 \\ 0.01 & 0.200 & 7.21 & -7.10 \end{array}\right]$$

$$x_3 = \frac{-7.10}{7.21} = -0.984$$

Back Substitution

$$\left[\begin{array}{ccc|c} -1.56 & 5.17 & -1.68 & 2.71 \\ 0.01 & 15900 & 10.1 & 7950 \\ 0.01 & 0.200 & 7.21 & -7.10 \end{array}\right]$$

$$\begin{aligned} x_3 &= \frac{-7.10}{7.21} = -0.984 \\ x_2 &= \frac{7950 - (10.1)(-0.984)}{15900} \\ &= \frac{7950 + 9.93}{15900} = \frac{7950}{15900} = 0.500 \end{aligned}$$

Back Substitution

$$\left[\begin{array}{ccc|c} -1.56 & 5.17 & -1.68 & 2.71 \\ 0.01 & 15900 & 10.1 & 7950 \\ 0.01 & 0.200 & 7.21 & -7.10 \end{array}\right]$$

$$\begin{aligned} x_3 &= \frac{-7.10}{7.21} = -0.984 \\ x_2 &= \frac{7950 - (10.1)(-0.984)}{15900} \\ &= \frac{7950 + 9.93}{15900} = \frac{7950}{15900} = 0.500 \\ x_1 &= \frac{2.71 - (-1.68)(-0.984) - (5.17)(0.500)}{-1.56} \\ &= \frac{2.71 - 1.65 - 2.58}{-1.56} = \frac{1.06 - 2.58}{-1.56} = \frac{-1.52}{-1.56} = 0.974 \end{aligned}$$

Back Substitution

$$\left[\begin{array}{ccc|c} -1.56 & 5.17 & -1.68 & 2.71 \\ 0.01 & 15900 & 10.1 & 7950 \\ 0.01 & 0.200 & 7.21 & -7.10 \end{array}\right]$$

$$\begin{aligned} x_3 &= \frac{-7.10}{7.21} = -0.984 \\ x_2 &= \frac{7950 - (10.1)(-0.984)}{15900} \\ &= \frac{7950 + 9.93}{15900} = \frac{7950}{15900} = 0.500 \\ x_1 &= \frac{2.71 - (-1.68)(-0.984) - (5.17)(0.500)}{-1.56} \\ &= \frac{2.71 - 1.65 - 2.58}{-1.56} = \frac{1.06 - 2.58}{-1.56} = \frac{-1.52}{-1.56} = 0.974 \end{aligned}$$

$$(x_1, x_2, x_3) = (0.974, 0.500, -0.984)$$

Comparison of Results

$$\begin{aligned} 3.3330x_1 + 15920x_2 - 10.333x_3 &= 7953 \\ 2.2220x_1 + 16.710x_2 + 9.6120x_3 &= 0.965 \\ -1.5611x_1 + 5.1792x_2 - 1.6855x_3 &= 2.714 \end{aligned}$$

- Exact solution: $(x_1, x_2, x_3) = (1, 0.5, -1)$
- Partial pivoting: $(x_1, x_2, x_3) = (-18.0, 0.503, -2.74)$
- Scaled column pivoting: $(x_1, x_2, x_3) = (0.974, 0.500, -0.984)$

Homework

- Read Section 6.2.
- Exercises: 1, 3, 5