Rational Function Approximation

MATH 375

J Robert Buchanan

Department of Mathematics

Spring 2022

Introduction

So far we have used **polynomials** to approximate arbitrary continuous functions.

- **Pros:**
  - easy to evaluate
  - easy to integrate and differentiate
- **Cons:**
  - tend to oscillate
  - error estimates can be much larger than the actual error

Rational Functions

Definition

A **rational function of degree** N is a function of the form

$$r(x) = \frac{p(x)}{q(x)}$$

where $p(x)$ and $q(x)$ are polynomials such that
$\deg p(x) + \deg q(x) = N$.

Rational Functions

Definition

A **rational function of degree** N is a function of the form

$$r(x) = \frac{p(x)}{q(x)}$$

where $p(x)$ and $q(x)$ are polynomials such that
$\deg p(x) + \deg q(x) = N$.

Remark: Every polynomial $p(x)$ can be rewritten as $\frac{p(x)}{1}$ then approximation by **rational** functions can perform no worse than approximation by **polynomial** functions.

$$r(x) = \frac{p_0 + p_1 x + \cdots + p_n x^n}{q_0 + q_1 x + \cdots + q_m x^m}$$

$r(x)$ is a rational function of degree $n + m$.

Comments

- Rational functions whose numerator and denominator have the same or nearly the same degree often perform better at approximation than a polynomial.
- If machine division requires approximately the same amount of effort as machine multiplication, rational function approximation is as computationally efficient as polynomial approximation.
- Use of rational functions will allow us to approximate functions with infinite discontinuities.

Comments

- Rational functions whose numerator and denominator have the same or nearly the same degree often perform better at approximation than a polynomial.
- If machine division requires approximately the same amount of effort as machine multiplication, rational function approximation is as computationally efficient as polynomial approximation.
- Use of rational functions will allow us to approximate functions with infinite discontinuities.
- First, we take up the procedure known as **Padé approximation**.

Assumptions

- Assume the interval over which we are approximating $f(x)$ contains $x = 0$.

$$r(0) = \frac{p(0)}{q(0)} = \frac{p_0}{q_0} \quad \Longrightarrow \quad q_0 \neq 0$$

- Assume $q_0 = 1$ for, if not,

$$\begin{aligned} r(x) &= \frac{p_0 + p_1 x + \cdots + p_n x^n}{q_0 + q_1 x + \cdots + q_m x^m} \\ &= \frac{(p_0 + p_1 x + \cdots + p_n x^n)/q_0}{(q_0 + q_1 x + \cdots + q_m x^m)/q_0} \\ &= \frac{\hat{p}(x)}{\hat{q}(x)} \quad \text{where } \hat{q}_0 = 1 \end{aligned}$$

Padé Technique

- If $r(x)$ is of degree N, then $r(x)$ contains $N+2$ coefficients.
- $q_0 = 1$ leaving $N+1$ coefficients undetermined.
- The Padé technique is an extension of **Taylor polynomial approximation** to rational functions.

Padé Technique

- If $r(x)$ is of degree N, then $r(x)$ contains $N+2$ coefficients.
- $q_0 = 1$ leaving $N+1$ coefficients undetermined.
- The Padé technique is an extension of **Taylor polynomial approximation** to rational functions.

Theorem (Padé Technique)

Given $f(x)$ and a rational function $r(x)$ with $q_0 = 1$, choose the remaining $N+1$ coefficients of $r(x)$ such that $f^{(n)}(0) = r^{(n)}(0)$ for $n = 0, 1, \ldots, N$.

Padé Technique

- If $r(x)$ is of degree N, then $r(x)$ contains $N+2$ coefficients.
- $q_0 = 1$ leaving $N+1$ coefficients undetermined.
- The Padé technique is an extension of **Taylor polynomial approximation** to rational functions.

Theorem (Padé Technique)

Given $f(x)$ and a rational function $r(x)$ with $q_0 = 1$, choose the remaining $N+1$ coefficients of $r(x)$ such that $f^{(n)}(0) = r^{(n)}(0)$ for $n = 0, 1, \ldots, N$.

- This technique will require solving a system of $N+1$ equations for $N+1$ unknowns.
- If the degree of the denominator is 0, this is equivalent to Taylor polynomial approximation.

Relationship to Taylor's Theorem

Suppose $f(x)$ has a convergent Taylor's Series expansion.

$$f(x) = \sum_{k=0}^{\infty} a_k x^k \quad \text{where } a_k = \frac{f^{(k)}(0)}{k!}$$

$$\begin{aligned}
f(x) - r(x) &= f(x) - \frac{p(x)}{q(x)} \\
&= \frac{f(x)q(x) - p(x)}{q(x)} \\
&= \frac{\left(\sum_{k=0}^{\infty} a_k x^k\right)\left(\sum_{k=0}^{m} q_k x^k\right) - \sum_{k=0}^{n} p_k x^k}{\sum_{k=0}^{m} q_k x^k}
\end{aligned}$$

Goal: We want

$$0 = f^{(n)}(0) - r^{(n)}(0) = \frac{d^n}{dx^n}\left(f(x) - r(x)\right)\Big|_{x=0} \quad \text{for } n = 0, 1, \ldots, N.$$

Multiple Root

Recall our earlier discussion of multiple roots of functions.

Definition

A solution p of $f(x) = 0$ is a **root of multiplicity** m of f if for $x \neq p$ we can write $f(x) = (x - p)^m q(x)$, where $\lim_{x \to p} q(x) \neq 0$.

Theorem

A function $f \in \mathcal{C}^m[a, b]$ has a root of multiplicity m at $p \in (a, b)$ iff

$$0 = f(p) = f'(p) = \cdots = f^{(m-1)}(p) \quad \textit{but} \quad f^{(m)}(p) \neq 0.$$

Multiple Roots and the Padé Technique

Remark: The design goal is to select the coefficients of $r(x)$ so that $f(x) - r(x)$ has a root of multiplicity $N+1$ at $x = 0$. Equivalently

$$\left(\sum_{k=0}^{\infty} a_k x^k\right)\left(\sum_{k=0}^{m} q_k x^k\right) - \sum_{k=0}^{n} p_k x^k$$

has no terms of degree less than $N+1$.

Coefficient List

Consider

$$\left(\sum_{k=0}^{\infty} a_k x^k\right)\left(\sum_{k=0}^{m} q_k x^k\right) - \sum_{k=0}^{n} p_k x^k =$$

$$(a_0 + a_1 x + a_2 x^2 \cdots)(1 + q_1 x + \cdots + q_m x^m)$$
$$- (p_0 + p_1 x + \cdots + p_n x^n)$$

n	Coefficient of x^n
0	$a_0 - p_0$
1	$a_0 q_1 + a_1 - p_1$
2	$a_0 q_2 + a_1 q_1 + a_2 - p_2$
$\vdots$	$\vdots$
k	$-p_k + \sum_{i=0}^{k} a_i q_{k-i}$

for $k = 0, 1, \ldots, N$.

Linear System

System of $N+1$ equations:

$$\begin{aligned} a_0 &= p_0 \\ a_0 q_1 + a_1 &= p_1 \\ a_0 q_2 + a_1 q_1 + a_2 &= p_2 \\ &\vdots \\ \sum_{i=0}^{k} a_i q_{k-i} &= p_k \end{aligned}$$

for $k = 0, 1, \ldots, N$.

Linear System

System of $N + 1$ equations:

$$\begin{aligned} a_0 &= p_0 \\ a_0 q_1 + a_1 &= p_1 \\ a_0 q_2 + a_1 q_1 + a_2 &= p_2 \\ &\vdots \\ \sum_{i=0}^{k} a_i q_{k-i} &= p_k \end{aligned}$$

for $k = 0, 1, \ldots, N$.

Note:

- For convenience if $k > m$ then $q_k = 0$ and if $k > n$ then $p_k = 0$.
- We must solve for $q_1, q_2, \ldots, q_m, p_0, p_1, \ldots, p_n$.

Example (1 of 5)

Consider $f(x) = x \ln(x + 1)$. Compare the Maclaurin polynomial approximation of degree 6 to the Padé rational approximation of degree 6.

Example (1 of 5)

Consider $f(x) = x \ln(x+1)$. Compare the Maclaurin polynomial approximation of degree 6 to the Padé rational approximation of degree 6.

Maclaurin Polynomial:

$$P_6(x) = x^2 - \frac{1}{2}x^3 + \frac{1}{3}x^4 - \frac{1}{4}x^5 + \frac{1}{5}x^6$$

Example (1 of 5)

Consider $f(x) = x\ln(x+1)$. Compare the Maclaurin polynomial approximation of degree 6 to the Padé rational approximation of degree 6.

Maclaurin Polynomial:

$$P_6(x) = x^2 - \frac{1}{2}x^3 + \frac{1}{3}x^4 - \frac{1}{4}x^5 + \frac{1}{5}x^6$$

Rational function: let $m = n = 3$,

$$(x^2 - \tfrac{1}{2}x^3 + \tfrac{1}{3}x^4 - \tfrac{1}{4}x^5 + \tfrac{1}{5}x^6)(1 + q_1 x + q_2 x^2 + q_3 x^3) - (p_0 + p_1 x + p_2 x^2 + p_3 x^3)$$

should have no terms of degree less than or equal to 6.

Example (2 of 5): Linear System

$$
\begin{aligned}
-p_0 &= 0 \quad \text{(degree 0)} \\
-p_1 &= 0 \quad \text{(degree 1)} \\
1 - p_2 &= 0 \quad \text{(degree 2)} \\
-\frac{1}{2} - p_3 + q_1 &= 0 \quad \text{(degree 3)} \\
\frac{1}{3} - \frac{1}{2}q_1 + q_2 &= 0 \quad \text{(degree 4)} \\
-\frac{1}{4} + \frac{1}{3}q_1 - \frac{1}{2}q_2 + q_3 &= 0 \quad \text{(degree 5)} \\
\frac{1}{5} - \frac{1}{4}q_1 + \frac{1}{3}q_2 - \frac{1}{2} - q_3 &= 0 \quad \text{(degree 6)}
\end{aligned}
$$

Example (3 of 5)

Solution:

$$p_0 = 0,\ p_1 = 0,\ p_2 = 1,\ p_3 = \frac{19}{30},\ q_1 = \frac{17}{15},\ q_2 = \frac{7}{30},\ q_3 = -\frac{1}{90}$$

$$\begin{aligned} P_6(x) &= x^2\left[x\left[x\left[x\left[\frac{1}{5}x - \frac{1}{4}\right] + \frac{1}{3}\right] - \frac{1}{2}\right] + 1\right] \\ r(x) &= \frac{x^2 + \frac{19}{30}x^3}{1 + \frac{17}{15}x + \frac{7}{30}x^2 - \frac{1}{90}x^3} \end{aligned}$$

Example (3 of 5)

Solution:

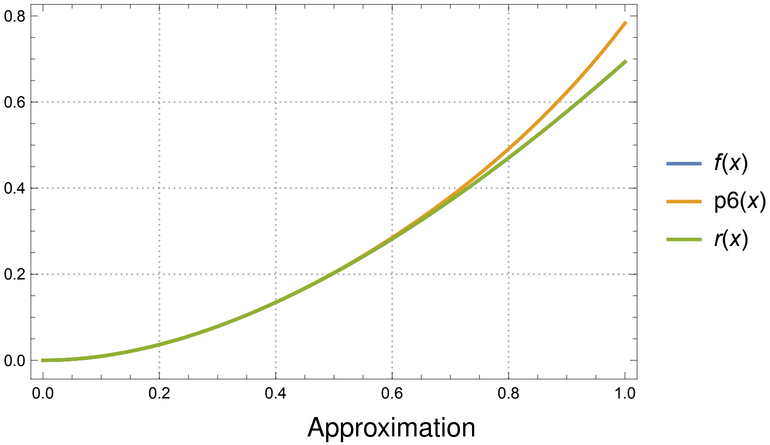
$$p_0 = 0,\ p_1 = 0,\ p_2 = 1,\ p_3 = \frac{19}{30},\ q_1 = \frac{17}{15},\ q_2 = \frac{7}{30},\ q_3 = -\frac{1}{90}$$

$$\begin{aligned} P_6(x) &= x^2\left[x\left[x\left[x\left[\frac{1}{5}x - \frac{1}{4}\right] + \frac{1}{3}\right] - \frac{1}{2}\right] + 1\right] \\ r(x) &= \frac{x^2 + \frac{19}{30}x^3}{1 + \frac{17}{15}x + \frac{7}{30}x^2 - \frac{1}{90}x^3} \end{aligned}$$

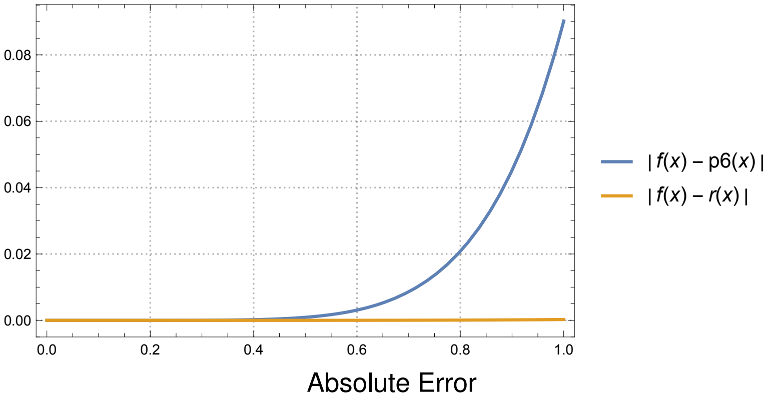
Remarks:

- $P_6(x)$ requires 6 MUL/DIV and 4 ADD/SUB.
- $r(x)$ requires 7 MUL/DIV and 4 ADD/SUB.

Example (4 of 5)



Example (4 of 5)



Example (5 of 5)

Approximation absolute error:

x	**Maclaurin**	**Rational**
0.0	0.0	0.0
0.1	1.535290×10^{-8}	1.553920×10^{-10}
0.2	1.821975×10^{-6}	1.545364×10^{-8}
0.3	0.00002902	2.084234×10^{-7}
0.4	0.00020364	1.249726×10^{-6}
0.5	0.00091328	4.828324×10^{-6}
0.6	0.0030890	0.000014171
0.7	0.0086059	0.000034484
0.8	0.0208128	0.000073278
0.9	0.0451972	0.000140581
1.0	0.0901862	0.000249046

Continued Fraction (1 of 4)

We can reduce the number of multiplications/divisions required for the evaluation of the rational function by using polynomial division.

$$r(x) = \frac{x^2 + \frac{19}{30}x^3}{1 + \frac{17}{15}x + \frac{7}{30}x^2 - \frac{1}{90}x^3}$$

Continued Fraction (1 of 4)

We can reduce the number of multiplications/divisions required for the evaluation of the rational function by using polynomial division.

$$
\begin{aligned}
r(x) &= \frac{x^2 + \frac{19}{30}x^3}{1 + \frac{17}{15}x + \frac{7}{30}x^2 - \frac{1}{90}x^3} \\
&= \frac{\left(x^2 + \frac{19}{30}x^3\right) 90}{\left(1 + \frac{17}{15}x + \frac{7}{30}x^2 - \frac{1}{90}x^3\right) 90} \\
&= \frac{90x^2 + 57x^3}{90 + 102x + 21x^2 - x^3}
\end{aligned}
$$

Continued Fraction (1 of 4)

We can reduce the number of multiplications/divisions required for the evaluation of the rational function by using polynomial division.

$$
\begin{aligned}
r(x) &= \frac{x^2 + \frac{19}{30}x^3}{1 + \frac{17}{15}x + \frac{7}{30}x^2 - \frac{1}{90}x^3} \\
&= \frac{\left(x^2 + \frac{19}{30}x^3\right) 90}{\left(1 + \frac{17}{15}x + \frac{7}{30}x^2 - \frac{1}{90}x^3\right) 90} \\
&= \frac{90x^2 + 57x^3}{90 + 102x + 21x^2 - x^3} \\
&= -57 + \frac{5130 + 5814x + 1287x^2}{90 + 102x + 21x^2 - x^3} \\
&= -57 + \frac{1}{\frac{90+102x+21x^2-x^3}{5130+5814x+1287x^2}}
\end{aligned}
$$

Continued Fraction (2 of 4)

Continue the polynomial division.

$$
\begin{aligned}
r(x) &= -57 + \cfrac{1}{\frac{90+102x+21x^2-x^3}{5130+5814x+1287x^2}} \\
&= -57 + \cfrac{1}{-\frac{1}{1287}x + \frac{3649}{184041} - \frac{239520+189946x}{184041(570+646x+143x^2)}}
\end{aligned}
$$

Continued Fraction (2 of 4)

Continue the polynomial division.

$$
\begin{aligned}
r(x) &= -57 + \cfrac{1}{\frac{90+102x+21x^2-x^3}{5130+5814x+1287x^2}} \\
&= -57 + \cfrac{1}{-\frac{1}{1287}x + \frac{3649}{184041} - \frac{239520+189946x}{184041(570+646x+143x^2)}} \\
&= -57 - \cfrac{1287}{x - \frac{3649}{143} + \frac{239520+189946x}{81510+92378x+20449x^2}}
\end{aligned}
$$

Continued Fraction (2 of 4)

Continue the polynomial division.

$$
\begin{aligned}
r(x) &= -57 + \cfrac{1}{\frac{90+102x+21x^2-x^3}{5130+5814x+1287x^2}} \\
&= -57 + \cfrac{1}{-\frac{1}{1287}x + \frac{3649}{184041} - \frac{239520+189946x}{184041(570+646x+143x^2)}} \\
&= -57 - \cfrac{1287}{x - \frac{3649}{143} + \frac{239520+189946x}{81510+92378x+20449x^2}} \\
&= -57 - \cfrac{1287}{x - \frac{3649}{143} + \cfrac{1}{\frac{81510+92378x+20449x^2}{239520+189946x}}}
\end{aligned}
$$

Continued Fraction (3 of 4)

$$\begin{aligned} r(x) &= -57 - \cfrac{1287}{x - \frac{3649}{143} + \cfrac{1}{\frac{81510+92378x+20449x^2}{239520+189946x}}} \\ &= -57 - \cfrac{1287}{x - \frac{3649}{143} + \cfrac{1}{\frac{3162221777}{9019870729} + \frac{20449}{189946}x - \frac{11102848453125}{9019870729(119760+94973x)}}} \end{aligned}$$

Continued Fraction (3 of 4)

$$
\begin{aligned}
r(x) &= -57 - \cfrac{1287}{x - \frac{3649}{143} + \cfrac{1}{\frac{81510+92378x+20449x^2}{239520+189946x}}} \\
&= -57 - \cfrac{1287}{x - \frac{3649}{143} + \cfrac{1}{\frac{3162221777}{9019870729} + \frac{20449}{189946}x - \frac{11102848453125}{9019870729(119760+94973x)}}} \\
&= -57 - \cfrac{1287}{x - \frac{3649}{143} + \cfrac{\frac{189946}{20449}}{x + \frac{44226878}{13581139} - \frac{1085906250}{94973(119760+94973x)}}}
\end{aligned}
$$

Continued Fraction (3 of 4)

$$
\begin{aligned}
r(x) &= -57 - \cfrac{1287}{x - \cfrac{3649}{143} + \cfrac{1}{\frac{81510+92378x+20449x^2}{239520+189946x}}} \\
&= -57 - \cfrac{1287}{x - \cfrac{3649}{143} + \cfrac{1}{\frac{3162221777}{9019870729} + \frac{20449}{189946}x - \frac{11102848453125}{9019870729(119760+94973x)}}} \\
&= -57 - \cfrac{1287}{x - \cfrac{3649}{143} + \cfrac{\frac{189946}{20449}}{x + \frac{44226878}{13581139} - \frac{1085906250}{94973(119760+94973x)}}} \\
&= -57 - \cfrac{1287}{x - \cfrac{3649}{143} + \cfrac{\frac{189946}{20449}}{x + \frac{44226878}{13581139} - \frac{1}{\frac{94973(119760+94973x)}{1085906250}}}}
\end{aligned}
$$

Continued Fraction (4 of 4)

$$
\begin{aligned}
r(x) &= -57 - \cfrac{1287}{x - \frac{3649}{143} + \cfrac{\frac{189946}{20449}}{x + \frac{44226878}{13581139} - \cfrac{1}{\frac{94973(119760+94973x)}{1085906250}}}} \\
&= -57 - \cfrac{1287}{x - \frac{3649}{143} + \cfrac{\frac{189946}{20449}}{x + \frac{44226878}{13581139} - \cfrac{1}{\frac{379132216}{36196875} + \frac{9019870729}{1085906250}x}}}
\end{aligned}
$$

Continued Fraction (4 of 4)

$$
\begin{aligned}
r(x) &= -57 - \cfrac{1287}{x - \frac{3649}{143} + \cfrac{\frac{189946}{20449}}{x + \frac{44226878}{13581139} - \cfrac{1}{\frac{94973(119760+94973x)}{1085906250}}}} \\
&= -57 - \cfrac{1287}{x - \frac{3649}{143} + \cfrac{\frac{189946}{20449}}{x + \frac{44226878}{13581139} - \cfrac{1}{\frac{379132216}{36196875} + \frac{9019870729}{1085906250}x}}} \\
&= -57 - \cfrac{1287}{x - \frac{3649}{143} + \cfrac{\frac{189946}{20449}}{x + \frac{44226878}{13581139} - \cfrac{\frac{1085906250}{9019870729}}{x + \frac{119760}{94973}}}}
\end{aligned}
$$

Continued Fraction (4 of 4)

$$
\begin{aligned}
r(x) &= -57 - \cfrac{1287}{x - \frac{3649}{143} + \cfrac{\frac{189946}{20449}}{x + \frac{44226878}{13581139} - \cfrac{1}{\frac{94973(119760+94973x)}{1085906250}}}} \\
&= -57 - \cfrac{1287}{x - \frac{3649}{143} + \cfrac{\frac{189946}{20449}}{x + \frac{44226878}{13581139} - \cfrac{1}{\frac{379132216}{36196875} + \frac{9019870729}{1085906250}x}}} \\
&= -57 - \cfrac{1287}{x - \frac{3649}{143} + \cfrac{\frac{189946}{20449}}{x + \frac{44226878}{13581139} - \cfrac{\frac{1085906250}{9019870729}}{x + \frac{119760}{94973}}}}
\end{aligned}
$$

Remark: this form requires only 3 MUL/DIV and 6 ADD/SUB.

Chebyshev Rational Function Approximation

Given a function $f(x)$,

- the **Chebyshev** rational function approximation is similar to the previous rational function approximation.
- In place of the x^n used in the Padé approximation, we will use the nth Chebyshev function, $T_n(x)$.

Chebyshev Rational Function Approximation

Given a function $f(x)$,

- the **Chebyshev** rational function approximation is similar to the previous rational function approximation.
- In place of the x^n used in the Padé approximation, we will use the nth Chebyshev function, $T_n(x)$.

Assumptions:

- Write $f(x)$ as a Chebyshev series:

$$f(x) = \sum_{k=0}^{\infty} a_k T_k(x).$$

- Write the Nth degree rational function as

$$r(x) = \frac{\sum_{k=0}^{n} p_k T_k(x)}{\sum_{k=0}^{m} q_k T_k(x)}$$

where $N = n + m$ and $q_0 = 1$.

Difference

Consider

$$f(x) - r(x) \quad = \quad \sum_{k=0}^{\infty} a_k T_k(x) - \frac{\sum_{k=0}^{n} p_k T_k(x)}{\sum_{k=0}^{m} q_k T_k(x)}$$

Difference

Consider

$$\begin{aligned} f(x) - r(x) &= \sum_{k=0}^{\infty} a_k T_k(x) - \frac{\sum_{k=0}^{n} p_k T_k(x)}{\sum_{k=0}^{m} q_k T_k(x)} \\ &= \frac{\sum_{k=0}^{\infty} a_k T_k(x) \sum_{k=0}^{m} q_k T_k(x) - \sum_{k=0}^{n} p_k T_k(x)}{\sum_{k=0}^{m} q_k T_k(x)}. \end{aligned}$$

Difference

Consider

$$\begin{aligned} f(x) - r(x) &= \sum_{k=0}^{\infty} a_k T_k(x) - \frac{\sum_{k=0}^{n} p_k T_k(x)}{\sum_{k=0}^{m} q_k T_k(x)} \\ &= \frac{\sum_{k=0}^{\infty} a_k T_k(x) \sum_{k=0}^{m} q_k T_k(x) - \sum_{k=0}^{n} p_k T_k(x)}{\sum_{k=0}^{m} q_k T_k(x)}. \end{aligned}$$

We wish to choose coefficients $q_1, q_2, \ldots, q_m, p_0, p_1, \ldots, p_n$ so that the right-hand side has zero coefficients for $T_k(x)$ for $k = 0, 1, \ldots, N = n + m$.

Numerator

The right-hand side has zero coefficients for $T_k(x)$ for $k = 0, 1, \ldots, N = n + m$ implies

$$(a_0 T_0(x) + a_1 T_1(x) + \cdots)(T_0(x) + q_1 T_1(x) + \cdots + q_m T_m(x)) - (p_0 T_0(x) + p_1 T_1(x) + \cdots + p_n T_n(x))$$

has no terms of degree less than or equal to N.

Challenges (1 of 2)

- The expression

$$(a_0 T_0(x) + a_1 T_1(x) + \cdots)(T_0(x) + q_1 T_1(x) + \cdots + q_m T_m(x))$$

requires that we form the product of Chebyshev functions.

Challenges (1 of 2)

- The expression

$$(a_0 T_0(x) + a_1 T_1(x) + \cdots)(T_0(x) + q_1 T_1(x) + \cdots + q_m T_m(x))$$

requires that we form the product of Chebyshev functions.

- Fortunately there is a Chebyshev product-to-sum formula which helps:

$$T_i(x) T_j(x) = \frac{1}{2} \left(T_{i+j}(x) + T_{|i-j|}(x) \right).$$

Challenges (2 of 2)

- Finding the Chebyshev coefficients for the representation of $f(x)$:

$$f(x) = \sum_{k=0}^{\infty} a_k T_k(x)$$

requires that we evaluate the definite integrals

$$\begin{aligned} a_0 &= \frac{1}{\pi} \int_{-1}^{1} \frac{f(x) T_0(x)}{\sqrt{1 - x^2}} \, dx \\ a_k &= \frac{2}{\pi} \int_{-1}^{1} \frac{f(x) T_k(x)}{\sqrt{1 - x^2}} \, dx \quad (k \geq 1). \end{aligned}$$

Challenges (2 of 2)

- Finding the Chebyshev coefficients for the representation of $f(x)$:

$$f(x) = \sum_{k=0}^{\infty} a_k T_k(x)$$

requires that we evaluate the definite integrals

$$\begin{aligned} a_0 &= \frac{1}{\pi} \int_{-1}^{1} \frac{f(x) T_0(x)}{\sqrt{1-x^2}}\, dx \\ a_k &= \frac{2}{\pi} \int_{-1}^{1} \frac{f(x) T_k(x)}{\sqrt{1-x^2}}\, dx \quad (k \geq 1). \end{aligned}$$

- At worst we may numerically approximate these integrals.

Example (1 of 7)

Find the Chebyshev rational approximation of degree 4 with $n = m = 2$ to $f(x) = \cos \pi x$.

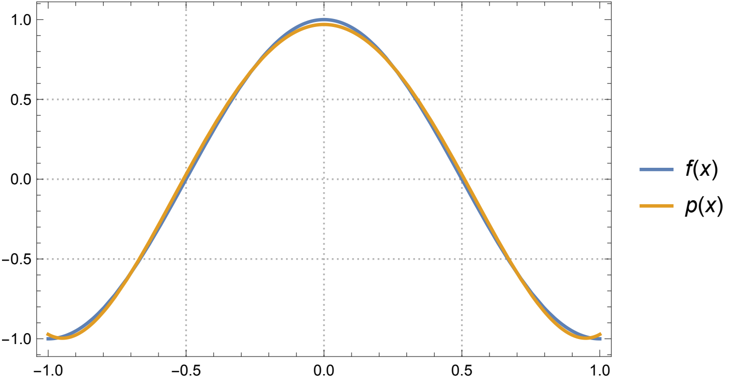
Example (1 of 7)

Find the Chebyshev rational approximation of degree 4 with $n = m = 2$ to $f(x) = \cos \pi x$.

$$
\begin{aligned}
a_0 &= \frac{1}{\pi} \int_{-1}^{1} \frac{\cos \pi x}{\sqrt{1 - x^2}}\, dx = -0.304242 \\
a_1 &= \frac{2}{\pi} \int_{-1}^{1} \frac{T_1(x) \cos \pi x}{\sqrt{1 - x^2}}\, dx = 0 \\
a_2 &= \frac{2}{\pi} \int_{-1}^{1} \frac{T_2(x) \cos \pi x}{\sqrt{1 - x^2}}\, dx = -0.970868 \\
a_3 &= \frac{2}{\pi} \int_{-1}^{1} \frac{T_3(x) \cos \pi x}{\sqrt{1 - x^2}}\, dx = 0 \\
a_4 &= \frac{2}{\pi} \int_{-1}^{1} \frac{T_4(x) \cos \pi x}{\sqrt{1 - x^2}}\, dx = 0.302849
\end{aligned}
$$

Example (2 of 7)

$$f(x) = \cos \pi x \approx p(x) = -0.304242 - 0.970868\, T_2(x) + 0.302849\, T_4(x)$$



Example (3 of 7)

Now we must choose p_0, p_1, p_2, q_1, q_2 so that the coefficients of $T_k(x)$ are zero in the expanded form of

$$(a_0T_0(x) + a_2T_2(x) + a_4T_4(x))(T_0(x) + q_1T_1(x) + q_2T_2(x)) - (p_0T_0(x) + p_1T_1(x) + p_2T_2(x)).$$

Example (3 of 7)

Now we must choose p_0, p_1, p_2, q_1, q_2 so that the coefficients of $T_k(x)$ are zero in the expanded form of

$$(a_0 T_0(x) + a_2 T_2(x) + a_4 T_4(x))(T_0(x) + q_1 T_1(x) + q_2 T_2(x)) - (p_0 T_0(x) + p_1 T_1(x) + p_2 T_2(x)).$$

Expanding the expression yields:

$$\begin{aligned} a_0 T_0(x) &- p_0 T_0(x) - p_1 T_1(x) + a_0 q_1 T_1(x) - p_2 T_2(x) \\ &+ a_0 q_2 T_2(x) + a_2 T_0(x) T_2(x) + a_2 q_1 T_1(x) T_2(x) + a_2 q_2 T_2^2(x) \\ &+ a_4 T_0(x) T_4(x) + a_4 q_1 T_1(x) T_4(x) + a_4 q_2 T_2(x) T_4(x) \end{aligned}$$

Now we may apply the product to sum formula.

Example (4 of 7)

$$\begin{aligned}
& a_0 T_0(x) - p_0 T_0(x) - p_1 T_1(x) + a_0 q_1 T_1(x) - p_2 T_2(x) \\
& \quad + a_0 q_2 T_2(x) + a_2 T_0(x) T_2(x) + a_2 q_1 T_1(x) T_2(x) + a_2 q_2 T_2^2(x) \\
& \quad + a_4 T_0(x) T_4(x) + a_4 q_1 T_1(x) T_4(x) + a_4 q_2 T_2(x) T_4(x) \\
= & \quad a_0 - p_0 - p_1 T_1(x) + a_0 q_1 T_1(x) - p_2 T_2(x) + a_0 q_2 T_2(x) \\
& \quad + a_2 T_2(x) + \frac{1}{2} a_2 q_1 T_3(x) + \frac{1}{2} a_2 q_1 T_1(x) + \frac{1}{2} a_2 q_2 T_4(x) \\
& \quad + \frac{1}{2} a_2 q_2 + a_4 T_4(x) + \frac{1}{2} a_4 q_1 T_5(x) + \frac{1}{2} a_4 q_1 T_3(x) \\
& \quad + \frac{1}{2} a_4 q_2 T_6(x) + \frac{1}{2} a_4 q_2 T_2(x)
\end{aligned}$$

Now we may collect the coefficients of $T_k(x)$ for $k = 0, 1, \ldots, 4$.

Example (5 of 7) Coefficients

$$
\begin{aligned}
T_0 &: & a_0 + \tfrac{a_2 q_2}{2} &= p_0 \\
T_1 &: & a_0 q_1 + \tfrac{a_2 q_1}{2} &= p_1 \\
T_2 &: & a_0 q_2 + a_2 + \tfrac{a_4 q_2}{2} &= p_2 \\
T_3 &: & a_2 q_1 + a_4 q_1 &= 0 \\
T_4 &: & a_4 + \tfrac{a_2 q_2}{2} &= 0
\end{aligned}
$$

Example (6 of 7)

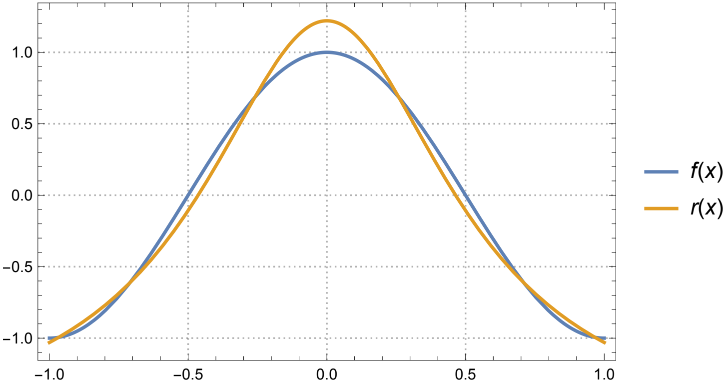
$$
\begin{aligned}
p_0 &= a_0 - a_4 = -0.607091 \\
p_1 &= 0 \\
p_2 &= \frac{a_2^2 - 2a_0a_4 - a_4^2}{a_2} = -1.06621 \\
q_1 &= 0 \\
q_2 &= -\frac{2a_4}{a_2} = 0.623873
\end{aligned}
$$

$$
\begin{aligned}
p_0 &= a_0 - a_4 = -0.607091 \\
p_1 &= 0 \\
p_2 &= \frac{a_2^2 - 2a_0a_4 - a_4^2}{a_2} = -1.06621 \\
q_1 &= 0 \\
q_2 &= -\frac{2a_4}{a_2} = 0.623873
\end{aligned}
$$

$$
r(x) = \frac{-0.607091\,T_0(x) - 1.06621\,T_2(x)}{T_0(x) + 0.623873\,T_2(x)}
$$

Example (7 of 7)

$$r(x) = \frac{-0.607091\,T_0(x) - 1.06621\,T_2(x)}{T_0(x) + 0.623873\,T_2(x)}$$



Homework

- Read Section 8.4
- Exercises: 1, 3, 8ac, 11, 14