# Relaxation Techniques for Solving Linear Systems

## MATH 375 *Numerical Analysis*

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# Objectives

In this lesson we will learn to

- accelerate the convergence of an iterative method for solving a linear system, and
- select the the iterative method with the most rapid convergence.

# Residual Vector

## Definition

Suppose vector $\tilde{\mathbf{x}} \in \mathbb{R}^n$ is an approximation to the solution of the linear system $A\mathbf{x} = \mathbf{b}$. The **residual vector** for $\tilde{\mathbf{x}}$ with respect to this system is $\mathbf{r} = \mathbf{b} - A\tilde{\mathbf{x}}$.

# Residual Vector for Gauss-Seidel

We would like the residual vectors to converge as rapidly as possible to the zero vector. Let the approximate solution vector to the Gauss-Seidel method be expressed as

$$\mathbf{x}_i^{(k)} = \begin{bmatrix} x_1^{(k)} \\ x_2^{(k)} \\ \vdots \\ x_{i-1}^{(k)} \\ x_i^{(k-1)} \\ \vdots \\ x_n^{(k-1)} \end{bmatrix}$$

Express the residual vector $\mathbf{r} = \mathbf{b} - A\tilde{\mathbf{x}}$.

$$\mathbf{r}_i^{(k)} = \mathbf{b} - A\mathbf{x}_i^{(k)}$$

$$\begin{bmatrix} r_{1i}^{(k)} \\ r_{2i}^{(k)} \\ \vdots \\ r_{mi}^{(k)} \\ \vdots \\ r_{ni}^{(k)} \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_m \\ \vdots \\ b_n \end{bmatrix} - \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \\ \vdots & \vdots & \ddots & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{bmatrix} \begin{bmatrix} x_1^{(k)} \\ x_2^{(k)} \\ \vdots \\ x_{i-1}^{(k)} \\ x_i^{(k-1)} \\ \vdots \\ x_n^{(k-1)} \end{bmatrix}$$

$$\mathbf{r}_i^{(k)} = \mathbf{b} - A\mathbf{x}_i^{(k)}$$

$$\begin{bmatrix} r_{1i}^{(k)} \\ r_{2i}^{(k)} \\ \vdots \\ r_{mi}^{(k)} \\ \vdots \\ r_{ni}^{(k)} \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_m \\ \vdots \\ b_n \end{bmatrix} - \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \\ \vdots & \vdots & \ddots & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{bmatrix} \begin{bmatrix} x_1^{(k)} \\ x_2^{(k)} \\ \vdots \\ x_{i-1}^{(k)} \\ x_i^{(k-1)} \\ \vdots \\ x_n^{(k-1)} \end{bmatrix}$$

**Note**: for $1 \leq m \leq n$

$$r_{mi}^{(k)} = b_m - \sum_{j=1}^{i-1} a_{mj} x_j^{(k)} - \sum_{j=i}^{n} a_{mj} x_j^{(k-1)}.$$

$$
\begin{aligned}
r_{mi}^{(k)} &= b_m - \sum_{j=1}^{i-1} a_{mj} x_j^{(k)} - \sum_{j=i}^{n} a_{mj} x_j^{(k-1)} \\
&= b_m - \sum_{j=1}^{i-1} a_{mj} x_j^{(k)} - a_{mi} x_i^{(k-1)} - \sum_{j=i+1}^{n} a_{mj} x_j^{(k-1)}
\end{aligned}
$$

---

$$
\begin{aligned}
r_{mi}^{(k)} &= b_m - \sum_{j=1}^{i-1} a_{mj} x_j^{(k)} - \sum_{j=i}^{n} a_{mj} x_j^{(k-1)} \\
&= b_m - \sum_{j=1}^{i-1} a_{mj} x_j^{(k)} - a_{mi} x_i^{(k-1)} - \sum_{j=i+1}^{n} a_{mj} x_j^{(k-1)}
\end{aligned}
$$

---

$$
\begin{aligned}
r_{ii}^{(k)} &= b_i - \sum_{j=1}^{i-1} a_{ij} x_j^{(k)} - a_{ii} x_i^{(k-1)} - \sum_{j=i+1}^{n} a_{ij} x_j^{(k-1)} \\
a_{ii} x_i^{(k-1)} + r_{ii}^{(k)} &= b_i - \sum_{j=1}^{i-1} a_{ij} x_j^{(k)} - \sum_{j=i+1}^{n} a_{ij} x_j^{(k-1)}
\end{aligned}
$$

$$
\begin{aligned}
r_{mi}^{(k)} &= b_m - \sum_{j=1}^{i-1} a_{mj} x_j^{(k)} - \sum_{j=i}^{n} a_{mj} x_j^{(k-1)} \\
&= b_m - \sum_{j=1}^{i-1} a_{mj} x_j^{(k)} - a_{mi} x_i^{(k-1)} - \sum_{j=i+1}^{n} a_{mj} x_j^{(k-1)}
\end{aligned}
$$

---

$$
\begin{aligned}
r_{ii}^{(k)} &= b_i - \sum_{j=1}^{i-1} a_{ij} x_j^{(k)} - a_{ii} x_i^{(k-1)} - \sum_{j=i+1}^{n} a_{ij} x_j^{(k-1)} \\
a_{ii} x_i^{(k-1)} + r_{ii}^{(k)} &= b_i - \sum_{j=1}^{i-1} a_{ij} x_j^{(k)} - \sum_{j=i+1}^{n} a_{ij} x_j^{(k-1)} \\
a_{ii} x_i^{(k-1)} + r_{ii}^{(k)} &= a_{ii} x_i^{(k)} \text{ (for Gauss-Seidel method)} \\
x_i^{(k)} &= x_i^{(k-1)} + \frac{r_{ii}^{(k)}}{a_{ii}}
\end{aligned}
$$

Thus the Gauss-Seidel method can be thought of as choosing $x_i^{(k)}$ such that

$$x_i^{(k)} = x_i^{(k-1)} + \frac{r_{ii}^{(k)}}{a_{ii}}.$$

Now consider the residual vector $\mathbf{r}_{i+1}^{(k)}$ associated with the approximate solution

$$\mathbf{x}_{i+1}^{(k)} = \begin{bmatrix} x_1^{(k)} \\ x_2^{(k)} \\ \vdots \\ x_i^{(k)} \\ x_{i+1}^{(k-1)} \\ \vdots \\ x_n^{(k-1)} \end{bmatrix}.$$

$$
\begin{aligned}
\mathbf{r}_{i+1}^{(k)} &= \mathbf{b} - A\,\mathbf{x}_{i+1}^{(k)} \\
\begin{bmatrix} r_{1,i+1}^{(k)} \\ r_{2,i+1}^{(k)} \\ \vdots \\ r_{m,i+1}^{(k)} \\ \vdots \\ r_{n,i+1}^{(k)} \end{bmatrix} &= \begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_m \\ \vdots \\ b_n \end{bmatrix} - \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \\ \vdots & \vdots & \ddots & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{bmatrix} \begin{bmatrix} x_1^{(k)} \\ x_2^{(k)} \\ \vdots \\ x_i^{(k)} \\ x_{i+1}^{(k-1)} \\ \vdots \\ x_n^{(k-1)} \end{bmatrix}
\end{aligned}
$$

$$\mathbf{r}_{i+1}^{(k)} = \mathbf{b} - A\mathbf{x}_{i+1}^{(k)}$$

$$\begin{bmatrix} r_{1,i+1}^{(k)} \\ r_{2,i+1}^{(k)} \\ \vdots \\ r_{m,i+1}^{(k)} \\ \vdots \\ r_{n,i+1}^{(k)} \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_m \\ \vdots \\ b_n \end{bmatrix} - \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \\ \vdots & \vdots & \ddots & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{bmatrix} \begin{bmatrix} x_1^{(k)} \\ x_2^{(k)} \\ \vdots \\ x_i^{(k)} \\ x_{i+1}^{(k-1)} \\ \vdots \\ x_n^{(k-1)} \end{bmatrix}$$

**Note**: for $1 \leq m \leq n$

$$r_{m,i+1}^{(k)} = b_m - \sum_{j=1}^{i} a_{mj} x_j^{(k)} - \sum_{j=i+1}^{n} a_{mj} x_j^{(k-1)}.$$

$$
\begin{aligned}
r_{m,i+1}^{(k)} &= b_m - \sum_{j=1}^{i} a_{mj} x_j^{(k)} - \sum_{j=i+1}^{n} a_{mj} x_j^{(k-1)} \\
&= b_m - \sum_{j=1}^{i-1} a_{mj} x_j^{(k)} - a_{mi} x_i^{(k)} - \sum_{j=i+1}^{n} a_{mj} x_j^{(k-1)}
\end{aligned}
$$

$$
\begin{aligned}
r_{m,i+1}^{(k)} &= b_m - \sum_{j=1}^{i} a_{mj} x_j^{(k)} - \sum_{j=i+1}^{n} a_{mj} x_j^{(k-1)} \\
&= b_m - \sum_{j=1}^{i-1} a_{mj} x_j^{(k)} - a_{mi} x_i^{(k)} - \sum_{j=i+1}^{n} a_{mj} x_j^{(k-1)}
\end{aligned}
$$

---

$$
\begin{aligned}
r_{i,i+1}^{(k)} &= b_i - \sum_{j=1}^{i-1} a_{ij} x_j^{(k)} - \sum_{j=i+1}^{n} a_{ij} x_j^{(k-1)} - a_{ii} x_i^{(k)} \\
&= 0 \text{ (for Gauss-Seidel method)}
\end{aligned}
$$

The Gauss-Seidel method chooses $x_{i+1}^{(k)}$ so that $r_{i,i+1}^{(k)} = 0$.

# Reducing the Norm of the Residual Vector

We make the following modification to the earlier equation:

$$x_i^{(k)} = x_i^{(k-1)} + \frac{r_{ii}^{(k)}}{a_{ii}}$$

in order the reduce the norm of the residual vector most efficiently. Let

$$x_i^{(k)} = x_i^{(k-1)} + \omega \frac{r_{ii}^{(k)}}{a_{ii}}$$

where $\omega > 0$.

Such a method is called a **relaxation method**.

- For $0 < \omega < 1$ these are called **under-relaxation methods.**
- For $1 < \omega$ these are called **over-relaxation methods.**

# Successive Over-Relaxation (SOR)

Let $1 < \omega$, then

$$\begin{aligned}
x_i^{(k)} &= x_i^{(k-1)} + \omega \frac{r_{ii}^{(k)}}{a_{ii}} \\
&= x_i^{(k-1)} + \frac{\omega}{a_{ii}} \left[ b_i - \sum_{j=1}^{i-1} a_{ij} x_j^{(k)} - a_{ii} x_i^{(k-1)} - \sum_{j=i+1}^{n} a_{ij} x_j^{(k-1)} \right] \\
&= (1-\omega) x_i^{(k-1)} + \frac{\omega}{a_{ii}} \left[ b_i - \sum_{j=1}^{i-1} a_{ij} x_j^{(k)} - \sum_{j=i+1}^{n} a_{ij} x_j^{(k-1)} \right]
\end{aligned}$$

# Matrix Formulation

The equation

$$x_i^{(k)} = (1-\omega)x_i^{(k-1)} + \frac{\omega}{a_{ii}}\left[b_i - \sum_{j=1}^{i-1} a_{ij}x_j^{(k)} - \sum_{j=i+1}^{n} a_{ij}x_j^{(k-1)}\right]$$

is equivalent to

$$a_{ii}x_i^{(k)} + \omega\sum_{j=1}^{i-1} a_{ij}x_j^{(k)} \quad = \quad (1-\omega)a_{ii}x_i^{(k-1)} - \omega\sum_{j=i+1}^{n} a_{ij}x_j^{(k-1)} + \omega b_i$$

# Matrix Formulation

The equation

$$x_i^{(k)} = (1-\omega)x_i^{(k-1)} + \frac{\omega}{a_{ii}}\left[b_i - \sum_{j=1}^{i-1} a_{ij}x_j^{(k)} - \sum_{j=i+1}^{n} a_{ij}x_j^{(k-1)}\right]$$

is equivalent to

$$\begin{aligned}
a_{ii}x_i^{(k)} + \omega\sum_{j=1}^{i-1} a_{ij}x_j^{(k)} &= (1-\omega)a_{ii}x_i^{(k-1)} - \omega\sum_{j=i+1}^{n} a_{ij}x_j^{(k-1)} + \omega b_i \\
(D-\omega L)\mathbf{x}^{(k)} &= [(1-\omega)D + \omega U]\,\mathbf{x}^{(k-1)} + \omega\mathbf{b} \\
\mathbf{x}^{(k)} &= (D-\omega L)^{-1}\,[(1-\omega)D + \omega U]\,\mathbf{x}^{(k-1)} + \omega(D-\omega L)^{-1}\mathbf{b}
\end{aligned}$$

# Matrix Formulation

The equation

$$x_i^{(k)} = (1-\omega)x_i^{(k-1)} + \frac{\omega}{a_{ii}}\left[b_i - \sum_{j=1}^{i-1} a_{ij}x_j^{(k)} - \sum_{j=i+1}^{n} a_{ij}x_j^{(k-1)}\right]$$

is equivalent to

$$\begin{aligned}
a_{ii}x_i^{(k)} + \omega\sum_{j=1}^{i-1} a_{ij}x_j^{(k)} &= (1-\omega)a_{ii}x_i^{(k-1)} - \omega\sum_{j=i+1}^{n} a_{ij}x_j^{(k-1)} + \omega b_i \\
(D-\omega L)\mathbf{x}^{(k)} &= \left[(1-\omega)D + \omega U\right]\mathbf{x}^{(k-1)} + \omega\mathbf{b} \\
\mathbf{x}^{(k)} &= (D-\omega L)^{-1}\left[(1-\omega)D + \omega U\right]\mathbf{x}^{(k-1)} + \omega(D-\omega L)^{-1}\mathbf{b} \\
\mathbf{x}^{(k)} &= T_\omega\mathbf{x}^{(k-1)} + \mathbf{c}_\omega
\end{aligned}$$

# Example

Compare the Gauss-Seidel iterative method to the SOR method with $\omega = 1.25$ for solving the following linear system. Use $\mathbf{x}^{(0)} = \mathbf{0}$ and let $\epsilon = 10^{-3}$.

$$\begin{aligned} -2x_1 + x_2 + \frac{1}{2}x_3 &= 4 \\ x_1 - 2x_2 - \frac{1}{2}x_3 &= -4 \\ x_2 + 2x_3 &= 0 \end{aligned}$$

The exact solution is

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} -16/11 \\ 16/11 \\ -8/11 \end{bmatrix} \approx \begin{bmatrix} -1.454545 \\ 1.454545 \\ -0.727273 \end{bmatrix}.$$

# Solution (1 of 5)

The Gauss-Seidel method can be expressed in matrix form as

$$\mathbf{x}^{(k)} = T_g \mathbf{x}^{(k-1)} + \mathbf{c}_g = \begin{bmatrix} 0 & 1/2 & 1/4 \\ 0 & 1/4 & -1/8 \\ 0 & -1/8 & 1/16 \end{bmatrix} \begin{bmatrix} x_1^{(k-1)} \\ x_2^{(k-1)} \\ x_3^{(k-1)} \end{bmatrix} + \begin{bmatrix} -2 \\ 1 \\ -1/2 \end{bmatrix}.$$

# Solution (2 of 5)

With $\omega = 1.25$ then

$$D - \omega L = \begin{bmatrix} -2 & 0 & 0 \\ 0 & -2 & 0 \\ 0 & 0 & 2 \end{bmatrix} - \frac{5}{4} \begin{bmatrix} 0 & 0 & 0 \\ -1 & 0 & 0 \\ 0 & -1 & 0 \end{bmatrix} = \begin{bmatrix} -2 & 0 & 0 \\ 5/4 & -2 & 0 \\ 0 & 5/4 & 2 \end{bmatrix}$$

$$(D - \omega L)^{-1} = \begin{bmatrix} -1/2 & 0 & 0 \\ -5/16 & -1/2 & 0 \\ 25/128 & 5/16 & 1/2 \end{bmatrix}$$

$$(1-\omega)D + \omega U = \frac{-1}{4} \begin{bmatrix} -2 & 0 & 0 \\ 0 & -2 & 0 \\ 0 & 0 & 2 \end{bmatrix} + \frac{5}{4} \begin{bmatrix} 0 & -1 & -1/2 \\ 0 & 0 & 1/2 \\ 0 & 0 & 0 \end{bmatrix}$$

$$= \begin{bmatrix} 1/2 & -5/4 & -5/8 \\ 0 & 1/2 & 5/8 \\ 0 & 0 & -1/2 \end{bmatrix}$$

# Solution (3 of 5)

$$
\begin{aligned}
T_\omega &= (D - \omega L)^{-1}\left[(1-\omega)D + \omega U\right] \\
&= \begin{bmatrix} -1/2 & 0 & 0 \\ -5/16 & -1/2 & 0 \\ 25/128 & 5/16 & 1/2 \end{bmatrix} \begin{bmatrix} 1/2 & -5/4 & -5/8 \\ 0 & 1/2 & 5/8 \\ 0 & 0 & -1/2 \end{bmatrix} \\
&= \frac{1}{1024} \begin{bmatrix} -256 & 640 & 320 \\ -160 & 144 & -120 \\ 100 & -90 & -181 \end{bmatrix} \\
\mathbf{c}_\omega &= \omega (D - \omega L)^{-1} \mathbf{b} \\
&= \frac{5}{4} \begin{bmatrix} -1/2 & 0 & 0 \\ -5/16 & -1/2 & 0 \\ 25/128 & 5/16 & 1/2 \end{bmatrix} \begin{bmatrix} 4 \\ -4 \\ 0 \end{bmatrix} \\
&= \frac{1}{128} \begin{bmatrix} -320 \\ 120 \\ -75 \end{bmatrix}
\end{aligned}
$$

# Solution (4 of 5)

The SOR method (with $\omega = 1.25$) can be expressed in matrix form as

$$\mathbf{x}^{(k)} = T_\omega \mathbf{x}^{(k-1)} + \mathbf{c}_\omega = \frac{1}{1024}\begin{bmatrix} -256 & 640 & 320 \\ -160 & 144 & -120 \\ 100 & -90 & -181 \end{bmatrix}\begin{bmatrix} x_1^{(k-1)} \\ x_2^{(k-1)} \\ x_3^{(k-1)} \end{bmatrix} + \frac{1}{128}\begin{bmatrix} -320 \\ 120 \\ -75 \end{bmatrix}$$

# Solution (5 of 5)

| $k$ | Gauss-Seidel $x_1^{(k)}$ | Gauss-Seidel $x_2^{(k)}$ | Gauss-Seidel $x_3^{(k)}$ | SOR ($\omega = 1.25$) $x_1^{(k)}$ | SOR ($\omega = 1.25$) $x_2^{(k)}$ | SOR ($\omega = 1.25$) $x_3^{(k)}$ |
|---|---|---|---|---|---|---|
| 0 | 0.0000 | 0.0000 | 0.0000 | 0.0000 | 0.0000 | 0.0000 |
| 1 | −2.0000 | 1.0000 | −0.5000 | −2.5000 | 0.9375 | −0.5859 |
| 2 | −1.6250 | 1.3125 | −0.6523 | −1.4722 | 1.5286 | −0.8089 |
| 3 | −1.5078 | 1.4102 | −0.7051 | −1.4294 | 1.4773 | −0.7211 |
| 4 | −1.4712 | 1.4407 | −0.7203 | −1.4447 | 1.4531 | −0.7279 |
| 5 | −1.4598 | 1.4502 | −0.7251 | −1.4581 | 1.4529 | −0.7261 |
| 6 | −1.4562 | 1.4532 | −0.7266 | −1.4543 | 1.4547 | −0.7277 |
| 7 | −1.4551 | 1.4541 | −0.7271 | −1.4546 | 1.4546 | −0.7272 |

# Choosing $\omega$ Optimally

**Comment**: there is no known method for choosing the best value of $\omega$ for the general $n \times n$ matrix $A$. For some commonly occurring special cases the optimal $\omega$ can be calculated.

# Choosing $\omega$ Optimally

**Comment**: there is no known method for choosing the best value of $\omega$ for the general $n \times n$ matrix $A$. For some commonly occurring special cases the optimal $\omega$ can be calculated.

### Theorem (Kahan)

*If $a_{ii} \neq 0$ for $i = 1, 2, \ldots, n$ then $\rho(T_\omega) \geq |\omega - 1|$. Thus the SOR method can converge only if $0 < \omega < 2$.*

# Positive Definite $A$

### Theorem (Ostrowski-Reich)

*If $A$ is a positive definite matrix and $0 < \omega < 2$ then the SOR method will converge for any choice of initial approximation vector $\mathbf{x}^{(0)}$.*

# Positive Definite $A$

## Theorem (Ostrowski-Reich)

*If $A$ is a positive definite matrix and $0 < \omega < 2$ then the SOR method will converge for any choice of initial approximation vector $\mathbf{x}^{(0)}$.*

## Theorem

*If $A$ is positive definite and tridiagonal, then $\rho(T_g) = [\rho(T_j)]^2 < 1$ and the optimal choice of $\omega$ for the SOR method is*

$$\omega = \frac{2}{1 + \sqrt{1 - [\rho(T_j)]^2}}.$$

*For this choice of $\omega$, then $\rho(T_\omega) = \omega - 1$.*

# Example

Find the optimal choice of $\omega$ for solving the following linear system.

$$\begin{aligned} 4x_1 + x_2 - x_3 &= 5 \\ -x_1 + 3x_2 + x_3 &= -4 \\ 2x_1 + 2x_2 + 5x_3 &= 1 \end{aligned}$$

The exact solution is

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 97/67 \\ -56/67 \\ -3/67 \end{bmatrix} \approx \begin{bmatrix} 1.447761 \\ -0.835821 \\ -0.044776 \end{bmatrix}.$$

# Solution (1 of 4)

In this positive definite, tridiagonal system

$$
\begin{aligned}
A &= \begin{bmatrix} 4 & 1 & -1 \\ -1 & 3 & 1 \\ 2 & 2 & 5 \end{bmatrix} \\
D &= \begin{bmatrix} 4 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 5 \end{bmatrix} \\
L &= \begin{bmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ -2 & -2 & 0 \end{bmatrix} \\
U &= \begin{bmatrix} 0 & -1 & 1 \\ 0 & 0 & -1 \\ 0 & 0 & 0 \end{bmatrix}.
\end{aligned}
$$

# Solution (2 of 4)

$$
\begin{aligned}
T_j &= D^{-1}(L+U) = \begin{bmatrix} 1/4 & 0 & 0 \\ 0 & 1/3 & 0 \\ 0 & 0 & 1/5 \end{bmatrix} \begin{bmatrix} 0 & -1 & 1 \\ 1 & 0 & -1 \\ -2 & -2 & 0 \end{bmatrix} \\
&= \begin{bmatrix} 0 & -1/4 & 1/4 \\ 1/3 & 0 & -1/3 \\ -2/5 & -2/5 & 0 \end{bmatrix}
\end{aligned}
$$

The eigenvalues of $T_j$ are

$$
\begin{aligned}
\lambda_1 &\approx 0.182266 + 0.386862i \\
\lambda_2 &\approx 0.182266 - 0.386862i \\
\lambda_3 &\approx -0.364531.
\end{aligned}
$$

The magnitudes of the eigenvalues of $T_j$ are

$$
\begin{aligned}
|\lambda_1| &\approx \sqrt{(0.182266)^2 + (0.386862)^2} \approx 0.427648 \\
|\lambda_2| &\approx \sqrt{(0.182266)^2 + (-0.386862)^2} \approx 0.427648 \\
|\lambda_3| &\approx |0.364531|.
\end{aligned}
$$

The spectral radius of $T_j$ is $\rho(T_j) \approx 0.427648$.

# Solution (3 of 4)

The optimal choice of $\omega$ for SOR is

$$\omega = \frac{2}{1+\sqrt{1-[\rho(T_j)]^2}} \approx \frac{2}{1+\sqrt{1-(0.427648)^2}} = 1.05045.$$

# Solution (3 of 4)

The optimal choice of $\omega$ for SOR is

$$\omega = \frac{2}{1+\sqrt{1-[\rho(T_j)]^2}} \approx \frac{2}{1+\sqrt{1-(0.427648)^2}} = 1.05045.$$

The SOR method for solving this linear system then takes the form:

$$\begin{aligned}
\mathbf{x}^{(k)} &= T_\omega \mathbf{x}^{(k-1)} + \mathbf{c}_\omega \\
\begin{bmatrix} x_1^{(k)} \\ x_3^{(k)} \\ x_3^{(k)} \end{bmatrix} &\approx \begin{bmatrix} -0.0504504 & -0.262613 & 0.262613 \\ -0.0176652 & -0.142404 & -0.258196 \\ 0.0286208 & 0.17018 & -0.0523061 \end{bmatrix} \begin{bmatrix} x_1^{(k-1)} \\ x_3^{(k-1)} \\ x_3^{(k-1)} \end{bmatrix} \\
&\quad + \begin{bmatrix} 1.31306 \\ -0.940831 \\ 0.0536857 \end{bmatrix}
\end{aligned}$$

# Solution (4 of 4)

| $k$ | $x_1^{(k)}$ | $x_2^{(k)}$ | $x_3^{(k)}$ |
|---|---|---|---|
| 0 | 0.0000 | 0.0000 | 0.0000 |
| 1 | 1.31306 | −0.940831 | 0.0536857 |
| 2 | 1.50799 | −0.84391 | −0.0716521 |
| 3 | 1.43979 | −0.828794 | −0.0430231 |
| 4 | 1.44678 | −0.837133 | −0.0439001 |
| 5 | 1.44839 | −0.835843 | −0.0450734 |
| 6 | 1.44766 | −0.835752 | −0.0447464 |
| 7 | 1.44776 | −0.835837 | −0.0447689 |
| 8 | 1.44777 | −0.83582 | −0.0447793 |
| 9 | 1.44776 | −0.83582 | −0.0447757 |
| 10 | 1.44776 | −0.835821 | −0.0447761 |

# Homework

- Read Section 7.4.
- Exercises: 1ac, 3ac, 7ac, 13