Review of Calculus

MATH 375 *Numerical Analysis*

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- The subject of this course is numerical **analysis**.
- Numerical analysis includes, but is not limited to, a study of numerical **methods**.
- In addition to numerical methods we will use the tools of calculus (real **analysis**) to understand the limitations and errors present in given numerical methods.
- We begin with a review of single variable calculus, particularly **Taylor's Theorem**.

Limits

Definition

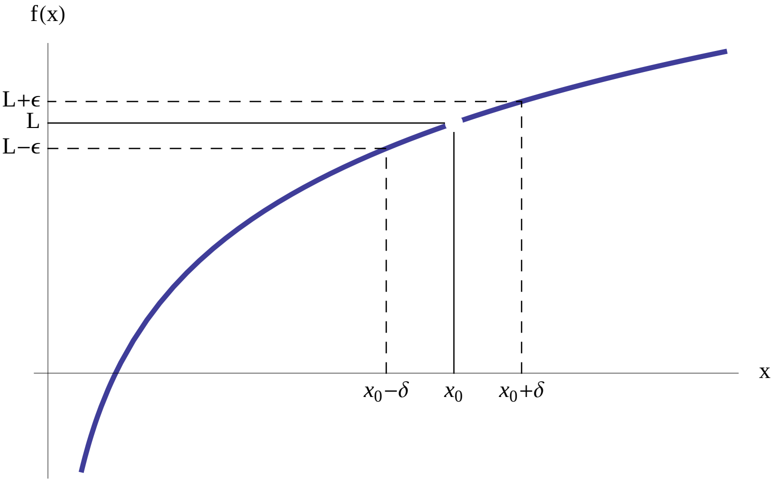
A function f defined on a set S of real numbers has a **limit** L at x_0, denoted

$$\lim_{x \to x_0} f(x) = L,$$

if for every $\epsilon > 0$, there exists a real number $\delta > 0$ such that

$$|f(x) - L| < \epsilon, \quad \text{whenever} \quad x \in S \quad \text{and} \quad 0 < |x - x_0| < \delta.$$

Graphical Interpretation



Limit of a Sequence

Definition

Let $\{x_n\}_{n=1}^{\infty}$ be an infinite sequence of real numbers. The sequence has the **limit** L (or is said to **converge** to L) if, for every $\epsilon > 0$ there exists a positive integer $N(\epsilon)$ such that whenever $n > N(\epsilon)$ then $|x_n - L| < \epsilon$.

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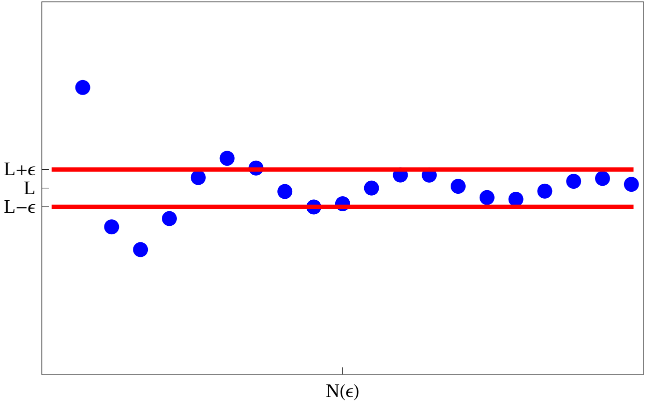
The notations

$$\begin{aligned} \lim_{n\to\infty} x_n &= L, \quad \text{or} \\ x_n &\to L \quad \text{as } n \to \infty \end{aligned}$$

denote the convergence of $\{x_n\}_{n=1}^{\infty}$ to L.

Graphical Interpretation



Continuity

Definition

Let function f be defined on a set S of real numbers and suppose $x_0 \in S$. We say that f is **continuous** at x_0 if

$$\lim_{x \to x_0} f(x) = f(x_0).$$

The function f is **continuous on the set** S if it is continuous at each $x \in S$.

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Definition

The set of functions continuous on set S will be denoted $C(S)$.

Continuity and Sequences

Theorem

Let f be a function defined on a set S and let $x_0 \in S$, then the following statements are equivalent.

1. *Function f is continuous at x_0.*
2. *If $\{x_n\}_{n=1}^{\infty}$ is any sequence in S converging to x_0, then $\lim_{n\to\infty} f(x_n) = f(x_0)$.*

Differentiability

Definition

Let f be a function defined in an open interval containing x_0. Then f is **differentiable** at x_0 if

$$\lim_{x \to x_0} \frac{f(x) - f(x_0)}{x - x_0}$$

exists. Provided the limit exists we denote it $f'(x_0)$ and call it the **derivative** of f at x_0. If function f has a derivative at each x in a set S, we say f is **differentiable** on S.

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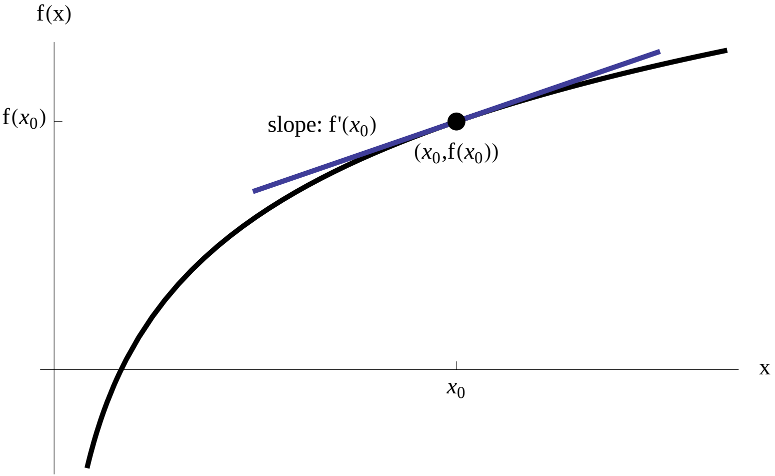
Theorem

If function f is differentiable at x_0, then f is continuous at x_0.

Graphical Interpretation



Sets of Differentiable Functions

We will frequently use the following notation.

- $C^n(S)$: the set of all functions that have n continuous derivatives on set S.
- $C^\infty(S)$: the set of all functions that have derivatives of all orders on set S.

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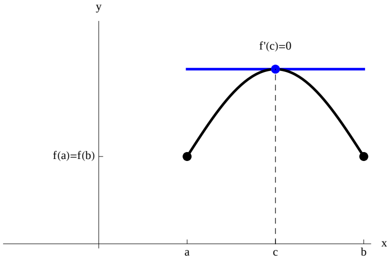
Example

1. If $p(x)$ is a polynomial, then $p \in C^\infty(\mathbb{R})$.
2. If $f(x) = \ln x$, then $f \in C^\infty(0, \infty)$.
3. If $f(x) = \displaystyle\int_0^x |t|\, dt$, then $f \in C^1(\mathbb{R})$.

Rolle's Theorem

Theorem

Suppose $f \in C[a,b]$ *and* f *is differentiable on* (a,b). *If* $f(a) = f(b)$, *then there exists* $c \in (a,b)$ *such that* $f'(c) = 0$.

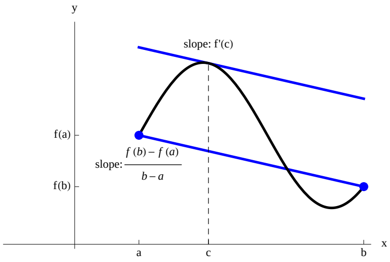


Mean Value Theorem

Theorem

Suppose $f \in C[a,b]$ and f is differentiable on (a,b). There exists $c \in (a,b)$ such that

$$f'(c) = \frac{f(b) - f(a)}{b - a}.$$



Extreme Value Theorem

Theorem

If $f \in C[a, b]$ *then there exist* $c_1 \in [a, b]$ *and* $c_2 \in [a, b]$ *such that* $f(c_1) \leq f(x) \leq f(c_2)$ *for all* $x \in [a, b]$. *If additionally* f *is differentiable on* (a, b), *then the numbers* c_1 *and* c_2 *occur either at the endpoints of* $[a, b]$ *or where* $f'(c) = 0$.

Extreme Value Theorem

Theorem

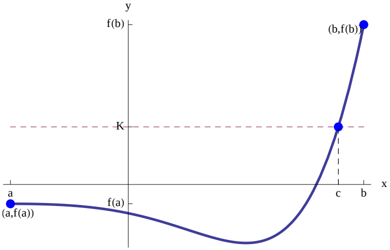
If $f \in C[a,b]$ then there exist $c_1 \in [a,b]$ and $c_2 \in [a,b]$ such that $f(c_1) \leq f(x) \leq f(c_2)$ for all $x \in [a,b]$. If additionally f is differentiable on (a,b), then the numbers c_1 and c_2 occur either at the endpoints of $[a,b]$ or where $f'(c) = 0$.

In the definition above $f(c_1)$ and $f(c_2)$ are referred to as the **absolute minimum** and **absolute maximum** of f on $[a,b]$ respectively.

Intermediate Value Theorem

Theorem

If $f \in C[a, b]$ and K is any number between $f(a)$ and $f(b)$, then there exists a number $c \in (a, b)$ for which $f(c) = K$.



Example

Show that $p(x) = x^5 - 3x^3 + 2x - 1$ has a root in the interval $[0, 2]$.

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Function p is a polynomial and therefore is continuous on $[0, 2]$.

$$\begin{aligned} p(0) &= -1 < 0 \\ p(2) &= 11 > 0 \end{aligned}$$

Since $p(0) < 0 < p(2)$ then by the IVT there exists $c \in (0, 2)$ for which $p(c) = 0$.

Generalized Rolle's Theorem

On occasion we will need a stronger version of Rolle's Theorem.

Theorem

Suppose $f \in C[a, b]$ is n times differentiable on (a, b). If f has $n + 1$ distinct roots

$$a \leq x_0 < x_1 < \cdots < x_n \leq b,$$

then there exists $c \in (x_0, x_n) \subset (a, b)$ such that $f^{(n)}(c) = 0$.

Riemann Integrals (1 of 2)

Definition

Let $P = \{x_0, x_1, \ldots, x_n\}$ be a set of numbers in $[a, b]$ such that

$$a = x_0 \leq x_1 \leq \cdots \leq x_n = b.$$

Set P is called a **partition** of $[a, b]$. If $\Delta x_i = x_i - x_{i-1}$ then $\|P\| = \max_{i=1,\ldots,n} \{\Delta x_i\}$ is called the **norm** of P.

Riemann Integrals (2 of 2)

Definition

Let function f be defined on $[a, b]$ and let P be any partition of $[a, b]$. If $z_i \in [x_{i-1}, x_i]$ for $i = 1, \ldots, n$ then the **Riemann integral** of f on $[a, b]$ is

$$\lim_{\|P\| \to 0} \sum_{i=1}^{n} f(z_i) \Delta x_i,$$

provided the limit exists and is the same for every partition and choice of z_i.

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provided the limit exists and is the same for every partition and choice of z_i.

If the Riemann integral exists, it will be denoted $\displaystyle\int_a^b f(x)\, dx$.

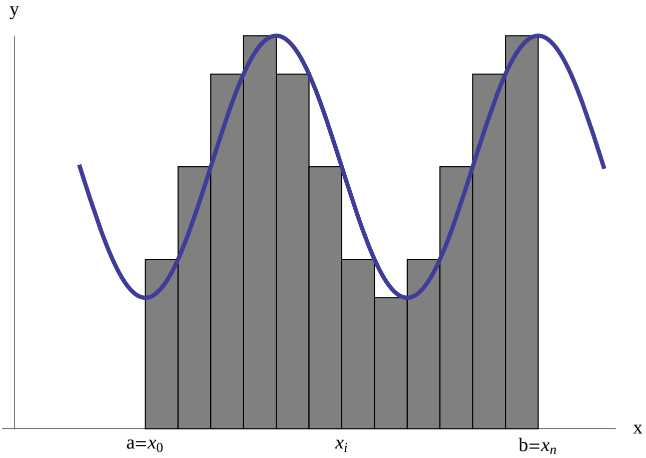
Riemann Integral and Continuity

Theorem

If $f \in C[a,b]$ then the Riemann integral of f on $[a,b]$ exists and can be evaluated as

$$\int_a^b f(x)\,dx = \lim_{n\to\infty} \frac{b-a}{n} \sum_{i=1}^{n} f\left(a + \frac{b-a}{n}\, i\right)$$

Graphical Interpretation



Weighted Mean Value Theorem for Integrals

Theorem

Suppose function $f \in C[a, b]$ and suppose function g is Riemann integrable on $[a, b]$ and that $g(x)$ does not change sign on $[a, b]$. There exists $c \in (a, b)$ such that

$$\int_a^b f(x)\, g(x)\, dx = f(c) \int_a^b g(x)\, dx.$$

Weighted Mean Value Theorem for Integrals

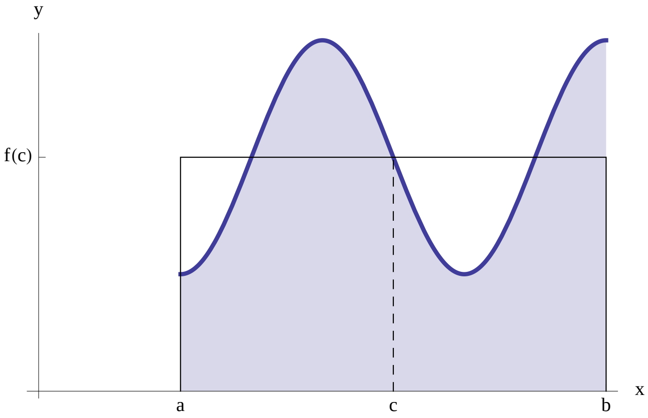
Theorem

Suppose function $f \in C[a,b]$ *and suppose function* g *is Riemann integrable on* $[a,b]$ *and that* $g(x)$ *does not change sign on* $[a,b]$*. There exists* $c \in (a,b)$ *such that*

$$\int_a^b f(x)\,g(x)\,dx = f(c)\int_a^b g(x)\,dx.$$

If $g(x) = 1$ on $[a,b]$ then $f(c)$ can be called the **average value** of f on $[a,b]$.

Average Value: Graphical Interpretation



Taylor's Theorem

Theorem

Suppose function $f \in C^n[a,b]$ and that $f^{(n+1)}$ exists on $[a,b]$ and let $x_0 \in [a,b]$. For every $x \in [a,b]$, there exists a number $z(x)$ between x_0 and x such that $f(x) = P_n(x) + R_n(x)$, where

$$P_n(x) = \sum_{k=0}^{n} \frac{f^{(k)}(x_0)}{k!}(x - x_0)^k$$

*is called the n**th Taylor polynomial** for f about x_0, and*

$$R_n(x) = \frac{f^{(n+1)}(z(x))}{(n+1)!}(x - x_0)^{n+1}$$

*is called the n**th Taylor remainder**.*

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$$R_n(x) = \frac{f^{(n+1)}(z(x))}{(n+1)!}(x - x_0)^{n+1}$$

*is called the n**th Taylor remainder**.*

In the case where $x_0 = 0$ these are sometimes called **Maclaurin polynomials**.

Example

Let $f(x) = \ln x$ and $x_0 = 1$.

1. Find the second Taylor polynomial and remainder for f about x_0. Use this polynomial to approximate $\ln 1.1$ and estimate the error in the approximation from the remainder.
2. Find the third Taylor polynomial and remainder for f about x_0. Use this polynomial to approximate $\ln 1.1$ and estimate the error in the approximation from the remainder.

Solution: $n = 2$

$$
\begin{aligned}
\ln x &= P_2(x) + R_2(x) \\
&= 0 + \frac{1/1}{1!}(x-1) - \frac{1/1^2}{2!}(x-1)^2 + \frac{2/(z(x))^3}{3!}(x-1)^3 \\
&= (x-1) - \frac{1}{2}(x-1)^2 + \frac{1}{3(z(x))^3}(x-1)^3
\end{aligned}
$$

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If $x = 1.1$ then $P_2(1.1) = 0.095$ and

$$|\ln 1.1 - P_2(1.1)| = |R_2(1.1)| = 0.000\overline{3}\,\frac{1}{(z(1.1))^3} \leq 0.000\overline{3}$$

since $1 \leq z(1.1) \leq 1.1$.

Solution: $n = 3$

$$
\begin{aligned}
\ln x &= P_3(x) + R_3(x) \\
&= 0 + \frac{1/1}{1!}(x-1) - \frac{1/1^2}{2!}(x-1)^2 + \frac{2/1^3}{3!}(x-1)^3 \\
&\quad - \frac{6/(z(x))^4}{4!}(x-1)^4 \\
&= (x-1) - \frac{1}{2}(x-1)^2 + \frac{1}{3}(x-1)^3 - \frac{1}{4(z(x))^4}(x-1)^4
\end{aligned}
$$

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\end{aligned}
$$

If $x = 1.1$ then $P_3(1.1) = 0.095\overline{3}$ and

$$|\ln 1.1 - P_3(1.1)| = |R_3(1.1)| = 0.000025\,\frac{1}{(z(1.1))^4} \leq 0.000025$$

since $1 \leq z(1.1) \leq 1.1$.

Return to Course Objectives

Our use of Taylor's Theorem illustrates two of the objectives of *Numerical Analysis*.

1. Given a problem, find an approximation to the solution to the problem.
2. Determine a bound for the accuracy of the approximated solution.

Example

Use the third Taylor polynomial and remainder for $f(x) = \ln x$ about $x_0 = 1$ to approximate

$$\int_1^{1.1} \ln x \, dx$$

and determine a bound for the error in the approximation.

Solution (1 of 2)

$$\begin{aligned}
\int_1^{1.1} \ln x \, dx &= \int_1^{1.1} P_3(x)\, dx + \int_1^{1.1} R_3(x)\, dx \\
&= \int_1^{1.1} \left[(x-1) - \frac{1}{2}(x-1)^2 + \frac{1}{3}(x-1)^3 \right] dx \\
&\quad - \int_1^{1.1} \frac{1}{4(z(x))^4}(x-1)^4 \, dx \\
&= \left[\frac{(x-1)^2}{2} - \frac{(x-1)^3}{6} + \frac{(x-1)^4}{12} \right]_{x=1}^{x=1.1} \\
&\quad - \frac{1}{4} \int_1^{1.1} \frac{(x-1)^4}{(z(x))^4}\, dx \\
&= \frac{(0.1)^2}{2} - \frac{(0.1)^3}{6} + \frac{(0.1)^4}{12} - \frac{1}{4}\int_1^{1.1} \frac{(x-1)^4}{(z(x))^4}\, dx \\
&= 0.004841\overline{6} - \frac{1}{4}\int_1^{1.1} \frac{(x-1)^4}{(z(x))^4}\, dx
\end{aligned}$$

Solution (2 of 2)

$$\int_1^{1.1} \ln x \, dx \approx 0.004841\overline{6}$$

and the error is

$$\begin{aligned}
\left| \int_1^{1.1} (\ln x - P_3(x)) \, dx \right| &= \left| \int_1^{1.1} R_3(x) \, dx \right| \\
&= \frac{1}{4} \left| \int_1^{1.1} \frac{(x-1)^4}{(z(x))^4} \, dx \right| \\
&\leq \frac{1}{4} \int_1^{1.1} \left| \frac{(x-1)^4}{(z(x))^4} \right| dx \\
&\leq \frac{1}{4} \int_1^{1.1} (x-1)^4 \, dx \\
&= 5 \times 10^{-7}.
\end{aligned}$$

Taylor Series

If function $f \in C^{\infty}[a, b]$ and $\lim_{n \to \infty} R_n(x) = 0$ then we may call

$$\lim_{n \to \infty} P_n(x) = \sum_{k=0}^{\infty} \frac{f^{(k)}(x_0)}{k!}(x - x_0)^k$$

the Taylor series for f about x_0 provided the infinite series converges.

Homework

- Read Section 1.1.
- Exercises: 1, 4, 11, 16, 17, 19, 22, 28