Richardson's Extrapolation

MATH 375 *Numerical Analysis*

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Objectives

Recall the centered-difference formula for $f'(x_0)$:

$$f'(x_0) = \frac{1}{2h}\left[f(x_0+h) - f(x_0-h)\right] - \underbrace{\frac{h^2}{6} f^{(3)}(z)}_{\text{truncation error}}$$

In today's lesson we will learn to create higher-accuracy approximations while using lower-order formulas.

The technique, known as **extrapolation** can be used whenever the truncation error has a predictable form (as above) and depends on a parameter such as h, the step size.

General Setting

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for some unknown constants $K_1, K_2, K_3, \ldots$.

Note that

$$\begin{aligned} M - N_1(0.1) &= K_1(0.1) + K_2(0.1)^2 + \cdots \approx (0.1)K_1 \\ M - N_1(0.01) &= K_1(0.01) + K_2(0.01)^2 + \cdots \approx (0.01)K_1 \end{aligned}$$

and in general $M - N_1(h) \approx K_1 h$.

Order of the Truncation Error

Question: Since the truncation error is $O(h)$, can we combine several $O(h)$ approximations to create an $O(h^n)$ approximation where $n \geq 2$?

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M &= N_1\left(\frac{h}{2}\right) + K_1 \frac{h}{2} + K_2 \frac{h^2}{4} + K_3 \frac{h^3}{8} + \cdots
\end{aligned}
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\end{aligned}
$$

Multiply the 2nd equation by 2 and subtract the 1st equation.

$$
\begin{aligned}
M &= 2N_1\left(\frac{h}{2}\right) - N_1(h) + K_2\left[\frac{h^2}{2} - h^2\right] + K_3\left[\frac{h^3}{4} - h^3\right] + \cdots \\
&= N_1\left(\frac{h}{2}\right) + \left[N_1\left(\frac{h}{2}\right) - N_1(h)\right] - \frac{K_2}{2} h^2 - \frac{3K_3}{4} h^3 - \cdots
\end{aligned}
$$

Note: the $O(h)$ truncation terms have vanished.

A Second Approximation

Define $N_2(h) = N_1\left(\frac{h}{2}\right) + \left[N_1\left(\frac{h}{2}\right) - N_1(h)\right]$ and then we have

$$M = N_2(h) - \frac{K_2}{2}h^2 - \frac{3K_3}{4}h^3 - \cdots$$

which has $O(h^2)$ truncation error.

Note: we have combined multiple $O(h)$ approximations to generate an $O(h^2)$ approximation.

Example

Recall the 2-point forward-difference formula for $f'(x_0)$:

$$f'(x_0) = \frac{1}{h}(f(x_0 + h) - f(x_0)) - \frac{f''(z)}{2}h$$

This is an $O(h)$ truncation error.

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Let $f(x) = x \sin x$ then we have:

h	$f'(1) \approx N_1(h)$	**Abs. Err.**
0.10	1.38857	0.00679782
0.05	1.38647	0.00469475

Applying the extrapolation formula gives us another approximation:

$$N_2(0.1) = N_1(0.05) + (N_1(0.05) - N_1(0.1)) = 1.38436$$

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Note that $|N_2(0.1) - f'(1)| \approx 0.00259168$.

Example

Use Richardson's Extrapolation and the 2-point forward difference formula for $f'(x_0)$ to develop an $O(h^2)$ approximation to $f'(2)$ where $f(x) = x^2 \cos x$ using $h = 0.1$.

Example

Use Richardson's Extrapolation and the 2-point forward difference formula for $f'(x_0)$ to develop an $O(h^2)$ approximation to $f'(2)$ where $f(x) = x^2 \cos x$ using $h = 0.1$.

h	$f'(2) \approx N_1(h)$	**Abs. Err.**
0.10	-5.61784	0.316063
0.05	-5.46141	0.159636

Applying the extrapolation formula gives us another approximation:

$$N_2(0.1) = N_1(0.05) + (N_1(0.05) - N_1(0.1)) = -5.30499$$

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Note that $|N_2(0.1) - f'(2)| \approx 0.00320877$.

Improving Richardson's Extrapolation

Remark: If the truncation error contains only even powers of h, the extrapolation is more effective.

Suppose

$$\begin{aligned} M &= N_1(h) + K_1 h^2 + K_2 h^4 + K_3 h^6 + \cdots \\ M &= N_1\left(\frac{h}{2}\right) + K_1 \frac{h^2}{4} + K_2 \frac{h^4}{16} + K_3 \frac{h^6}{64} + \cdots \end{aligned}$$

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\end{aligned}
$$

Multiply the 2nd equation by 4 and subtract the 1st equation.

$$
3M = \left[4N_1\left(\frac{h}{2}\right) - N_1(h)\right] + K_2\left[\frac{h^4}{4} - h^4\right] + K_3\left[\frac{h^6}{16} - h^6\right] + \cdots
$$

$O(h^4)$ Truncation Error

If we multiply the previous equation by $1/3$ we obtain

$$M = \frac{1}{3}\left[4N_1\left(\frac{h}{2}\right) - N_1(h)\right] + \frac{K_2}{3}\left[\frac{h^4}{4} - h^4\right] + \frac{K_3}{3}\left[\frac{h^6}{16} - h^6\right] + \cdots$$

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Define

$$N_2(h) = \frac{1}{3}\left[4N_1\left(\frac{h}{2}\right) - N_1(h)\right] = N_1\left(\frac{h}{2}\right) + \frac{1}{3}\left[N_1\left(\frac{h}{2}\right) - N_1(h)\right].$$

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If we multiply the previous equation by $1/3$ we obtain

$$M = \frac{1}{3}\left[4N_1\left(\frac{h}{2}\right) - N_1(h)\right] + \frac{K_2}{3}\left[\frac{h^4}{4} - h^4\right] + \frac{K_3}{3}\left[\frac{h^6}{16} - h^6\right] + \cdots$$

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This is an approximation formula with truncation error $O(h^4)$.

$$M = N_2(h) - \frac{K_2}{4}h^4 - \frac{5K_3}{16}h^6 + \cdots$$

Example

Recall the 3-point centered-difference formula for $f'(x_0)$:

$$f'(x_0) = \frac{1}{2h}(f(x_0 + h) - f(x_0 - h)) - \frac{f'''(z)}{6}h^2$$

This is an $O(h^2)$ truncation error.

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Recall the 3-point centered-difference formula for $f'(x_0)$:

$$f'(x_0) = \frac{1}{2h}(f(x_0+h) - f(x_0-h)) - \frac{f'''(z)}{6}h^2$$

This is an $O(h^2)$ truncation error.

Let $f(x) = x \sin x$ then we have:

h	$f'(1) \approx N_1(h)$	**Abs. Err.**
0.10	1.37667	0.0051039
0.05	1.38050	0.0012767

Applying the extrapolation formula gives us another approximation:

$$N_2(0.1) = N_1(0.05) + \frac{1}{3}(N_1(0.05) - N_1(0.1)) = 1.38177$$

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Note that $|N_2(0.1) - f'(1)| \approx 9.88697 \times 10^{-7}$.

Example

Use Richardson's Extrapolation and the 3-point centered-difference formula for $f'(x_0)$ to develop an $O(h^4)$ approximation to $f'(2)$ where $f(x) = x^2 \cos x$ using $h = 0.1$.

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Use Richardson's Extrapolation and the 3-point centered-difference formula for $f'(x_0)$ to develop an $O(h^4)$ approximation to $f'(2)$ where $f(x) = x^2 \cos x$ using $h = 0.1$.

h	$f'(2) \approx N_1(h)$	**Abs. Err.**
0.10	-5.29648	0.00529713
0.05	-5.30045	0.00132331

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Note that $|N_2(0.1) - f'(2)| \approx 1.29563 \times 10^{-6}$.

Return to the $O(h^4)$ Formula

Recall:

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Replace h by $h/2$:

$$M = N_2\left(\frac{h}{2}\right) - \frac{K_2}{64}h^4 - \frac{5K_3}{1024}h^6 + \cdots$$

which is also has an $O(h^4)$ truncation error.

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which is also has an $O(h^4)$ truncation error.

Multiply the 2nd equation by 16 and subtract the first equation from it.

$$15M = \left[16N_2\left(\frac{h}{2}\right) - N_2(h)\right] + \frac{15K_3}{64}h^6 + \cdots$$

$O(h^6)$ Truncation Error

Multiplying both sides of the last equation by $1/15$ yields:

$$M = \frac{1}{15}\left[16N_2\left(\frac{h}{2}\right) - N_2(h)\right] + \frac{K_3}{64}h^6 + \cdots$$

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We can define

$$N_3(h) = N_2\left(\frac{h}{2}\right) + \frac{1}{15}\left[N_2\left(\frac{h}{2}\right) - N_2(h)\right].$$

This approximation formula has an $O(h^6)$ truncation error.

Example

We will use two approximations to $f'(1)$ where $f(x) = x \sin x$ with $O(h^4)$ truncation errors to develop an approximation with $O(h^6)$ truncation error.

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h	$f'(1) \approx N_2(h)$	**Abs. Err.**
0.10	1.38177	9.88697×10^{-7}
0.05	1.38177	6.18122×10^{-8}

Applying the extrapolation formula gives us another approximation:

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Note that $|N_3(0.1) - f'(1)| \approx 1.99358 \times 10^{-11}$.

Example

Use Richardson's Extrapolation and the 3-point centered-difference formula for $f'(x_0)$ to develop an $O(h^6)$ approximation to $f'(2)$ where $f(x) = x^2 \cos x$ using $h = 0.1$.

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h	$f'(2) \approx N_2(h)$	**Abs. Err.**
0.10	-5.30178	1.29563×10^{-6}
0.05	-5.30178	8.10432×10^{-8}

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Note that $|N_3(0.1) - f'(2)| \approx 7.09512 \times 10^{-11}$.

General Situation

For $j = 2, 3, \ldots$ the $O(h^{2j})$ truncation error approximation is given by the formula

$$N_j(h) = N_{j-1}\left(\frac{h}{2}\right) + \frac{1}{4^{j-1} - 1}\left[N_{j-1}\left(\frac{h}{2}\right) - N_{j-1}(h)\right].$$

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Multi-point Differentiation Formulas

Remark: Richardson's extrapolation provides a convenient means for developing the 5-point approximations to $f'(x_0)$.

Assuming $f \in \mathcal{C}^5[a, b]$ and $x_0 \in (a, b)$ expand $f(x)$ as a degree 4 Taylor polynomial about x_0.

$$
\begin{aligned}
f(x) \quad = \quad & f(x_0) + f'(x_0)(x - x_0) + \frac{1}{2} f''(x_0)(x - x_0)^2 \\
& + \frac{1}{6} f'''(x_0)(x - x_0)^3 + \frac{1}{24} f^{(4)}(x_0)(x - x_0)^4 \\
& + \frac{1}{120} f^{(5)}(z)(x - x_0)^5
\end{aligned}
$$

where z lies between x and x_0.

Five-Point Formula (1 of 5)

Evaluate the Taylor polynomial expansion at $x = x_0 \pm h$.

$$
\begin{aligned}
f(x_0 + h) &= f(x_0) + f'(x_0)h + \frac{1}{2}f''(x_0)h^2 + \frac{1}{6}f'''(x_0)h^3 \\
&\quad + \frac{1}{24}f^{(4)}(x_0)h^4 + \frac{1}{120}f^{(5)}(z_1)h^5 \\
f(x_0 - h) &= f(x_0) - f'(x_0)h + \frac{1}{2}f''(x_0)h^2 - \frac{1}{6}f'''(x_0)h^3 \\
&\quad + \frac{1}{24}f^{(4)}(x_0)h^4 - \frac{1}{120}f^{(5)}(z_2)h^5
\end{aligned}
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with $x_0 - h \leq z_2 \leq x_0 \leq z_1 \leq x_0 + h$.

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&\quad + \frac{1}{24}f^{(4)}(x_0)h^4 + \frac{1}{120}f^{(5)}(z_1)h^5 \\
f(x_0 - h) &= f(x_0) - f'(x_0)h + \frac{1}{2}f''(x_0)h^2 - \frac{1}{6}f'''(x_0)h^3 \\
&\quad + \frac{1}{24}f^{(4)}(x_0)h^4 - \frac{1}{120}f^{(5)}(z_2)h^5
\end{aligned}$$

with $x_0 - h \leq z_2 \leq x_0 \leq z_1 \leq x_0 + h$.

Now subtract the 2nd equation from the 1st equation.

Five-Point Formula (2 of 5)

$$f(x_0 + h) - f(x_0 - h) = 2hf'(x_0) + \frac{h^3}{3} f'''(x_0) + \frac{h^5}{120} \left[f^{(5)}(z_1) + f^{(5)}(z_2) \right]$$

Five-Point Formula (2 of 5)

$$f(x_0+h)-f(x_0-h)=2hf'(x_0)+\frac{h^3}{3}f'''(x_0)+\frac{h^5}{120}\left[f^{(5)}(z_1)+f^{(5)}(z_2)\right]$$

By assuming that $f \in \mathcal{C}^5[a,b]$ we know $f^{(5)}(x)$ is continuous on $[a,b]$.

Note that

$$\frac{1}{2}\left[f^{(5)}(z_1)+f^{(5)}(z_2)\right]$$

lies between $f^{(5)}(z_1)$ and $f^{(5)}(z_2)$.

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lies between $f^{(5)}(z_1)$ and $f^{(5)}(z_2)$.

According to the Intermediate Value Theorem there exists w between z_1 and z_2 for which

$$\begin{aligned} f^{(5)}(w) &= \frac{1}{2}\left[f^{(5)}(z_1) + f^{(5)}(z_2)\right] \\ 2f^{(5)}(w) &= f^{(5)}(z_1) + f^{(5)}(z_2) \end{aligned}$$

Five-Point Formula (3 of 5)

Thus we may write the Taylor polynomial difference as

$$f(x_0+h)-f(x_0-h)=2hf'(x_0)+\frac{h^3}{3}f'''(x_0)+\frac{h^5}{60}f^{(5)}(w)$$

for some $x_0-h \leq w \leq x_0+h$.

Solve this equation for $f'(x_0)$.

Five-Point Formula (3 of 5)

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$$f(x_0 + h) - f(x_0 - h) = 2hf'(x_0) + \frac{h^3}{3} f'''(x_0) + \frac{h^5}{60} f^{(5)}(w)$$

for some $x_0 - h \leq w \leq x_0 + h$.

Solve this equation for $f'(x_0)$.

$$f'(x_0) = \frac{1}{2h} [f(x_0 + h) - f(x_0 - h)] - \frac{h^2}{6} f'''(x_0) - \frac{h^4}{120} f^{(5)}(w)$$

Now apply the Richardson's extrapolation technique to this approximation.

Five-Point Formula (4 of 5)

$$f'(x_0) = \frac{1}{2h}\left[f(x_0+h) - f(x_0-h)\right] - \frac{h^2}{6}f'''(x_0) - \frac{h^4}{120}f^{(5)}(w)$$

If we replace h by $2h$ we get

$$f'(x_0) = \frac{1}{4h}\left[f(x_0+2h) - f(x_0-2h)\right] - \frac{4h^2}{6}f'''(x_0) - \frac{16h^4}{120}f^{(5)}(\tilde{w})$$

where $\tilde{w}$ lies between $x_0 - 2h$ and $x_0 + 2h$.

Five-Point Formula (4 of 5)

$$f'(x_0) = \frac{1}{2h}\left[f(x_0+h) - f(x_0-h)\right] - \frac{h^2}{6}f'''(x_0) - \frac{h^4}{120}f^{(5)}(w)$$

If we replace h by $2h$ we get

$$f'(x_0) = \frac{1}{4h}\left[f(x_0+2h) - f(x_0-2h)\right] - \frac{4h^2}{6}f'''(x_0) - \frac{16h^4}{120}f^{(5)}(\tilde{w})$$

where $\tilde{w}$ lies between $x_0 - 2h$ and $x_0 + 2h$.

Multiply the 1st equation by 4:

$$4f'(x_0) = \frac{4}{2h}\left[f(x_0+h) - f(x_0-h)\right] - \frac{4h^2}{6}f'''(x_0) - \frac{4h^4}{120}f^{(5)}(w)$$

and subtract the 2nd equation.

Five-Point Formula (5 of 5)

$$\begin{aligned}
3f'(x_0) &= \frac{2}{h}\left[f(x_0+h)-f(x_0-h)\right]-\frac{1}{4h}\left[f(x_0+2h)-f(x_0-2h)\right] \\
&\quad -\frac{4h^4}{120}f^{(5)}(w)+\frac{16h^4}{120}f^{(5)}(\tilde{w}) \\
f'(x_0) &= \frac{2}{3h}\left[f(x_0+h)-f(x_0-h)\right]-\frac{1}{12h}\left[f(x_0+2h)-f(x_0-2h)\right] \\
&\quad -\frac{h^4}{90}f^{(5)}(w)+\frac{2h^4}{45}f^{(5)}(\tilde{w}) \\
&= \frac{1}{12h}\left[f(x_0-2h)-8f(x_0-h)+8f(x_0+h)-f(x_0+2h)\right] \\
&\quad -\frac{h^4}{30}f^{(5)}(\hat{w})
\end{aligned}$$

The form of the truncation error has not been justified.

Homework

- Read Section 4.2.
- Exercises: 1a, 7, 9