Romberg Integration

MATH 375 *Numerical Analysis*

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Objectives and Background

In this lesson we will learn to obtain high accuracy approximations to definite integrals using the **Composite Trapezoidal Rule**.

$$\int_a^b f(x)\,dx = \frac{h}{2}\left[f(a) + 2\sum_{j=1}^{n-1} f(x_j) + f(b)\right] - \frac{(b-a)f''(\mu)}{12}h^2$$

where $h = \dfrac{b-a}{n}$ and $x_j = a + jh$ for $j = 0, 1, \ldots, n$.

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where $h = \dfrac{b-a}{n}$ and $x_j = a + jh$ for $j = 0, 1, \ldots, n$.

We will use Richardson's Extrapolation to develop $O(h^{2m})$ approximations to the definite integral.

Alternative Truncation Error

If $f \in \mathcal{C}^\infty[a, b]$ then we may write the truncation error of the Composite Trapezoidal Rule as

$$\int_a^b f(x)\,dx = \frac{h}{2}\left[f(a) + 2\sum_{j=1}^{n-1} f(x_j) + f(b)\right] + K_1 h^2 + K_2 h^4 + K_3 h^6 + \cdots$$

where the coefficients K_i are constants.

The truncation error contains only even powers of h and we have seen Richardson's extrapolation be very effective in this situation.

Extrapolation applied to this quadrature formula is called **Romberg integration**.

Richardson's Extrapolation

For $j = 2, 3, \ldots$ the $O(h^{2j})$ truncation error approximation is given by the formula

$$N_j(h) = N_{j-1}\left(\frac{h}{2}\right) + \frac{1}{4^{j-1} - 1}\left[N_{j-1}\left(\frac{h}{2}\right) - N_{j-1}(h)\right].$$

Outline

Given $\int_a^b f(x)\,dx$, use the Composite Trapezoidal rule with $n = 1, 2, 4, 8, \ldots, 2^k, \ldots$ to form approximations labeled respectively

$$R_{1,1},\ R_{2,1},\ R_{3,1},\ R_{4,1},\ \ldots R_{k+1,1},\ \ldots$$

Since these are $O(h^2)$ approximations we can combine them using Richardson's extrapolation to form $O(h^4)$ approximations.

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For example,

$$R_{k,2} = R_{k,1} + \frac{1}{3}\left(R_{k,1} - R_{k-1,1}\right),$$

for $k = 2, 3, \ldots$.

Test Case

Throughout this presentation we will use the following definite integral to evaluate the performance of the various approximations.

$$\begin{aligned}\int_0^{\pi} x\cos 3x\,dx &= \left[\frac{1}{9}\cos 3x + \frac{1}{3}x\sin 3x\right]_{x=0}^{x=\pi} \\ &= -\frac{2}{9} \\ &\approx -0.222222\end{aligned}$$

Remark: the antiderivative was found using integration by parts.

Example (1 of 5)

Approximate the definite integral using the Composite Trapezoidal Rule.

n	$R_{k+1,1}$	**Abs. Err.**
2^0	-4.9348	4.71258
2^1	-2.4674	2.24518
2^2	-0.361343	0.13912
2^3	-0.249811	0.0275889
2^4	-0.228761	0.0063855

Example (2 of 5)

Now form the $O(h^4)$ approximations using the formula

$$R_{k,2} = R_{k,1} + \frac{1}{3}\left(R_{k,1} - R_{k-1,1}\right)$$

n	$R_{k+1,1}$	$R_{k+1,2}$	**Abs. Err.**
2^0	-4.9348		
2^1	-2.4674	-1.64493	1.42271
2^2	-0.361343	0.340677	0.562899
2^3	-0.249811	-0.212634	0.00958818
2^4	-0.228761	-0.221744	0.000478242

Note that

$$\begin{aligned} R_{2,2} &= R_{2,1} + \frac{1}{3}(R_{2,1} - R_{1,1}) \\ &= -2.4674 + \frac{1}{3}(-2.4674 - (-4.9348)) \\ &= -1.64493 \end{aligned}$$

Example (3 of 5)

Now form the $O(h^6)$ approximations using the formula

$$R_{k,3} = R_{k,2} + \frac{1}{15}\left(R_{k,2} - R_{k-1,2}\right)$$

n	$R_{k+1,1}$	$R_{k+1,2}$	$R_{k+1,3}$	**Abs. Err.**
2^0	-4.9348			
2^1	-2.4674	-1.64493		
2^2	-0.361343	0.340677	0.473051	0.695273
2^3	-0.249811	-0.212634	-0.249521	0.0272992
2^4	-0.228761	-0.221744	-0.222351	0.000129087

Note that

$$\begin{aligned}
R_{4,3} &= R_{4,2} + \frac{1}{15}(R_{4,2} - R_{3,2}) \\
&= -0.212634 + \frac{1}{15}(-0.212634 - 0.340677) \\
&= -0.249521
\end{aligned}$$

Example (4 of 5)

Now form the $O(h^8)$ approximations using the formula

$$R_{k,4} = R_{k,3} + \frac{1}{63}\left(R_{k,3} - R_{k-1,3}\right)$$

n	$R_{k+1,1}$	$R_{k+1,2}$	$R_{k+1,3}$	$R_{k+1,4}$	**Abs. Err.**
2^0	-4.9348				
2^1	-2.4674	-1.64493			
2^2	-0.361343	0.340677	0.473051		
2^3	-0.249811	-0.212634	-0.249521	-0.260991	0.0387686
2^4	-0.228761	-0.221744	-0.222351	-0.22192	0.000302185

Note that

$$\begin{aligned}
R_{4,4} &= R_{4,3} + \frac{1}{63}(R_{4,3} - R_{3,3}) \\
&= -0.249521 + \frac{1}{63}(-0.249521 - 0.473051) \\
&= -0.260991
\end{aligned}$$

Example (5 of 5)

Now form the $O(h^{10})$ approximations using the formula

$$R_{k,5} = R_{k,4} + \frac{1}{255}\left(R_{k,4} - R_{k-1,4}\right)$$

n	$R_{k+1,1}$	$R_{k+1,2}$	$R_{k+1,3}$	$R_{k+1,4}$	$R_{k+1,5}$	**Abs. Err.**
2^0	-4.9348					
2^1	-2.4674	-1.64493				
2^2	-0.361343	0.340677	0.473051			
2^3	-0.249811	-0.212634	-0.249521	-0.260991		
2^4	-0.228761	-0.221744	-0.222351	-0.22192	-0.221767	0.000455403

Note that

$$\begin{aligned} R_{5,5} &= R_{5,4} + \frac{1}{255}(R_{5,4} - R_{4,4}) \\ &= -0.22192 + \frac{1}{255}(-0.22192 - (-0.260991)) \\ &= -0.221767 \end{aligned}$$

Remarks

- Since n increases by a factor of 2 each time, $R_{k+1,1}$ uses all the function evaluations of $R_{k,1}$ and adds the function evaluations half-way between the previous nodes.
- The old function evaluations do not need to be re-computed, only the 2^{k-2} new values must be computed.

Remarks

- Since n increases by a factor of 2 each time, $R_{k+1,1}$ uses all the function evaluations of $R_{k,1}$ and adds the function evaluations half-way between the previous nodes.
- The old function evaluations do not need to be re-computed, only the 2^{k-2} new values must be computed.
- We can express the Composite Trapezoidal rule in a recursive manner.

Recursion

Define $h_k = \dfrac{b-a}{2^{k-1}}$, then

$$\begin{aligned} R_{1,1} &= \frac{b-a}{2}\left[f(a)+f(b)\right] \\ &= \frac{h_1}{2}\left[f(a)+f(b)\right] \end{aligned}$$

Recursion

Define $h_k = \dfrac{b-a}{2^{k-1}}$, then

$$
\begin{aligned}
R_{1,1} &= \frac{b-a}{2}\left[f(a)+f(b)\right] \\
&= \frac{h_1}{2}\left[f(a)+f(b)\right] \\
R_{2,1} &= \frac{b-a}{4}\left[f(a)+f(b)+2f\left(a+\frac{b-a}{2}\right)\right] \\
&= \frac{h_2}{2}\left[f(a)+f(b)+2f(a+h_2)\right] \\
&= \frac{1}{2}\left[R_{1,1}+h_1 f(a+h_2)\right]
\end{aligned}
$$

Recursion

Define $h_k = \dfrac{b-a}{2^{k-1}}$, then

$$
\begin{aligned}
R_{1,1} &= \frac{b-a}{2}\left[f(a)+f(b)\right] \\
&= \frac{h_1}{2}\left[f(a)+f(b)\right] \\
R_{2,1} &= \frac{b-a}{4}\left[f(a)+f(b)+2f\left(a+\frac{b-a}{2}\right)\right] \\
&= \frac{h_2}{2}\left[f(a)+f(b)+2f(a+h_2)\right] \\
&= \frac{1}{2}\left[R_{1,1}+h_1 f(a+h_2)\right] \\
R_{3,1} &= \frac{1}{2}\left\{R_{2,1}+h_2\left[f(a+h_3)+f(a+3h_3)\right]\right\} \\
&\vdots \\
R_{k,1} &= \frac{1}{2}\left[R_{k-1,1}+h_{k-1}\sum_{i=1}^{2^{k-2}} f(a+(2i-1)h_k)\right]
\end{aligned}
$$

Romberg Table

Once we have calculated $R_{k,1}$ for $k = 1, 2, \ldots, n$ we use the extrapolation formula below to produce the $O(h_k^{2j})$ approximation.

$$R_{k,j} = R_{k,j-1} + \frac{1}{4^{j-1} - 1}\left(R_{k,j-1} - R_{k-1,j-1}\right)$$

for $k = j, j+1, \ldots$.

k	$O(h_k^2)$	$O(h_k^4)$	$O(h_k^6)$	$O(h_k^8)$	$\cdots$	$O(h_k^{2n})$
1	$R_{1,1}$					
2	$R_{2,1}$	$R_{2,2}$				
3	$R_{3,1}$	$R_{3,2}$	$R_{3,3}$			
4	$R_{4,1}$	$R_{4,2}$	$R_{4,3}$	$R_{4,4}$		
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\ddots$	
n	$R_{n,1}$	$R_{n,2}$	$R_{n,3}$	$R_{n,4}$	$\cdots$	$R_{n,n}$

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$$R_{k,j} = R_{k,j-1} + \frac{1}{4^{j-1} - 1} \left(R_{k,j-1} - R_{k-1,j-1} \right)$$

for $k = j, j + 1, \ldots$.

k	$O(h_k^2)$	$O(h_k^4)$	$O(h_k^6)$	$O(h_k^8)$	$\cdots$	$O(h_k^{2n})$
1	$R_{1,1}$					
2	$R_{2,1}$	$R_{2,2}$				
3	$R_{3,1}$	$R_{3,2}$	$R_{3,3}$			
4	$R_{4,1}$	$R_{4,2}$	$R_{4,3}$	$R_{4,4}$		
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\ddots$	
n	$R_{n,1}$	$R_{n,2}$	$R_{n,3}$	$R_{n,4}$	$\cdots$	$R_{n,n}$

Remark: calculate the table one row at a time.

Example

Add another row to our previously calculated table.

k	$R_{k,1}$	$R_{k,2}$	$R_{k,3}$	$R_{k,4}$	$R_{k,5}$	$R_{k,6}$
1	−4.9348					
2	−2.4674	−1.64493				
3	−0.361343	0.340677	0.473051			
4	−0.249811	−0.212634	−0.249521	−0.260991		
5	−0.228761	−0.221744	−0.222351	−0.22192	−0.221767	

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k	$R_{k,1}$	$R_{k,2}$	$R_{k,3}$	$R_{k,4}$	$R_{k,5}$	$R_{k,6}$
1	−4.9348					
2	−2.4674	−1.64493				
3	−0.361343	0.340677	0.473051			
4	−0.249811	−0.212634	−0.249521	−0.260991		
5	−0.228761	−0.221744	−0.222351	−0.22192	−0.221767	
6	−0.223836	−0.222194	−0.222224	−0.222222	−0.222223	−0.222223

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k	$R_{k,1}$	$R_{k,2}$	$R_{k,3}$	$R_{k,4}$	$R_{k,5}$	$R_{k,6}$
1	−4.9348					
2	−2.4674	−1.64493				
3	−0.361343	0.340677	0.473051			
4	−0.249811	−0.212634	−0.249521	−0.260991		
5	−0.228761	−0.221744	−0.222351	−0.22192	−0.221767	
6	−0.223836	−0.222194	−0.222224	−0.222222	−0.222223	−0.222223

Remark: Romberg integration is efficient since only the entries in the $R_{k,1}$ column are calculated by evaluating $f(x)$. All others are found by weighted averaging.

Romberg Algorithm

INPUT Endpoints a, b and integer n.

STEP 1 Set $h = b - a$; set $R_{1,1} = \frac{h}{2}(f(a) + f(b))$.

STEP 2 OUTPUT $R_{1,1}$.

STEP 3 For $i = 2, 3, \ldots, n$ do STEPS 4–8.

STEP 4 Set $R_{2,1} = \frac{1}{2}\left[R_{1,1} + h\sum_{k=1}^{2^{i-2}} f(a + (k - 0.5)h)\right]$.

STEP 5 For $j = 2, 3, \ldots, i$ set $R_{2,j} = R_{2,j-1} + \frac{R_{2,j-1} - R_{1,j-1}}{4^{j-1} - 1}$.

STEP 6 For $j = 1, 2, \ldots, i$ OUTPUT $R_{2,j}$.

STEP 7 Set $h = \frac{h}{2}$.

STEP 8 For $j = 1, 2, \ldots, i$ set $R_{1,j} = R_{2,j}$.

STEP 9 STOP.

Homework

- Read Section 4.5.
- Exercises: 1ad, 3ad, 5ad, 18, 19