Trigonometric Polynomial Approximation

MATH 375 *Numerical Analysis*

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Background

Let $\mathcal{B} = \{\phi_0, \phi_1, \ldots, \phi_{2n}\}$ where

$$\begin{aligned} \phi_0(x) &= \frac{1}{2} \\ \phi_k(x) &= \cos kx \quad (\text{for } k = 1, 2, \ldots, n) \\ \phi_{n+k}(x) &= \sin kx \quad (\text{for } k = 1, 2, \ldots, n) \end{aligned}$$

and let $\mathcal{T}_n$ be the set of all linear combinations of elements of $\mathcal{B}$.

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\end{aligned}
$$

and let $\mathcal{T}_n$ be the set of all linear combinations of elements of $\mathcal{B}$.

Remarks:

- $\mathcal{B}$ is a linearly independent set on $[-\pi, \pi]$ with respect to weight function $w(x) = 1$.
- $\mathcal{T}_n$ is called the set of **trigonometric polynomials of degree less than or equal to** n.

Orthogonal Trigonometric Polynomials

If $f \in \mathcal{C}[-\pi, \pi]$ then the continuous least squares approximation of f by functions in $\mathcal{T}_n$ takes the form

$$S_n(x) = \frac{a_0}{2} + \sum_{k=1}^{n} (a_k \cos kx + b_k \sin kx)$$

where

$$\begin{aligned} a_k &= \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) \cos kx \, dx \quad (\text{for } k = 0, 1, \ldots, n), \\ b_k &= \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) \sin kx \, dx \quad (\text{for } k = 1, 2, \ldots, n). \end{aligned}$$

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If the $\lim_{n \to \infty} S_n(x)$ exists, it may be called the Fourier series representation of $f(x)$.

Example (1 of 2)

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b_k &= \frac{1}{\pi} \int_{-\pi}^{\pi} x^3 \sin kx \, dx \\
&= \frac{1}{\pi} \left(\left[\frac{6x}{k^3} - \frac{x}{k} \right] \cos kx + \left[\frac{3x^2}{k^2} - \frac{6}{k^4} \right] \sin kx \right) \Bigg|_{-\pi}^{\pi}
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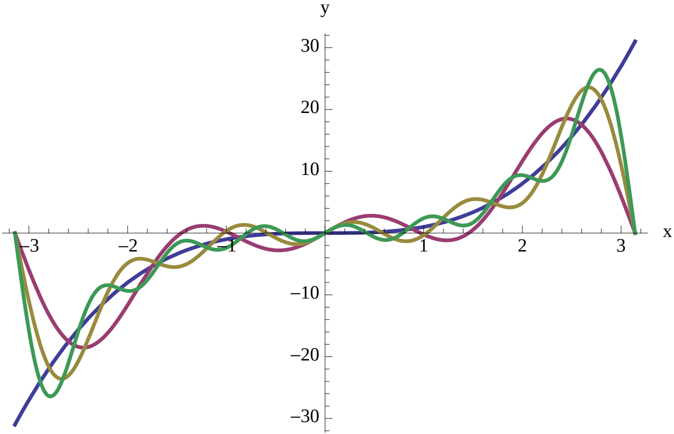
&= \left(\frac{12}{k^3} - \frac{2\pi^2}{k}\right)(-1)^k \quad (\text{for } k = 1, 2, \ldots, n)
\end{aligned}$$

Example (2 of 2)

$$S_n(x) = \sum_{k=1}^{n} \left(\frac{12\pi}{k^3} - \frac{2\pi^3}{k} \right) (-1)^k \sin kx$$



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a_0 &= \frac{1}{\pi}\int_{-\pi}^{\pi} x^2\, dx = \frac{2\pi^2}{3} \\
a_k &= \frac{1}{\pi}\int_{-\pi}^{\pi} x^2 \cos kx\, dx \\
&= \frac{1}{\pi}\left(\frac{2x}{k^2}\cos kx + \left[\frac{x^2}{k} - \frac{2}{k^3}\right]\sin kx\right)\Bigg|_{-\pi}^{\pi}
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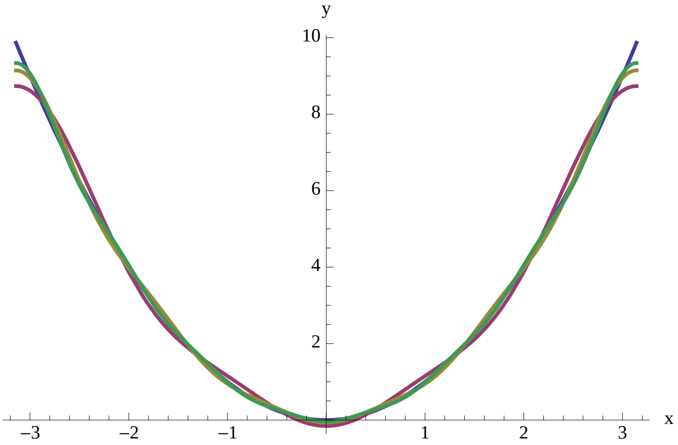
&= \frac{1}{\pi}\left(\frac{2x}{k^2}\cos kx + \left[\frac{x^2}{k} - \frac{2}{k^3}\right]\sin kx\right)\Bigg|_{-\pi}^{\pi} \\
&= \frac{4(-1)^k}{k^2} \quad (\text{for } k = 1, 2, \ldots, n)
\end{aligned}$$

Example (2 of 2)

$$S_n(x) = \frac{\pi^2}{3} + \sum_{k=1}^{n} \frac{4(-1)^k}{k^2} \cos kx$$



Discrete Trigonometric Approximation

Suppose we have $2m$ data points of the form $\{(x_j, y_j)\}_{j=0}^{2m-1}$ where the x_j's partition the interval $[-\pi, \pi]$.

Under this assumption, $x_j = -\pi + \dfrac{j\pi}{m}$ for $j = 0, 1, \ldots, 2m-1$.

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Objective: determine the trigonometric polynomial $S_n(x)$ in $\mathcal{T}_n$ that minimizes

$$E(S_n) = \sum_{j=0}^{2m-1} (y_j - S_n(x_j))^2 .$$

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This is analogous to fitting the discrete data with an ordinary polynomial.

Minimization

$$E(S_n) = \sum_{j=0}^{2m-1} \left(y_j - \left[\frac{a_0}{2} + \sum_{k=1}^{n} (a_k \cos kx_j + b_k \sin kx_j) \right] \right)^2$$

We must choose $a_0, a_1, \ldots, a_n, b_1, b_2, \ldots, b_n$ to minimize $E(S_n)$.

This minimization is made easier by an orthogonality property.

Preliminary Lemma

Lemma

Suppose that integer r is not a multiple of $2m$, then

$$\sum_{j=0}^{2m-1} \cos rx_j = 0 = \sum_{j=0}^{2m-1} \sin rx_j.$$

Furthermore, if r is not a multiple of m, then

$$\sum_{j=0}^{2m-1} \cos^2 rx_j = m = \sum_{j=0}^{2m-1} \sin^2 rx_j.$$

Proof (1 of 3)

Let $i = \sqrt{-1}$ then

$$
\begin{aligned}
\sum_{j=0}^{2m-1} \cos rx_j + i \sum_{j=0}^{2m-1} \sin rx_j &= \sum_{j=0}^{2m-1} (\cos rx_j + i \sin rx_j) \\
&= \sum_{j=0}^{2m-1} e^{irx_j} \quad \text{(Euler's Identity)} \\
&= \sum_{j=0}^{2m-1} e^{ir(-\pi + j\pi/m)} \quad \text{(def'n. of } x_j\text{)} \\
&= \sum_{j=0}^{2m-1} e^{-ir\pi} e^{irj\pi/m} \\
&= e^{-ir\pi} \sum_{j=0}^{2m-1} e^{irj\pi/m}.
\end{aligned}
$$

Proof (2 of 3)

$$\begin{aligned}
\sum_{j=0}^{2m-1} \cos r x_j + i \sum_{j=0}^{2m-1} \sin r x_j
&= e^{-ir\pi} \sum_{j=0}^{2m-1} e^{irj\pi/m} \\
&= e^{-ir\pi} \left(\frac{1 - (e^{ir\pi/m})^{2m}}{1 - e^{ir\pi/m}} \right) \qquad \text{(geometric series)} \\
&= e^{-ir\pi} \left(\frac{1 - e^{2ir\pi}}{1 - e^{ir\pi/m}} \right) \\
&= e^{-ir\pi} \left(\frac{1 - \cos 2r\pi - i \sin 2r\pi}{1 - e^{ir\pi/m}} \right) = 0
\end{aligned}$$

Thus $\displaystyle\sum_{j=0}^{2m-1} \cos r x_j = 0 = \sum_{j=0}^{2m-1} \sin r x_j$.

Proof (3 of 3)

$$
\begin{aligned}
\sum_{j=0}^{2m-1} \cos^2 rx_j &= \frac{1}{2}\sum_{j=0}^{2m-1}(1+\cos 2rx_j) \quad \text{(half angle)} \\
&= \frac{1}{2}\sum_{j=0}^{2m-1} 1 + \frac{1}{2}\sum_{j=0}^{2m-1}\cos 2rx_j \\
&= \frac{2m}{2} + \frac{0}{2} \\
&= m
\end{aligned}
$$

Similarly we can show

$$\sum_{j=0}^{2m-1} \sin^2 rx_j = m.$$

Orthogonality

We can use the previous lemma to show that for each x_j if $i \neq k$ then

$$\sum_{j=0}^{2m-1} \phi_k(x_j)\phi_i(x_j) = 0.$$

This implies the set $\{\phi_0, \phi_1, \ldots, \phi_{2n}\}$ is **orthogonal with respect to summation** over the equally spaced points $\{x_j\}_{j=0}^{2m-1}$.

Proof

$$
\begin{aligned}
\sum_{j=0}^{2m-1} \phi_k(x_j)\phi_i(x_j) &= \sum_{j=0}^{2m-1} (\cos kx_j)(\sin ix_j) \\
&= \frac{1}{2}\sum_{j=0}^{2m-1} (\sin(i+k)x_j + \sin(i-k)x_j) \\
&= \frac{1}{2}\sum_{j=0}^{2m-1} \sin(i+k)x_j + \frac{1}{2}\sum_{j=0}^{2m-1} \sin(i-k)x_j \\
&= 0 \quad \text{(by the Lemma)}
\end{aligned}
$$

Least Squares Coefficients

Theorem

The coefficients of the summation

$$S_n(x) = \frac{a_0}{2} + \sum_{k=1}^{n} (a_k \cos kx + b_k \sin kx)$$

that minimize the least squares error are

$$\begin{aligned} a_k &= \frac{1}{m} \sum_{j=0}^{2m-1} y_j \cos kx_j \quad \textit{(for } k = 0, 1, \ldots, n\textit{)} \\ b_k &= \frac{1}{m} \sum_{j=0}^{2m-1} y_j \sin kx_j \quad \textit{(for } k = 1, 2, \ldots, n\textit{)} \end{aligned}$$

Proof (1 of 2)

Setting the partial derivatives with respect to the coefficients equal to zero as in

$$
\begin{aligned}
0 &= \frac{\partial E}{\partial b_k} = -2 \sum_{j=0}^{2m-1} \sin kx_j (y_j - S_n(x_j)) \\
0 &= \sum_{j=0}^{2m-1} y_j \sin kx_j - \sum_{j=0}^{2m-1} S_n(x_j) \sin kx_j \\
&= \sum_{j=0}^{2m-1} y_j \sin kx_j - \frac{a_0}{2} \sum_{j=0}^{2m-1} \sin kx_j - b_k \sum_{j=0}^{2m-1} \sin^2 kx_j \\
&\quad - a_n \sum_{j=0}^{2m-1} \sin kx_j \cos nx_j - \sum_{l=1}^{n-1} a_l \sum_{j=0}^{2m-1} \sin kx_j \cos lx_j \\
&\quad - \sum_{l=1, l\neq k}^{n-1} b_l \sum_{j=0}^{2m-1} \sin kx_j \sin lx_j
\end{aligned}
$$

Proof (2 of 2)

$$
\begin{aligned}
0 &= \sum_{j=0}^{2m-1} y_j \sin kx_j - b_k \sum_{j=0}^{2m-1} \sin^2 kx_j \\
mb_k &= \sum_{j=0}^{2m-1} y_j \sin kx_j \\
b_k &= \frac{1}{m} \sum_{j=0}^{2m-1} y_j \sin kx_j
\end{aligned}
$$

Example (1 of 3)

Determine the discrete least squares trigonometric polynomial $S_3(x)$ approximating $f(x) = x^2 \cos x + \sin x$ on the interval $[-\pi, \pi]$ for $m = 3$.

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Determine the discrete least squares trigonometric polynomial $S_3(x)$ approximating $f(x) = x^2 \cos x + \sin x$ on the interval $[-\pi, \pi]$ for $m = 3$.

Let $x_j = -\pi + \frac{j\pi}{m}$ for $j = 0, 1, \ldots, 2m - 1 = 5$ and $y_j = f(x_j)$.

$$
\begin{aligned}
a_0 &= \frac{1}{3}\sum_{j=0}^{5} y_j \cos(0x_j) = -4.38649 \\
a_1 &= \frac{1}{3}\sum_{j=0}^{5} y_j \cos x_j = 4.20372 \\
a_2 &= \frac{1}{3}\sum_{j=0}^{5} y_j \cos 2x_j = -2.74156 \\
a_3 &= \frac{1}{3}\sum_{j=0}^{5} y_j \cos 3x_j = 1.46216
\end{aligned}
$$

Example (2 of 3)

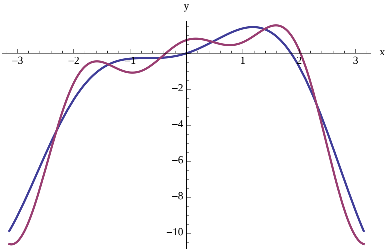
The remaining coefficients are

$$\begin{aligned} b_1 &= \frac{1}{3}\sum_{j=0}^{5} y_j \sin x_j = 1.0, \\ b_2 &= \frac{1}{3}\sum_{j=0}^{5} y_j \sin 2x_j = 0, \\ b_3 &= \frac{1}{3}\sum_{j=0}^{5} y_j \sin 3x_j = 0. \end{aligned}$$

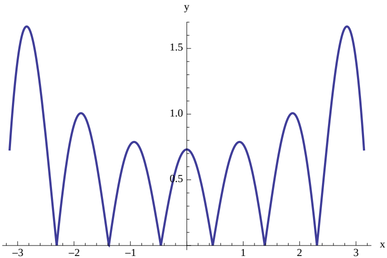
Thus the approximating trigonometric polynomial is

$$\begin{aligned} S_3(x) &= \frac{-4.38649}{2} + \sin x + 4.20372\cos x - 2.74156\cos 2x \\ &\quad + 1.46216\cos 3x \\ &= -2.19325 + \sin x + 4.20372\cos x - 2.74156\cos 2x \\ &\quad + 1.46216\cos 3x \end{aligned}$$

Example (3 of 3)



Approximation



Absolute Error

The sum of squares error is $E(S_3) = 3.20688$.

Example (1 of 5)

Find the discrete least squares error approximation $S_3(x)$ for

$$f(x) = 2\sec x - \tan 2x$$

on the interval $[2, 4]$ using $m = 6$.

Example (1 of 5)

Find the discrete least squares error approximation $S_3(x)$ for

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on the interval $[2, 4]$ using $m = 6$.

- We must linearly transform the interval from $[2, 4]$ to $[-\pi, \pi]$.
- If $x \in [2, 4]$ then $z = \pi(x - 3) \in [-\pi, \pi]$.
- The transformed data consists of ordered pairs of the form

$$\left\{ \left(z_j, f\left(3 + \frac{z}{\pi}\right) \right) \right\}_{j=0}^{11}.$$

Example (2 of 5)

$$
\begin{aligned}
z_j &= \pi\left(2 + \frac{j}{6} - 3\right) \quad \text{for } j = 0, 1, \ldots, 11, \\
y_j &= f\left(2 + \frac{j}{6}\right) \quad \text{for } j = 0, 1, \ldots, 11.
\end{aligned}
$$

j	z_j	y_j	j	z_j	y_j
0	$-\pi$	-5.96382	6	0	-1.72921
1	$-5\pi/6$	-6.07416	7	$\pi/6$	-2.05082
2	$-2\pi/3$	-24.7513	8	$\pi/3$	-2.44079
3	$-\pi/2$	0.884084	9	$\pi/2$	-3.00716
4	$-\pi/3$	-0.850947	10	$2\pi/3$	-4.05529
5	$-\pi/6$	-1.39027	11	$5\pi/6$	-7.87296

Example (3 of 5)

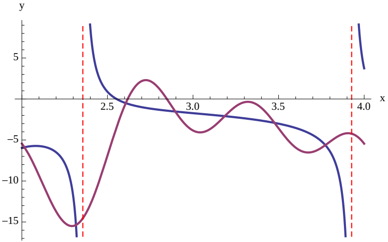
$$
\begin{aligned}
a_0 &= \frac{1}{6}\sum_{j=0}^{11} f\left(2+\frac{j}{6}\right)\cos(0z_j) = -9.88377 \\
a_1 &= \frac{1}{6}\sum_{j=0}^{11} f\left(2+\frac{j}{6}\right)\cos z_j = 4.34842 \\
a_2 &= \frac{1}{6}\sum_{j=0}^{11} f\left(2+\frac{j}{6}\right)\cos 2z_j = 0.297516 \\
a_3 &= \frac{1}{6}\sum_{j=0}^{11} f\left(2+\frac{j}{6}\right)\cos 3z_j = -3.5467
\end{aligned}
$$

Example (4 of 5)

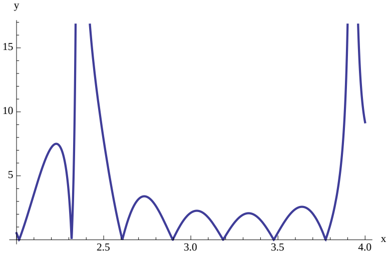
$$
\begin{aligned}
b_1 &= \frac{1}{6}\sum_{j=0}^{11} f\left(2+\frac{j}{6}\right)\sin z_j = 1.90425 \\
b_2 &= \frac{1}{6}\sum_{j=0}^{11} f\left(2+\frac{j}{6}\right)\sin 2z_j = -3.05239 \\
b_3 &= \frac{1}{6}\sum_{j=0}^{11} f\left(2+\frac{j}{6}\right)\sin 3z_j = 0.23865
\end{aligned}
$$

Example (5 of 5)

$$S_3(z) \quad = \quad -4.94189 + 4.34842 \cos z + 0.297516 \cos 2z - 3.5467 \cos 3z + 1$$
$$S_3(x) \quad = \quad S_3(3 + z/\pi)$$



Approximation



Absolute Error

Homework

- Read Section 8.5.
- Exercises: 1, 3, 5, 7ab, 9, 11