Introduction to Fourier Series

MATH 467 *Partial Differential Equations*

J Robert Buchanan

Department of Mathematics

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Objectives

In this lesson we will learn:

- the formal process for finding a Fourier series representation of a function,
- the orthogonality of the trigonometric functions,
- the Euler-Fourier formulas for finding Fourier series coefficients,
- properties of periodic functions,
- how to periodically extend a function,
- the properties of even and odd periodic extensions of functions, and
- practice finding the Fourier series representations of functions.

Informal Definition of a Fourier Series

The **Fourier series** expansion of a function $f(x)$ is a representation of $f(x)$ on an interval $[-L, L]$ as the sum of sine and cosine functions of the form

$$\frac{a_0}{2} + \sum_{n=1}^{\infty} \left(a_n \cos \frac{n\pi x}{L} + b_n \sin \frac{n\pi x}{L} \right)$$

where a_n and b_n are constants.

Issues Raised by Fourier Series

- What functions $f(x)$ can be written as a Fourier series?
- If $f(x)$ can be represented as a Fourier Series, what are the constants a_n and b_n?
- Will the Fourier series converge?
- Provided the Fourier series converges, does it converge to $f(x)$ at all points in the interval $[-L, L]$?
- Can Fourier series be differentiated and integrated?

Inner Product

Definition

If $u(x)$ and $v(x)$ are integrable on $[a, b]$, the **inner product** of u and v on $[a, b]$, denoted as $\langle u, v \rangle$, is defined as

$$\langle u, v \rangle = \int_a^b u(x)v(x)\,dx.$$

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Definition

The functions u and v are said to be **orthogonal** on $[a, b]$ if

$$\langle u, v \rangle = \int_a^b u(x)v(x)\,dx = 0.$$

A set S of integrable functions on $[a, b]$ is said to be a **mutually orthogonal set** if each pair of distinct functions in the set is orthogonal.

Trigonometric System

Let S be the infinite set of functions

$$\left\{1, \cos\frac{\pi x}{L}, \sin\frac{\pi x}{L}, \cos\frac{2\pi x}{L}, \sin\frac{2\pi x}{L}, \cdots, \cos\frac{n\pi x}{L}, \sin\frac{n\pi x}{L}, \cdots\right\}.$$

S is a mutually orthogonal set on $[-L, L]$.

Product-to-Sum Formulas

$$\cos\alpha\cos\beta = \frac{1}{2}\left(\cos(\alpha+\beta)+\cos(\alpha-\beta)\right)$$
$$\cos\alpha\sin\beta = \frac{1}{2}\left(\sin(\alpha+\beta)-\sin(\alpha-\beta)\right)$$
$$\sin\alpha\sin\beta = \frac{1}{2}\left(\cos(\alpha-\beta)-\cos(\alpha+\beta)\right)$$

Justification of Orthogonality

$$\int_{-L}^{L} \cos\frac{m\pi x}{L}\cos\frac{n\pi x}{L}\,dx$$

$$= \frac{1}{2}\int_{-L}^{L}\left[\cos\frac{(m+n)\pi x}{L} + \cos\frac{(m-n)\pi x}{L}\right]dx$$

$$= \begin{cases} \frac{L}{2\pi}\left[\frac{1}{m+n}\sin\frac{(m+n)\pi x}{L} + \frac{1}{m-n}\sin\frac{(m-n)\pi x}{L}\right]_{-L}^{L} & \text{if } m \neq n, \\ \frac{1}{2}\left[\frac{L}{2m\pi}\sin\frac{2m\pi x}{L} + x\right]_{-L}^{L} & \text{if } m = n \end{cases}$$

$$= \begin{cases} 0 & \text{if } m \neq n, \\ L & \text{if } m = n. \end{cases}$$

The orthogonality of $\sin(m\pi x/L)$, $\sin(n\pi x/L)$, and $\cos(k\pi x/L)$ is handled similarly.

Euler-Fourier Formulas

Assuming $f(x)$ defined on $[-L, L]$ can be represented as a Fourier series we write

$$f(x) \sim \frac{a_0}{2} + \sum_{n=1}^{\infty} \left(a_n \cos \frac{n\pi x}{L} + b_n \sin \frac{n\pi x}{L} \right),$$

where

$$a_0 = \frac{1}{L} \int_{-L}^{L} f(x)\, dx$$

$$a_n = \frac{1}{L} \int_{-L}^{L} f(x) \cos \frac{n\pi x}{L}\, dx$$

$$b_n = \frac{1}{L} \int_{-L}^{L} f(x) \sin \frac{n\pi x}{L}\, dx$$

for $n = 1, 2, \ldots$.

Justification (1 of 2)

Assuming $f(x)$ equals its Fourier representation on $[-L, L]$ and that the infinite series can be integrated term-by-term, multiply both sides of the equation

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} \left(a_n \cos \frac{n\pi x}{L} + b_n \sin \frac{n\pi x}{L} \right),$$

by $\sin(m\pi x/L)$ and integrate over $[-L, L]$.

$$\begin{aligned}
\int_{-L}^{L} f(x) \sin \frac{m\pi x}{L}\, dx &= \int_{-L}^{L} \left[\frac{a_0}{2} + \sum_{n=1}^{\infty} \left(a_n \cos \frac{n\pi x}{L} + b_n \sin \frac{n\pi x}{L} \right) \right] \sin \frac{m\pi x}{L}\, dx \\
&= \frac{a_0}{2} \int_{-L}^{L} \sin \frac{m\pi x}{L}\, dx + \sum_{n=1}^{\infty} a_n \int_{-L}^{L} \cos \frac{n\pi x}{L} \sin \frac{m\pi x}{L}\, dx \\
&\quad + \sum_{n=1}^{\infty} b_n \int_{-L}^{L} \sin \frac{n\pi x}{L} \sin \frac{m\pi x}{L}\, dx = b_m L
\end{aligned}$$

Justification (2 of 2)

Multiplying both sides of the earlier equation by $\cos(m\pi x/L)$ and integrating over $[-L, L]$ yields a_m for $m \in \mathbb{N}$.

Integrating both sides of

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} \left(a_n \cos \frac{n\pi x}{L} + b_n \sin \frac{n\pi x}{L} \right),$$

over $[-L, L]$ produces

$$\begin{aligned} \int_{-L}^{L} f(x)\,dx &= \int_{-L}^{L} \frac{a_0}{2}\,dx + \sum_{n=1}^{\infty} \left(a_n \int_{-L}^{L} \cos \frac{n\pi x}{L}\,dx + b_n \int_{-L}^{L} \sin \frac{n\pi x}{L}\,dx \right) \\ &= a_0 L. \end{aligned}$$

Remarks

- In general the symbol $\sim$ is used in place of $=$ since we do not yet know whether the infinite series converges, or if it does converge, that it converges to $f(x)$.
- The only assumption placed on $f(x)$ is that it be integrable on $[-L, L]$. It does not even need to be defined at all points in $[-L, L]$.
- If the infinite series converges, it does so to a $2L$-periodic function, which can be thought of as the $2L$-periodic extension of $f(x)$.

Periodic Functions

Definition

A function $f(x)$ is said to be **periodic** if there exists a constant $T > 0$ such that, for any x in the domain of f, $x + T$ is in its domain and $f(x + T) = f(x)$ holds for all such x. In this case, T is called a **period** of $f(x)$ and, often $f(x)$ is said to be $T-$ **periodic** or **periodic with period** T.

Properties of Periodic Functions

- Any constant function is periodic and any $T > 0$ is a period.
- If T is a period of function $f(x)$, so is $k\,T$ for any $k \in \mathbb{N}$.
- If $f(x)$ and $g(x)$ are periodic with common period T, then for any constant c, $cf(x)$, $f(x) \pm g(x)$, $f(x) \cdot g(x)$, and $f(x)/g(x)$ are all periodic with period T on their respective domains.
- If $f(x)$ is periodic with period T, then so is $f'(x)$ on its domain.
- If $f(x)$ is T–periodic, integrable and $\int_0^T f(x)\,dx = 0$, then $\int_0^x f(t)\,dt$ is T–periodic.
- If $f(x)$ is an integrable, periodic function with period T defined on $(-\infty, \infty)$, then for any $a \in \mathbb{R}$,

$$\int_a^{a+T} f(x)\,dx = \int_0^T f(x)\,dx.$$

Periodic Extensions

Suppose $f(x)$ is defined on $[-L, L]$ where $L > 0$. A periodic function $F(x)$ can be defined on $(-\infty, \infty)$ in the following way:

- If $x \in (-L, L]$, then $F(x) = f(x)$.
- If $x \notin (-L, L]$ and k is an integer such that $x + k(2L) \in (-L, L]$, then $F(x) = f(x + k(2L))$.

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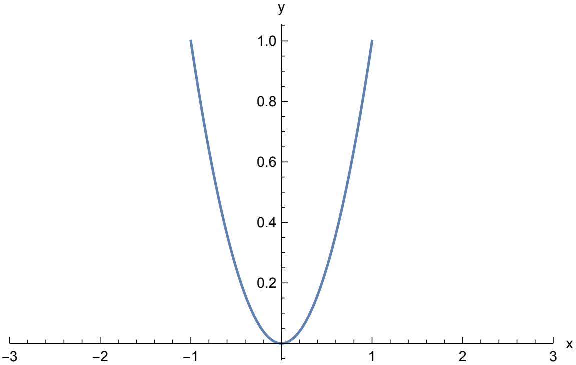
Remarks:

- $F(x)$ is periodic with period $2L$.
- If no confusion results, $f(x)$ is used to denote its own periodic extension.
- $F(x)$ as defined not a "true" extension of $f(x)$ unless $f(-L) = f(L)$.

Example (1 of 2)

Function $f(x) = x^2$ is continuous on $[-1, 1]$. Sketch its 2–periodic extension.



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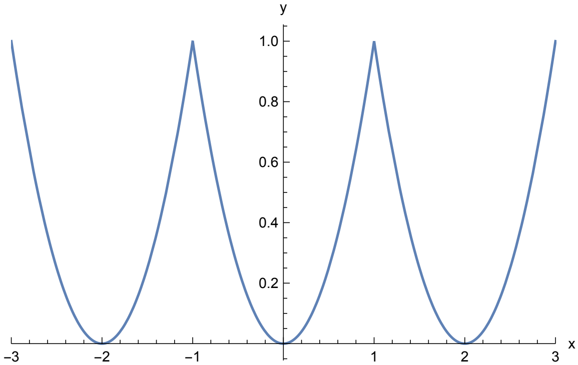
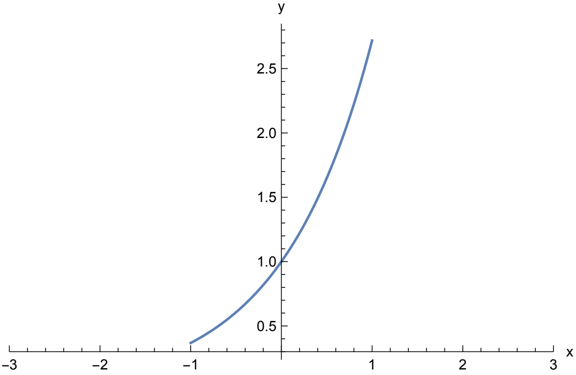


Example (2 of 2)

Function $f(x) = e^x$ is continuous on $[-1, 1]$. Sketch its 2–periodic extension.



Example (2 of 2)

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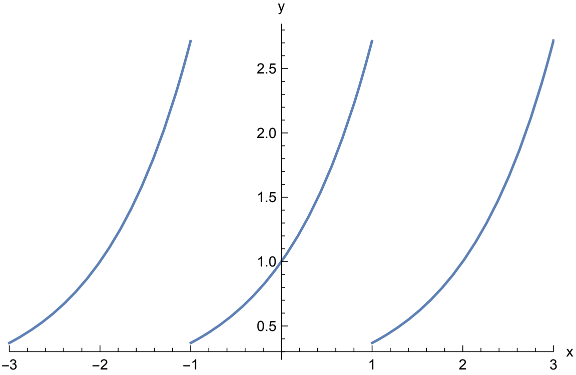


Find the Fourier Coefficients

Consider the piecewise-defined function

$$f(x) = \begin{cases} x & \text{if } -1 \leq x < 0, \\ 0 & \text{if } 0 \leq x < 1. \end{cases}$$

1. Write down the Fourier series of $f(x)$.
2. Sketch the 2-periodic extension of $f(x)$.

Coefficients

$$a_0 = \frac{1}{1}\int_{-1}^{1} f(x)\,dx = \int_{-1}^{0} x\,dx = -\frac{1}{2}$$

$$a_n = \frac{1}{1}\int_{-1}^{1} f(x)\cos\frac{n\pi x}{1}\,dx = \int_{-1}^{0} x\cos(n\pi x)\,dx$$

$$= \frac{1-(-1)^n}{n^2\pi^2} = \begin{cases} 2/(n\pi)^2 & \text{if } n \text{ is odd,} \\ 0 & \text{if } n \text{ is even.} \end{cases}$$

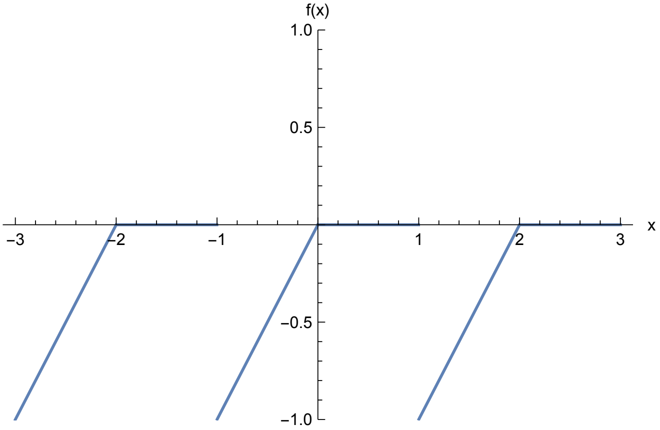
$$b_n = \frac{1}{1}\int_{-1}^{1} f(x)\sin\frac{n\pi x}{1}\,dx = \int_{-1}^{0} x\sin(n\pi x)\,dx$$

$$= \frac{(-1)^{n+1}}{n\pi}$$

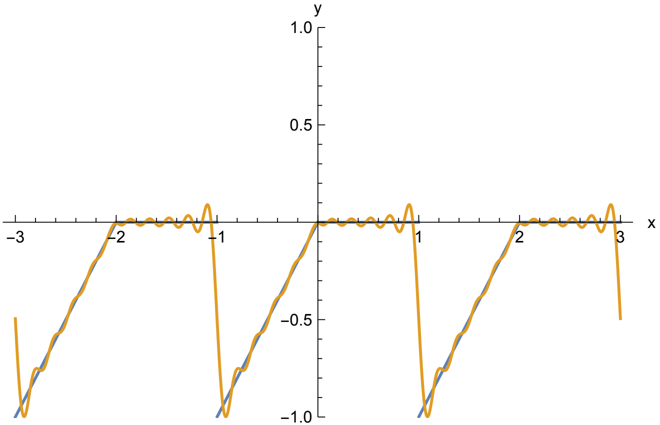
Fourier Representation

$$f(x) \sim -\frac{1}{4} + \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n\pi} \sin(n\pi x) + \sum_{n=1}^{\infty} \frac{2}{(2n-1)^2\pi^2} \cos((2n-1)\pi x)$$

2-Periodic Extension



Fourier Series (truncated to 10 terms)



Find the Fourier Coefficients

Consider the function $f(x) = x^2$.

1. Write down the Fourier series of $f(x)$ valid for $[-\pi, \pi]$.
2. Sketch the 2π-periodic extension of $f(x)$.

Coefficients

$$a_0 = \frac{1}{\pi}\int_{-\pi}^{\pi} x^2\,dx = \frac{2}{3}\pi^2$$

$$a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} x^2\cos(nx)\,dx = \frac{4(-1)^n}{n^2}$$

$$b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} x^2\sin(nx)\,dx = 0$$

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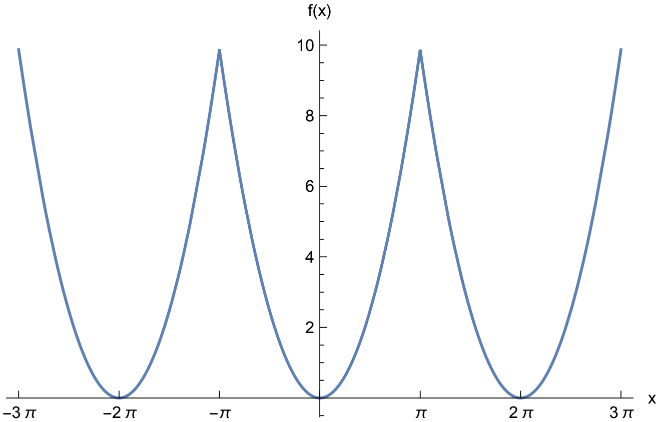
$$b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} x^2\sin(nx)\,dx = 0$$

$$f(x) \sim \frac{\pi^2}{3} + \sum_{n=1}^{\infty} \frac{4(-1)^n}{n^2}\cos(nx)$$

2π-Periodic Extension



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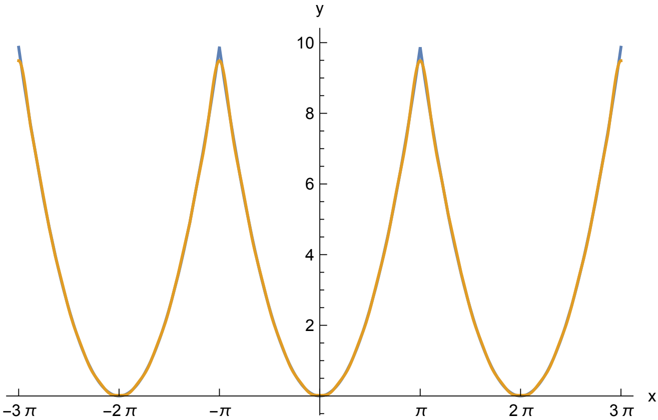

Find the Fourier Coefficients

Consider the function

$$f(x) = \begin{cases} 0 & \text{if } -\pi \le x \le 0, \\ \sin x & \text{if } 0 < x < \pi. \end{cases}$$

1. Write down the Fourier series of $f(x)$ valid for $[-\pi, \pi]$.
2. Sketch the 2π-periodic extension of $f(x)$.

Coefficients

$$a_0 = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\,dx = \frac{1}{\pi}\int_0^{\pi} \sin x\,dx = \frac{2}{\pi}$$

$$a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos(nx)\,dx = \frac{1}{\pi}\int_0^{\pi} \sin x\cos(nx)\,dx$$

$$= \begin{cases} -2/(\pi(n^2-1)) & \text{if } n \text{ is even,} \\ 0 & \text{if } n \text{ is odd.} \end{cases}$$

$$b_1 = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\sin x\,dx = \frac{1}{\pi}\int_0^{\pi} \sin^2 x\,dx = \frac{1}{2}$$

$$b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\sin(nx)\,dx = \frac{1}{\pi}\int_0^{\pi} \sin x\sin(nx)\,dx = 0$$

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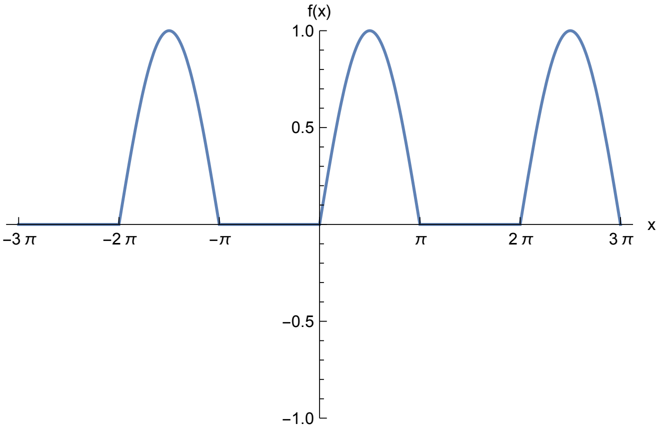
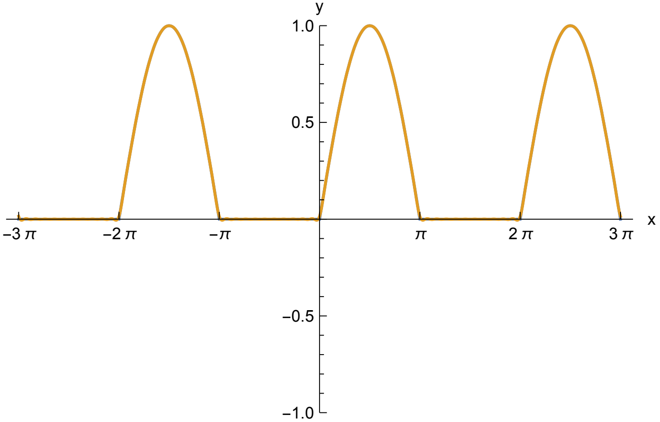
$$b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\sin(nx)\,dx = \frac{1}{\pi}\int_0^{\pi} \sin x\sin(nx)\,dx = 0$$

$$f(x) \sim \frac{1}{\pi} + \frac{1}{2}\sin x - \sum_{n=1}^{\infty} \frac{2}{\pi(4n^2-1)}\cos(2nx)$$

2π-Periodic Extension



Fourier Series (truncated to 10 terms)



Find the Fourier Coefficients

Find the Fourier series representation of $g(x) = |\sin x|$ on $[-\pi, \pi]$.

Solution

- Note that
$$|\sin x| = -\sin x + \begin{cases} 0 & \text{if } -\pi \le x \le 0, \\ 2\sin x & \text{if } 0 \le x \le \pi. \end{cases}$$
- The Fourier series for $\sin x$ is merely $\sin x$.
- The Fourier series for the piecewise-defined function was found in the previous example.

Solution

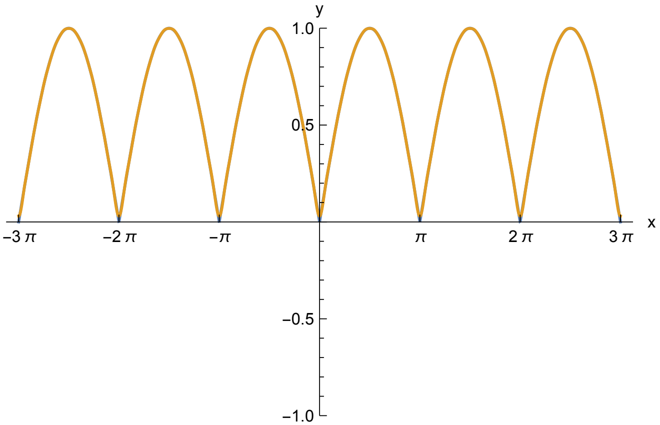
- Note that

$$|\sin x| = -\sin x + \begin{cases} 0 & \text{if } -\pi \leq x \leq 0, \\ 2\sin x & \text{if } 0 \leq x \leq \pi. \end{cases}$$

- The Fourier series for $\sin x$ is merely $\sin x$.
- The Fourier series for the piecewise-defined function was found in the previous example.

$$f(x) \sim \frac{2}{\pi} - \sum_{n=1}^{\infty} \frac{4}{\pi(4n^2 - 1)} \cos(2nx)$$

Fourier Series (truncated to 10 terms)



Even, Odd, Periodic Extensions

Comment: the spatial domain of many of the PDEs we study (*e.g.*, the heat equation and wave equation) is the interval $[0, L]$, not $[-L, L]$. If an initial condition is specified on $[0, L]$ we may extend it to $[-L, L]$ (and thence to $(-\infty, \infty)$) in any way that it remains integrable. Options include:

Even Extension

$$f_e(x) = \begin{cases} f(-x) & \text{if } -L \leq x < 0, \\ f(x) & \text{if } 0 \leq x \leq L. \end{cases}$$

Odd Extension

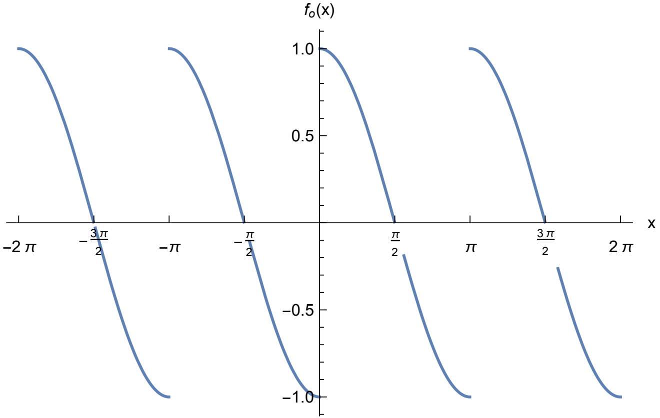
$$f_o(x) = \begin{cases} -f(-x) & \text{if } -L \leq x < 0, \\ f(x) & \text{if } 0 \leq x \leq L. \end{cases}$$

Example

Consider the function $f(x) = \cos x$ on $[0, \pi/2]$.

1. Sketch the odd π-periodic extension of $f(x)$.
2. Find the Fourier series representation for the odd π-periodic extension of $f(x)$.

Graph of $f_o(x)$



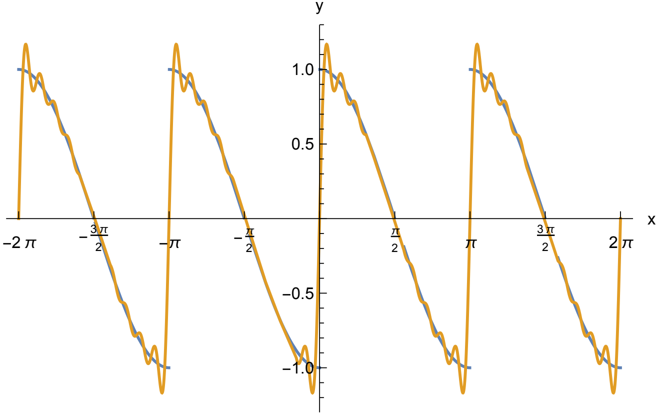
Fourier Series Coefficients

$$
\begin{aligned}
a_0 &= \frac{2}{\pi}\int_{-\pi/2}^{\pi/2} f_o(x)\,dx = 0 \\
a_n &= \frac{2}{\pi}\int_{-\pi/2}^{\pi/2} f_o(x)\cos\frac{n\pi x}{\pi/2}\,dx = 0 \\
b_n &= \frac{2}{\pi}\int_{-\pi/2}^{\pi/2} f_o(x)\sin\frac{n\pi x}{\pi/2}\,dx \\
&= -\frac{2}{\pi}\int_{-\pi/2}^{0} \cos(-x)\sin\frac{n\pi x}{\pi/2}\,dx + \frac{2}{\pi}\int_{0}^{\pi/2} \cos(x)\sin\frac{n\pi x}{\pi/2}\,dx \\
&= \frac{4}{\pi}\int_{0}^{\pi/2} \cos(x)\sin(2nx)\,dx = \frac{8n}{(4n^2-1)\pi}
\end{aligned}
$$

Since only the b_n coefficients are nonzero, this is called a **Fourier sine series**.

Fourier Series Representation

$$f_o(x) \sim \sum_{n=1}^{\infty} \frac{8n}{(4n^2-1)\pi} \sin(2nx)$$

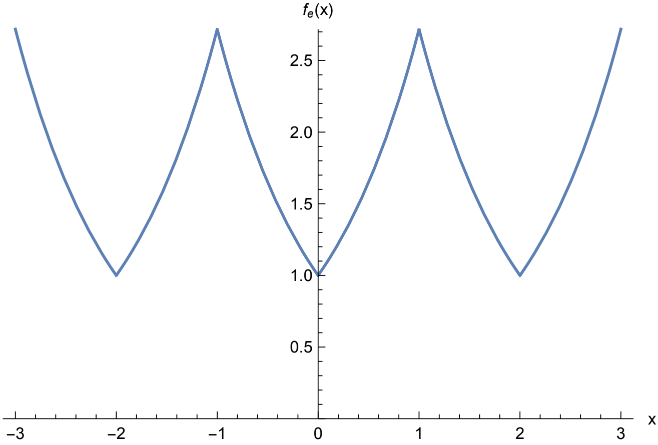


Example

Consider the function $f(x) = e^x$ on $[0, 1]$.

1. Sketch the even 2-periodic extension of $f(x)$.
2. Find the Fourier series representation for the even 2-periodic extension of $f(x)$.

Graph of $f_e(x)$



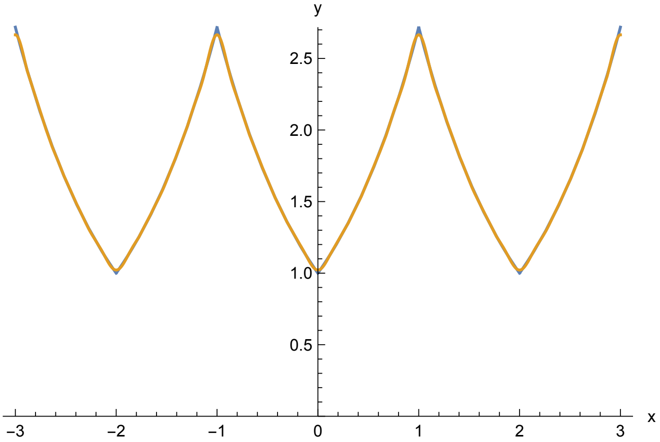
Fourier Series Coefficients

$$
\begin{aligned}
a_0 &= \int_{-1}^{1} f_e(x)\, dx \\
&= 2\int_{0}^{1} e^x\, dx = 2(e-1) \\
a_n &= \int_{-1}^{1} f_e(x) \cos(n\pi x)\, dx \\
&= 2\int_{0}^{1} e^x \cos(n\pi x)\, dx = \frac{2((-1)^n e - 1)}{n^2\pi^2 + 1} \\
b_n &= \int_{-1}^{1} f_e(x) \sin(n\pi x)\, dx = 0
\end{aligned}
$$

Since only the a_n coefficients are nonzero, this is called a **Fourier cosine series**.

Fourier Series Representation

$$f_e(x) \sim e - 1 + \sum_{n=1}^{\infty} \frac{2((-1)^n e - 1)}{n^2\pi^2 + 1} \cos(n\pi x)$$



Remark

Any function $f(x)$ defined on $(-\infty, \infty)$ can be written as the sum of an even function and an odd function. In fact,

$$f(x) = \frac{f(x) + f(-x)}{2} + \frac{f(x) - f(-x)}{2}$$

where $(f(x) + f(-x))/2$ is even (sometimes called the **even part** of f) and $(f(x) - f(-x))/2$ is odd (likewise called the **odd part** of f).

Homework

- Read Sections 3.1–3.5
- Exercises: 1–9