Neumann Problems on Rectangles

MATH 467 Partial Differential Equations

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Objectives

In this lesson we will learn:

- to solve Laplace’s equation on two-dimensional domains with Neumann boundary conditions,
- to compare the solutions on domains with Dirichlet boundary conditions to solution on domains with Neumann boundary conditions.
Boundary Value Problem

Consider Laplace’s equation on the rectangle 
\( \Omega = \{(x, y) \mid 0 < x < a, 0 < y < b\} \) with Neumann boundary conditions:

\[
\begin{align*}
\triangle u &= 0 \text{ for } (x, y) \in \Omega \\
u_y(x, 0) &= u_y(x, b) = 0 \text{ for } 0 < x < a \\
u_x(0, y) &= 0 \text{ for } 0 < y < b \\
u_x(a, y) &= f(y) \text{ for } 0 < y < b.
\end{align*}
\]

**Physical Interpretation**: the steady-state heat distribution in \( \Omega \) when the rectangular region is insulated along its bottom, top, and left edges and there is a flow of heat on the right edge.
Product Solutions

Assume $u(x, y) = X(x) Y(y)$ solves Laplace’s equation. Separation of variables induces the following ODEs for $X(x)$ and $Y(y)$.

$$X''(x) - \sigma X(x) = 0 \text{ with } X'(0) = 0$$
$$Y''(y) + \sigma Y(y) = 0 \text{ with } Y'(0) = 0 = Y'(b),$$

where $\sigma$ is a constant.

Taking the second BVP, the only nontrivial solutions are:

$$Y_n(y) = \cos \frac{n\pi y}{b}$$

$$\sigma_n = \frac{n^2 \pi^2}{b^2}$$

for $n = 0, 1, 2, \ldots$.

This implies $X_n(x) = \cosh \frac{n\pi x}{b}$ for $n = 0, 1, 2, \ldots$. 
Series Solution

The product solutions:

\[ u_n(x, y) = \cosh \frac{n\pi x}{b} \cos \frac{n\pi y}{b} \] for \( n = 0, 1, 2, \ldots \)

solve Laplace’s equation and satisfy the three homogeneous boundary conditions.

By the Principle of Superposition,

\[ u(x, y) = b_0 + \sum_{n=1}^{\infty} b_n \cosh \frac{n\pi x}{b} \cos \frac{n\pi y}{b}. \]

The coefficients \( b_n \) can be determined from the remaining nonhomogeneous boundary condition.
Determining the Coefficients

Differentiate the formal series solution and let \( x = a \).

\[
ux(a, y) = \sum_{n=1}^{\infty} b_n \left( \frac{n\pi}{b} \right) \sinh \frac{n\pi a}{b} \cos \frac{n\pi y}{b} = f(y)
\]

Remarks:

- The coefficient \( b_0 \) was lost during the differentiation.
- The infinite series can be regarded as a cosine series for \( f(y) \) if the integral of \( f \) over \([0, b]\) vanishes, i.e., if

\[
\int_{0}^{b} f(y) \, dy = 0.
\]
Further Remarks

- If $\int_0^b f(y) \, dy \neq 0$ then a solution to the BVP does not exist.

- Consider the physics:
  - If the definite integral vanishes then there is no net flux of heat across the boundary at $x = a$ and hence a steady-state (time independent) heat distribution can evolve.
  - If the definite integral does not vanish, then there is a net flux of heat in or out of $\Omega$ and no time independent temperature distribution can exist.

- Even if the definite integral vanishes, the solution can be determined only up to the addition of an arbitrary constant. Thus Laplace’s equation on a rectangle with Neumann boundary conditions on all four edges has no unique solution.

- This type of boundary value problem is ill-posed.
Assuming \( \int_0^b f(y) \, dy = 0 \)

\[
b_n = \frac{2}{n\pi \sinh \frac{n\pi a}{b}} \int_0^b f(y) \cos \frac{n\pi y}{b} \, dy,
\]

for \( n \in \mathbb{N} \).

\[
u(x, y) = b_0 + \sum_{n=1}^{\infty} b_n \cosh \frac{n\pi x}{b} \cos \frac{n\pi y}{b}
\]

where \( b_0 \) is an arbitrary constant.
Example

Find a solution to the Neumann boundary value problem on the unit square:

\[ \triangle u = 0 \text{ for } 0 < x < 1 \text{ and } 0 < y < 1 \]
\[ u_y(x, 0) = u_y(x, 1) = 0 \text{ for } 0 < x < 1 \]
\[ u_x(0, y) = 0 \text{ for } 0 < y < 1 \]
\[ u_x(1, y) = y - 1/2 \text{ for } 0 < y < 1. \]
Solution (1 of 2)

Check: \[ \int_0^1 \left( y - \frac{1}{2} \right) \, dy = \left[ \frac{y^2}{2} - \frac{y}{2} \right]_{y=0}^{y=1} = 0. \]

Using the Euler-Fourier coefficient formula:

\[
b_n = \frac{2}{n\pi \sinh(n\pi)} \int_0^1 \left( y - \frac{1}{2} \right) \cos(n\pi y) \, dy
\]

\[= \frac{-2}{n^2\pi^2 \sinh(n\pi)} \int_0^1 \sin(n\pi y) \, dy
\]

\[= \frac{2((-1)^n - 1)}{n^3\pi^3 \sinh(n\pi)}.\]

\[u(x, y) = b_0 - \frac{4}{\pi^3} \sum_{n=1}^{\infty} \frac{\cosh((2n-1)\pi x) \cos((2n-1)\pi y)}{(2n-1)^3 \sinh((2n-1)\pi)}\]

where \(b_0\) is an arbitrary constant.
General Case

Consider Laplace’s equation on a rectangle with Neumann BCs on all four edges.

\[ u_{xx} + u_{yy} = 0 \text{ for } (x, y) \in R \]
\[ u_x(0, y) = g_1(y) \text{ for } 0 < y < b \]
\[ u_x(a, y) = g_2(y) \text{ for } 0 < y < b \]
\[ u_y(x, 0) = f_1(x) \text{ for } 0 < x < a \]
\[ u_y(x, b) = f_2(x) \text{ for } 0 < x < a \]

This BVP can be decomposed into four sub-problems with homogeneous boundary conditions on three edges.
Sub-Problems

\[ \Delta u_1 = 0 \text{ for } (x, y) \in R \]
\[ (u_1)_x(0, y) = g_1(y) \text{ for } y \in (0, b) \]
\[ (u_1)_x(a, y) = 0 \text{ for } y \in (0, b) \]
\[ (u_1)_y(x, 0) = 0 \text{ for } x \in (0, a) \]
\[ (u_1)_y(x, b) = 0 \text{ for } x \in (0, a) \]

\[ \Delta u_2 = 0 \text{ for } (x, y) \in R \]
\[ (u_2)_x(0, y) = 0 \text{ for } y \in (0, b) \]
\[ (u_2)_x(a, y) = g_2(y) \text{ for } y \in (0, b) \]
\[ (u_2)_y(x, 0) = 0 \text{ for } x \in (0, a) \]
\[ (u_2)_y(x, b) = 0 \text{ for } x \in (0, a) \]

\[ \Delta u_3 = 0 \text{ for } (x, y) \in R \]
\[ (u_3)_x(0, y) = 0 \text{ for } y \in (0, b) \]
\[ (u_3)_x(a, y) = 0 \text{ for } y \in (0, b) \]
\[ (u_3)_y(x, 0) = f_1(y) \text{ for } x \in (0, a) \]
\[ (u_3)_y(x, b) = 0 \text{ for } x \in (0, a) \]

\[ \Delta u_4 = 0 \text{ for } (x, y) \in R \]
\[ (u_4)_x(0, y) = 0 \text{ for } y \in (0, b) \]
\[ (u_4)_x(a, y) = 0 \text{ for } y \in (0, b) \]
\[ (u_4)_y(x, 0) = 0 \text{ for } x \in (0, a) \]
\[ (u_4)_y(x, b) = f_2(y) \text{ for } x \in (0, a) \]
Solutions to the Sub-Problems

\[ u_1(x, y) = a_0 + \sum_{n=1}^{\infty} a_n \cosh \frac{n\pi(a - x)}{b} \cos \frac{n\pi y}{b} \]

\[ u_2(x, y) = b_0 + \sum_{n=1}^{\infty} b_n \cosh \frac{n\pi x}{b} \cos \frac{n\pi y}{b} \]

\[ u_3(x, y) = c_0 + \sum_{n=1}^{\infty} c_n \cos \frac{n\pi x}{a} \cosh \frac{n\pi(b - y)}{a} \]

\[ u_4(x, y) = d_0 + \sum_{n=1}^{\infty} d_n \cos \frac{n\pi x}{a} \cosh \frac{n\pi y}{a} \]
Series Coefficients

Provided \( \int_0^b g_1(y) \, dy = \int_0^b g_2(y) \, dy = 0 \) and \( \int_0^a f_1(x) \, dx = \int_0^a f_2(x) \, dx = 0 \), then

\[
a_n = \frac{-2}{n\pi \sinh \frac{n\pi a}{b}} \int_0^b g_1(y) \cos \frac{n\pi y}{b} \, dy
\]

\[
b_n = \frac{2}{n\pi \sinh \frac{n\pi a}{b}} \int_0^b g_2(y) \cos \frac{n\pi y}{b} \, dy
\]

\[
c_n = \frac{-2}{n\pi \sinh \frac{n\pi b}{a}} \int_0^a f_1(x) \cos \frac{n\pi x}{a} \, dx
\]

\[
d_n = \frac{2}{n\pi \sinh \frac{n\pi b}{a}} \int_0^a f_2(x) \cos \frac{n\pi x}{a} \, dx.
\]

The solution to the original BVP is then

\[u(x, y) = u_1(x, y) + u_2(x, y) + u_3(x, y) + u_4(x, y).\]
Neumann Problems on Disks

Consider Laplace’s equation on the disk of radius $a > 0$:

\[ \Delta u = 0 \text{ for } x^2 + y^2 < a^2 \]

\[ \frac{\partial u}{\partial n}(x, y) = \phi(x, y) \text{ for } x^2 + y^2 = a^2. \]

$\partial u/\partial n$ denotes the derivative in the direction of the unit outward normal vector to the boundary.
Consider Laplace’s equation on the disk of radius \( a > 0 \):

\[ \triangle u = 0 \text{ for } x^2 + y^2 < a^2 \]

\[ \frac{\partial u}{\partial n}(x, y) = \phi(x, y) \text{ for } x^2 + y^2 = a^2. \]

\( \frac{\partial u}{\partial n} \) denotes the derivative in the direction of the unit outward normal vector to the boundary.

Convert to polar coordinates.

\[ v_{rr} + \frac{1}{r} v_r + \frac{1}{r^2} v_{\theta\theta} = 0 \text{ for } 0 < r < a \text{ and } -\infty < \theta < \infty \]

\[ \frac{\partial v}{\partial r}(a, \theta) = \phi(a \cos \theta, a \sin \theta) = f(\theta) \text{ for } -\infty < \theta < \infty. \]
Series Solution

The formal series solution can be written as

\[ v(r, \theta) = d_0 + \sum_{n=1}^{\infty} r^n [c_n \cos(n\theta) + d_n \sin(n\theta)], \]

with coefficients \( d_0, c_n, \) and \( d_n \) chosen such that

\[ v_r(a, \theta) = \sum_{n=1}^{\infty} na^{n-1} [c_n \cos(n\theta) + d_n \sin(n\theta)] = f(\theta). \]

A necessary condition for the solution to exist is

\[ \int_{-\pi}^{\pi} f(\theta) \, d\theta = \int_{-\pi}^{\pi} \phi(a \cos \theta, a \sin \theta) \, d\theta = 0. \]
Series Coefficients

\[ c_n = \frac{a^{1-n}}{n\pi} \int_{-\pi}^{\pi} f(\theta) \cos(n\theta) \, d\theta \]

\[ d_n = \frac{a^{1-n}}{n\pi} \int_{-\pi}^{\pi} f(\theta) \sin(n\theta) \, d\theta. \]

**Remark:** Coefficient \( d_0 \) can be chosen arbitrarily and thus the solution to Laplace’s equation on a disk with Neumann boundary conditions is not unique.
Find a bounded solution to Laplace’s equation on 
\( \Omega = \{(r, \theta) \mid 0 \leq r < 1\} \) that satisfies the Neumann boundary condition,

\[
u_r(1, \theta) = f(\theta) = \theta,
\]
The solution can be written as

\[ u(r, \theta) = d_0 + \sum_{n=1}^{\infty} r^n [c_n \cos(n\theta) + d_n \sin(n\theta)]. \]

The boundary condition implies

\[ u_r(1, \theta) = \sum_{n=1}^{\infty} n c_n \cos(n\theta) + n d_n \sin(n\theta) = \theta. \]

Applying the Euler-Fourier formula:

\[ n c_n = \frac{1}{\pi} \int_{-\pi}^{\pi} \theta \cos(n\theta) \, d\theta = 0 \]

\[ n d_n = \frac{1}{\pi} \int_{-\pi}^{\pi} \theta \sin(n\theta) \, d\theta \]

\[ = \left[ \frac{-\theta}{n\pi} \cos(n\theta) \right]_{\theta=-\pi}^{\theta=\pi} + \frac{1}{n\pi} \int_{-\pi}^{\pi} \cos(n\theta) \, d\theta \]

\[ d_n = \frac{-2(-1)^n}{n^2}. \]
Solution (2 of )

\[ u(r, \theta) = d_0 - 2 \sum_{n=1}^{\infty} \frac{(-1)^n r^n}{n^2} \sin(n\theta). \]
Homework

- Read Sections 6.5 and 6.6
- Exercises: 20–23