# The Nonhomogeneous Wave Equation

MATH 467 *Partial Differential Equations*

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# Objectives

In this lesson we will learn:

- a decomposition approach to solving nonhomogeneous wave equations.

# General Nonhomogeneous Wave Equation

Consider the following initial boundary value problem:

$$\begin{aligned} u_{tt} &= c^2 u_{xx} + F(x,t) \text{ for } 0 < x < L \text{ and } t > 0 \\ u(0,t) = \phi(t) &\text{ and } u(L,t) = \psi(t) \text{ for } t > 0 \\ u(x,0) = f(x) &\text{ and } u_t(x,0) = g(x) \text{ for } 0 < x < L. \end{aligned}$$

The solution to the IBVP can be found by solving two simpler initial boundary value problems and using the Principle of Superposition to reconstruct the full solution.

# Two Sub-Problems

$$\begin{aligned} v_{tt} &= c^2 v_{xx} + F(x,t) \text{ for } 0 < x < L \text{ and } t > 0 \\ v(0,t) &= v(L,t) = 0 \text{ for } t > 0 \\ v(x,0) &= v_t(x,0) = 0 \text{ for } 0 < x < L \end{aligned}$$

The IBVP above contains the nonhomogeneous PDE.

$$\begin{aligned} w_{tt} &= c^2 w_{xx} \text{ for } 0 < x < L \text{ and } t > 0 \\ w(0,t) &= \phi(t) \text{ and } w(L,t) = \psi(t) \text{ for } t > 0 \\ w(x,0) &= f(x) \text{ and } w_t(x,0) = g(x) \text{ for } 0 < x < L. \end{aligned}$$

The IBVP above contains the nonhomogeneous BCs.

If $v(x,t)$ and $w(x,t)$ solve their respective IBVPs, then $u(x,t) = v(x,t) + w(x,t)$ solves the original IBVP.

# Homogeneous PDE with Nonhomogeneous BCs

Consider the IBVP with nonhomogeneous BCs:

$$\begin{aligned} w_{tt} &= c^2 w_{xx} \text{ for } 0 < x < L \text{ and } t > 0 \\ w(0,t) &= \phi(t) \text{ for } t > 0 \\ w(L,t) &= \psi(t) \text{ for } t > 0 \\ w(x,0) &= f(x) \text{ for } 0 < x < L \\ w_t(x,0) &= g(x) \text{ for } 0 < x < L. \end{aligned}$$

Assume the solution can be written as $w(x,t) = y(x,t) + r(x,t)$ where $r(x,t)$ is a **reference function** satisfying the nonhomogeneous BCs and $y(x,t)$ is an unknown function, to be found later.

# Reference Function

Find any function $r(x, t)$ which satisfies the nonhomogeneous BCs:

$$w(0, t) = \phi(t) \text{ for } t > 0$$
$$w(L, t) = \psi(t) \text{ for } t > 0.$$

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$$w(L,t) = \psi(t) \text{ for } t > 0.$$

Many solutions are possible, but a straightforward one is

$$r(x,t) = \frac{x}{L}\left[\psi(t) - \phi(t)\right] + \phi(t).$$

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Many solutions are possible, but a straightforward one is

$$r(x,t) = \frac{x}{L}\left[\psi(t) - \phi(t)\right] + \phi(t).$$

If $w(x,t) = y(x,t) + r(x,t)$ solves the IBVP given earlier with nonhomogeneous BCs, find the IBVP which $y(x,t)$ solves.

# Related IBVP with Homogeneous BCs

$$\begin{aligned} w_{tt} &= y_{tt} + \frac{x}{L}\left[\psi''(t) - \phi''(t)\right] + \phi''(t) \\ w_{xx} &= y_{xx} \end{aligned}$$

# Related IBVP with Homogeneous BCs

$$
\begin{aligned}
w_{tt} &= y_{tt} + \frac{x}{L}\left[\psi''(t) - \phi''(t)\right] + \phi''(t) \\
w_{xx} &= y_{xx} \\
y_{tt} &= c^2 y_{xx} - \frac{x}{L}\left[\psi''(t) - \phi''(t)\right] - \phi''(t) \text{ for } 0 < x < L,\ t > 0 \\
y(0,t) &= y(L,t) = 0 \text{ for } t > 0 \\
y(x,0) &= f(x) - \frac{x}{L}\left[\psi(0) - \phi(0)\right] - \phi(0) \text{ for } 0 < x < L \\
y_t(x,0) &= g(x) - \frac{x}{L}\left[\psi'(0) - \phi'(0)\right] - \phi'(0) \text{ for } 0 < x < L.
\end{aligned}
$$

# Related IBVP with Homogeneous BCs

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w_{tt} &= y_{tt} + \frac{x}{L}\left[\psi''(t) - \phi''(t)\right] + \phi''(t) \\
w_{xx} &= y_{xx} \\
y_{tt} &= c^2 y_{xx} - \frac{x}{L}\left[\psi''(t) - \phi''(t)\right] - \phi''(t) \text{ for } 0 < x < L,\ t > 0 \\
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y_t(x,0) &= g(x) - \frac{x}{L}\left[\psi'(0) - \phi'(0)\right] - \phi'(0) \text{ for } 0 < x < L.
\end{aligned}$$

**Remark**: the IBVP for $y(x,t)$ has homogeneous Dirichlet boundary conditions, but a nonhomogeneous PDE. We must decompose it into two additional sub-problems in order to solve it.

# Two More Sub-Problems

**Note**: the dependent variables are again named $v$ and $w$ though they are not the same functions as mentioned earlier.

**IBVP with homogeneous PDE**:

$$\begin{aligned}
w_{tt} &= c^2 w_{xx} \text{ for } 0 < x < L,\ t > 0 \\
w(0,t) &= w(L,t) = 0 \text{ for } t > 0 \\
w(x,0) &= f(x) - \frac{x}{L}\left[\psi(0) - \phi(0)\right] - \phi(0) \text{ for } 0 < x < L \\
w_t(x,0) &= g(x) - \frac{x}{L}\left[\psi'(0) - \phi'(0)\right] - \phi'(0) \text{ for } 0 < x < L.
\end{aligned}$$

**IBVP with nonhomogeneous PDE**:

$$\begin{aligned}
v_{tt} &= c^2 v_{xx} - \frac{x}{L}\left[\psi''(t) - \phi''(t)\right] - \phi''(t) \text{ for } 0 < x < L,\ t > 0 \\
v(0,t) &= v(L,t) = 0 \text{ for } t > 0 \\
v(x,0) &= 0 \text{ for } 0 < x < L \\
v_t(x,0) &= 0 \text{ for } 0 < x < L.
\end{aligned}$$

# IBVP With Homogeneous PDE

$$\begin{aligned}
w_{tt} &= c^2 w_{xx} \text{ for } 0 < x < L,\ t > 0 \\
w(0,t) &= w(L,t) = 0 \text{ for } t > 0 \\
w(x,0) &= f(x) - \frac{x}{L}\left[\psi(0) - \phi(0)\right] - \phi(0) \text{ for } 0 < x < L \\
w_t(x,0) &= g(x) - \frac{x}{L}\left[\psi'(0) - \phi'(0)\right] - \phi'(0) \text{ for } 0 < x < L.
\end{aligned}$$

**Remark**: this IBVP has homogeneous Dirichlet BCs and thus we can express the solution as a Fourier series:

$$w(x,t) = \sum_{n=1}^{\infty}\left(a_n \cos\frac{cn\pi t}{L} + b_n \sin\frac{cn\pi t}{L}\right)\sin\frac{n\pi x}{L}.$$

# IBVP with nonhomogeneous PDE

$$
\begin{aligned}
v_{tt} &= c^2 v_{xx} - \frac{x}{L}\left[\psi''(t) - \phi''(t)\right] - \phi''(t) \text{ for } 0 < x < L,\ t > 0 \\
v(0,t) &= v(L,t) = 0 \text{ for } t > 0 \\
v(x,0) &= 0 \text{ for } 0 < x < L \\
v_t(x,0) &= 0 \text{ for } 0 < x < L.
\end{aligned}
$$

**Remark**: this IBVP has homogeneous Dirichlet BCs and ICs of zero. We will solve this nonhomogeneous PDE in more generality by assuming the nonhomogeneity is a function $F(x,t)$.

# Finding the Solution to the Nonhomogeneous PDE

$$
\begin{aligned}
u_{tt} &= c^2 u_{xx} + F(x,t) \text{ for } 0 < x < L,\ t > 0 \\
u(0,t) &= u(L,t) = 0 \text{ for } t > 0 \\
u(x,0) &= 0 \text{ for } 0 < x < L \\
u_t(x,0) &= 0 \text{ for } 0 < x < L.
\end{aligned}
$$

Assume the solution has the form

$$
u(x,t) = \sum_{n=1}^{\infty} T_n(t) \sin \frac{n\pi x}{L}.
$$

Differentiate and substitute into the PDE.

# Differentiating the Solution

$$u_{tt} = c^2 u_{xx} + F(x, t)$$

$$\sum_{n=1}^{\infty} T_n''(t) \sin \frac{n\pi x}{L} = -c^2 \sum_{n=1}^{\infty} \left(\frac{n\pi}{L}\right)^2 T_n(t) \sin \frac{n\pi x}{L} + F(x, t)$$

Rearrange terms to isolate the nonhomogeneous term.

$$\sum_{n=1}^{\infty} \left[ T_n''(t) + \left(\frac{n\pi c}{L}\right)^2 T_n(t) \right] \sin \frac{n\pi x}{L} = F(x, t)$$

Multiply both sides by $\sin(m\pi x)/L$ and integrate from $x = 0$ to $x = L$.

# Orthogonality

Since $\sin(n\pi x)/L$ and $\sin(m\pi x)/L$ are orthogonal when $n \neq m$ then

$$T_m''(t) + \left(\frac{m\pi c}{L}\right)^2 T_m(t) = \frac{2}{L}\int_0^L F(x,t)\sin\frac{m\pi x}{L}\,dx = F_m(t)$$

for $m \in \mathbb{N}$.

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for $m \in \mathbb{N}$.

For $n = 1, 2, \ldots$ we must solve the initial value problems:

$$\begin{aligned} T_n''(t) + \left(\frac{n\pi c}{L}\right)^2 T_n(t) &= F_n(t) \\ T_n(0) &= 0 \\ T_n'(0) &= 0. \end{aligned}$$

# Example

Find the solution to the following IBVP:

$$
\begin{aligned}
u_{tt} &= u_{xx} + t \sin x \text{ for } 0 < x < \pi \text{ and } t > 0 \\
u(0,t) &= u(\pi,t) = 0 \text{ for } t > 0 \\
u(x,0) &= \sin x \text{ for } 0 < x < \pi \\
u_t(x,0) &= \sin(3x) \text{ for } 0 < x < \pi.
\end{aligned}
$$

# Solution (1 of 7)

Find the solution to the homogeneous IBVP:

$$\begin{aligned} w_{tt} &= w_{xx} \text{ for } 0 < x < \pi \text{ and } t > 0 \\ w(0,t) &= w(\pi,t) = 0 \text{ for } t > 0 \\ w(x,0) &= \sin x \text{ for } 0 < x < \pi \\ w_t(x,0) &= \sin(3x) \text{ for } 0 < x < \pi. \end{aligned}$$

This can be found readily in d'Alembertian form.

# Solution (2 of 7)

$$
\begin{aligned}
w(x,t) &= \frac{1}{2}\left(\sin(x-t)+\sin(x+t)\right)+\frac{1}{2}\int_{x-t}^{x+t}\sin(3s)\,ds \\
&= \frac{1}{2}\left(\sin(x-t)+\sin(x+t)\right)+\frac{1}{6}\left(\cos 3(x-t)-\cos 3(x+t)\right) \\
&= \sin x\cos t+\frac{1}{3}\sin(3x)\sin(3t)
\end{aligned}
$$

# Solution (3 of 7)

Now find the solution to the IBVP containg the nonhomogeneous PDE and zero initial conditions.

$$\begin{aligned} v_{tt} &= v_{xx} + t \sin x \text{ for } 0 < x < \pi \text{ and } t > 0 \\ v(0,t) &= v(\pi,t) = 0 \text{ for } t > 0 \\ v(x,0) &= 0 \text{ for } 0 < x < \pi \\ v_t(x,0) &= 0 \text{ for } 0 < x < \pi. \end{aligned}$$

Make the assumption that the solution can be expressed as

$$v(x,t) = \sum_{n=1}^{\infty} T_n(t) \sin(n\,x).$$

# Solution (4 of 7)

Differentiating $v(x,t)$ and substituting into the PDE produce:

$$\sum_{n=1}^{\infty} T_n''(t)\sin(n\,x) = -\sum_{n=1}^{\infty} n^2\,T_n(t)\sin(n\,x) + t\,\sin x$$

$$\sum_{n=1}^{\infty} \left(T_n(t) + n^2\,T_n(t)\right)\sin(n\,x) = t\,\sin x.$$

# Solution (4 of 7)

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$$\sum_{n=1}^{\infty} T_n''(t)\sin(n\,x) = -\sum_{n=1}^{\infty} n^2\,T_n(t)\sin(n\,x) + t\,\sin x$$

$$\sum_{n=1}^{\infty} \left(T_n(t) + n^2\,T_n(t)\right)\sin(n\,x) = t\,\sin x.$$

Multiply both sides by $\sin(m\,x)$ and integrate over $[0,\pi]$.

# Solution (5 of 7)

$$
\begin{aligned}
T_m''(t) + m^2 T_m(t) &= \frac{2}{\pi} \int_0^{\pi} t \sin x \sin(m x)\, dx \\
&= \begin{cases} t & \text{if } m = 1 \\ 0 & \text{if } m = 2, 3, \ldots \end{cases}
\end{aligned}
$$

# Solution (5 of 7)

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Solve the initial value problems:

$$\begin{aligned} T_1''(t) + T_1(t) &= t \\ T_n''(t) + n^2 T_n(t) &= 0 \end{aligned}$$

for $n = 2, 3, \ldots$ with zero initial conditions.

# Solution (6 of 7)

We can immediately check that $T_n(t) = 0$ for $n = 2, 3, \ldots$.

$$T_1(t) = A_1 \cos t + B_1 \sin t + t$$

Making use of the initial conditions ($T_1(0) = T_1'(0) = 0$) reveals,

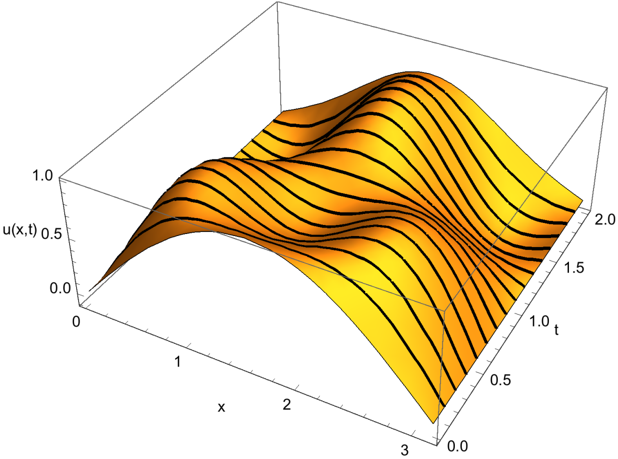
$$T_1(t) = t - \sin t.$$

This implies,

$$v(x,t) = T_1(t) \sin x = (t - \sin t) \sin x.$$

# Solution (7 of 7)

$$
\begin{aligned}
u(x,t) &= w(x,t) + v(x,t) \\
&= \sin x \cos t + \frac{1}{3}\sin(3x)\sin(3t) + (t - \sin t)\sin x.
\end{aligned}
$$



# Homework

- Read Sections 5.4
- Exercises: 18–20