Separation of Variables

MATH 467 *Partial Differential Equations*

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Objectives

In this lesson we will learn the approach of a fundamental technique for solving many PDEs, namely **separation of variables**.

This technique reduces the problem of finding the unknown dependent variable of the PDE u, which depends on n independent variables, to the problem of solving n ordinary differential equations each depending on a single independent variable.

We will assume the dependent variable is a **product solution**.

$$u(x, y, t) = X(x)Y(y)T(t)$$

Example

Apply the method of separation of variables to the equation

$$x^2 u_{xx} - 2y\, u_y = 0$$

and find a corresponding set of ordinary differential equations.

Solution (1 of 2)

- Assume $u(x, y) = X(x)Y(y)$ then

$$\begin{aligned} u_{xx} &= X''(x)Y(y) \\ u_y &= X(x)Y'(y). \end{aligned}$$

Solution (1 of 2)

- Assume $u(x, y) = X(x)Y(y)$ then

$$u_{xx} = X''(x)Y(y)$$
$$u_y = X(x)Y'(y).$$

- Substitute into the PDE.

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- Substitute into the PDE.

$$\begin{aligned} x^2 u_{xx} - 2y\, u_y &= 0 \\ x^2 X''(x)Y(y) - 2yX(x)Y'(y) &= 0 \end{aligned}$$

- Divide both sides by $u(x, y) = X(x)Y(y)$.

$$\frac{x^2 X''(x)Y(y)}{X(x)Y(y)} - \frac{2yX(x)Y'(y)}{X(x)Y(y)} = 0$$

Solution (1 of 2)

- Assume $u(x, y) = X(x)Y(y)$ then

$$\begin{aligned} u_{xx} &= X''(x)Y(y) \\ u_y &= X(x)Y'(y). \end{aligned}$$

- Substitute into the PDE.

$$\begin{aligned} x^2 u_{xx} - 2y\, u_y &= 0 \\ x^2 X''(x)Y(y) - 2yX(x)Y'(y) &= 0 \end{aligned}$$

- Divide both sides by $u(x, y) = X(x)Y(y)$.

$$\begin{aligned} \frac{x^2 X''(x)Y(y)}{X(x)Y(y)} - \frac{2yX(x)Y'(y)}{X(x)Y(y)} &= 0 \\ x^2 \frac{X''(x)}{X(x)} &= 2y\, \frac{Y'(y)}{Y(y)} \end{aligned}$$

Solution (2 of 2)

$$x^2 \frac{X''(x)}{X(x)} = 2y \frac{Y'(y)}{Y(y)}$$

Key observation: the left-hand side is a function of x while the right-hand side is a function of y. Since they are equal they must be constant.

Solution (2 of 2)

$$x^2 \frac{X''(x)}{X(x)} = 2y \frac{Y'(y)}{Y(y)}$$

Key observation: the left-hand side is a function of x while the right-hand side is a function of y. Since they are equal they must be constant.

$$x^2 \frac{X''(x)}{X(x)} = c = 2y \frac{Y'(y)}{Y(y)}$$

This implies

$$\begin{aligned} x^2 \frac{X''(x)}{X(x)} &= c \\ 2y \frac{Y'(y)}{Y(y)} &= c \end{aligned}$$

Solution (2 of 2)

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Key observation: the left-hand side is a function of x while the right-hand side is a function of y. Since they are equal they must be constant.

$$x^2 \frac{X''(x)}{X(x)} = c = 2y \, \frac{Y'(y)}{Y(y)}$$

This implies

$$\begin{aligned} x^2 \tfrac{X''(x)}{X(x)} &= c \\ 2y \, \tfrac{Y'(y)}{Y(y)} &= c \end{aligned} \iff \begin{aligned} x^2 X''(x) - c \, X(x) &= 0 \\ 2y \, Y'(y) - c \, Y(y) &= 0 \end{aligned}$$

Example

Apply the method of separation of variables to the following equation and determine the corresponding set of ordinary differential equations.

$$u_{xx} + u_x + 2u_y - u \sin x = 0$$

Solution (1 of 2)

- Assume $u(x, y) = X(x)Y(y)$ then

$$\begin{aligned} u_x &= X'(x)Y(y) \\ u_{xx} &= X''(x)Y(y) \\ u_y &= X(x)Y'(y). \end{aligned}$$

Solution (1 of 2)

- Assume $u(x,y) = X(x)Y(y)$ then

$$\begin{aligned} u_x &= X'(x)Y(y) \\ u_{xx} &= X''(x)Y(y) \\ u_y &= X(x)Y'(y). \end{aligned}$$

- Substitute into the PDE.

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Solution (1 of 2)

- Assume $u(x, y) = X(x)Y(y)$ then

$$\begin{aligned} u_x &= X'(x)Y(y) \\ u_{xx} &= X''(x)Y(y) \\ u_y &= X(x)Y'(y). \end{aligned}$$

- Substitute into the PDE.

$$\begin{aligned} u_{xx} + u_x + 2u_y - u \sin x &= 0 \\ X''(x)Y(y) + X'(x)Y(y) + 2X(x)Y'(y) - X(x)Y(y) \sin x &= 0 \end{aligned}$$

Solution (1 of 2)

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$$\begin{aligned} u_x &= X'(x)Y(y) \\ u_{xx} &= X''(x)Y(y) \\ u_y &= X(x)Y'(y). \end{aligned}$$

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- Divide both sides by $u(x,y) = X(x)Y(y)$.

$$\frac{X''(x)Y(y)}{X(x)Y(y)} + \frac{X'(x)Y(y)}{X(x)Y(y)} + \frac{2X(x)Y'(y)}{X(x)Y(y)} - \frac{X(x)Y(y)\sin x}{X(x)Y(y)} = 0$$

Solution (1 of 2)

- Assume $u(x, y) = X(x)Y(y)$ then

$$\begin{aligned} u_x &= X'(x)Y(y) \\ u_{xx} &= X''(x)Y(y) \\ u_y &= X(x)Y'(y). \end{aligned}$$

- Substitute into the PDE.

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- Divide both sides by $u(x, y) = X(x)Y(y)$.

$$\begin{aligned} \frac{X''(x)Y(y)}{X(x)Y(y)} + \frac{X'(x)Y(y)}{X(x)Y(y)} + \frac{2X(x)Y'(y)}{X(x)Y(y)} - \frac{X(x)Y(y) \sin x}{X(x)Y(y)} &= 0 \\ \frac{X''(x)}{X(x)} + \frac{X'(x)}{X(x)} - \sin x &= -2\frac{Y'(y)}{Y(y)} \end{aligned}$$

Solution (2 of 2)

$$\frac{X''(x)}{X(x)} + \frac{X'(x)}{X(x)} - \sin x = -2\,\frac{Y'(y)}{Y(y)}$$

Since the left-hand side is a function of x while the right-hand side is a function of y. Since they are equal they must be constant.

Solution (2 of 2)

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Since the left-hand side is a function of x while the right-hand side is a function of y. Since they are equal they must be constant.

$$\frac{X''(x)}{X(x)} + \frac{X'(x)}{X(x)} - \sin x = c = -2\,\frac{Y'(y)}{Y(y)}$$

This implies

$$\begin{aligned} \frac{X''(x)}{X(x)} + \frac{X'(x)}{X(x)} - \sin x &= c \\ -2\,\frac{Y'(y)}{Y(y)} &= c \end{aligned}$$

Solution (2 of 2)

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This implies

$$\begin{aligned} \tfrac{X''(x)}{X(x)} + \tfrac{X'(x)}{X(x)} - \sin x &= c \\ -2\,\tfrac{Y'(y)}{Y(y)} &= c \end{aligned} \iff \begin{aligned} X''(x) + X'(x) - X(x)\sin x &= c\,X(x) \\ 2\,Y'(y) + c\,Y(y) &= 0 \end{aligned}$$

Example

Determine if the method of separation of variables can be applied to the following partial differential equation. If so, determine the resulting ordinary differential equations.

$$u_x + (x + y)u_y = 0.$$

Solution

- Assume $u(x,y) = X(x)Y(y)$ then

$$u_x = X'(x)Y(y)$$
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- Assume $u(x, y) = X(x)Y(y)$ then

$$u_x = X'(x)Y(y)$$
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- Substitute into the PDE.

$$u_x + (x + y)u_y = 0$$
$$X'(x)Y(y) + (x + y)X(x)Y'(y) = 0$$

Solution

- Assume $u(x, y) = X(x)Y(y)$ then

$$\begin{aligned} u_x &= X'(x)Y(y) \\ u_y &= X(x)Y'(y). \end{aligned}$$

- Substitute into the PDE.

$$\begin{aligned} u_x + (x + y)u_y &= 0 \\ X'(x)Y(y) + (x + y)X(x)Y'(y) &= 0 \end{aligned}$$

- Divide both sides by $u(x, y) = X(x)Y(y)$.

$$\begin{aligned} \frac{X'(x)Y(y)}{X(x)Y(y)} + \frac{(x + y)X(x)Y'(y)}{X(x)Y(y)} &= 0 \\ \frac{X'(x)}{X(x)} + \frac{(x + y)Y'(y)}{Y(y)} &= 0 \end{aligned}$$

Solution

- Assume $u(x,y) = X(x)Y(y)$ then

$$\begin{aligned} u_x &= X'(x)Y(y) \\ u_y &= X(x)Y'(y). \end{aligned}$$

- Substitute into the PDE.

$$\begin{aligned} u_x + (x+y)u_y &= 0 \\ X'(x)Y(y) + (x+y)X(x)Y'(y) &= 0 \end{aligned}$$

- Divide both sides by $u(x,y) = X(x)Y(y)$.

$$\begin{aligned} \frac{X'(x)Y(y)}{X(x)Y(y)} + \frac{(x+y)X(x)Y'(y)}{X(x)Y(y)} &= 0 \\ \frac{X'(x)}{X(x)} + \frac{(x+y)Y'(y)}{Y(y)} &= 0 \end{aligned}$$

It is not possible to separate the variables in this example.

Example

Determine if the method of separation of variables can be applied to the partial differential equation below. If so, determine the ordinary differential equations in each variable which result.

$$u_{rr} + \frac{1}{r} u_r + \frac{1}{r^2} u_{\theta\theta} = 0 \quad \text{(for } r > 0\text{)}$$

The dependent variable is bounded as $r \to 0$ and is 2π-periodic in θ.

Solution (1 of 4)

- Let $u(r,\theta) = R(r)T(\theta)$, then

$$\begin{aligned} u_r &= R'(r)T(\theta) \\ u_{rr} &= R''(r)T(\theta) \\ u_{\theta\theta} &= R(r)T''(\theta). \end{aligned}$$

- Substitute into the PDE.

$$\begin{aligned} u_{rr} + \frac{1}{r}u_r + \frac{1}{r^2}u_{\theta\theta} &= 0 \\ R''(r)T(\theta) + \frac{1}{r}R'(r)T(\theta) + \frac{1}{r^2}R(r)T''(\theta) &= 0 \end{aligned}$$

- Multiply both sides of the equation by $r^2/(R(r)T(\theta))$.

$$r^2\frac{R''(r)}{R(r)} + r\frac{R'(r)}{R(r)} + \frac{T''(\theta)}{T(\theta)} = 0$$

- Separate the variables.

Solution (2 of 4)

$$r^2 \frac{R''(r)}{R(r)} + r \frac{R'(r)}{R(r)} = -\frac{T''(\theta)}{T(\theta)}$$

Solution (2 of 4)

$$r^2 \frac{R''(r)}{R(r)} + r\,\frac{R'(r)}{R(r)} = -\frac{T''(\theta)}{T(\theta)} = c$$

where c is a constant.

The ordinary differential equations for r and θ are thus

$$\begin{aligned} r^2 R''(r) + r\,R'(r) - c\,R(r) &= 0 \\ T''(\theta) + c\,T(\theta) &= 0. \end{aligned}$$

Solution (2 of 4)

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where c is a constant.

The ordinary differential equations for r and θ are thus

$$\begin{aligned} r^2 R''(r) + r R'(r) - c R(r) &= 0 \\ T''(\theta) + c T(\theta) &= 0. \end{aligned}$$

The equation in θ is a second-order linear, constant coefficient, linear, homogeneous ODE.

Solution (3 of 4)

If $c \geq 0$ then the ODE

$$T''(\theta) + c\,T(\theta) = 0$$

has solutions

$$T(\theta) = A\cos(\sqrt{c}\,\theta) + B\sin(\sqrt{c}\,\theta).$$

Solution (3 of 4)

If $c \geq 0$ then the ODE

$$T''(\theta) + c\,T(\theta) = 0$$

has solutions

$$T(\theta) = A\cos(\sqrt{c}\,\theta) + B\sin(\sqrt{c}\,\theta).$$

The solution $u(r, \theta)$ should be 2π-periodic in θ, thus

$$\frac{2\pi}{\sqrt{c}} = \frac{2\pi}{n} \iff c = n^2$$

where $n \in \mathbb{N}$ or $c = 0$.

Solution (3 of 4)

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The solution $u(r,\theta)$ should be 2π-periodic in θ, thus

$$\frac{2\pi}{\sqrt{c}} = \frac{2\pi}{n} \iff c = n^2$$

where $n \in \mathbb{N}$ or $c = 0$.

Thus the solutions for $T(\theta)$ can be summarized as

$$T_n(\theta) = A_n\cos(n\theta) + B_n\sin(n\theta)$$

for $n = 0, 1, 2, \ldots$.

Solution (4 of 4)

Since $c = n^2$ for $n = 0, 1, 2, \ldots$ then the ODE for r can be written as

$$r^2 R''(r) + r\,R'(r) - n^2 R(r) = 0$$

which is Euler's equation. The solutions are

$$R_0(r) = C_0 + D_0 \ln r \quad (\text{for } n = 0)$$
$$R_n(r) = C_n r^n + D_n r^{-n} \quad (\text{for } n = 1, 2, \ldots)$$

Solution (4 of 4)

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$$R_0(r) = C_0 + D_0 \ln r \quad (\text{for } n = 0)$$
$$R_n(r) = C_n r^n + D_n r^{-n} \quad (\text{for } n = 1, 2, \ldots)$$

The solutions are supposed to be bounded as $r \to 0$ and thus $D_0 = D_n = 0$ for $n = 1, 2, \ldots$. Consequently

$$u_n(r, \theta) = r^n \left[A_n \cos(n\theta) + B_n \sin(n\theta)\right]$$

for $n = 0, 1, 2, \ldots$.

Example: The Heat Equation

Consider the one-dimensional, homogeneous heat equation with Dirichlet boundary conditions and an initial condition as below.

$$\begin{aligned} u_t &= k\, u_{xx}, \quad 0 < x < L, \quad t > 0 \\ u(0,t) &= 0, \quad t > 0 \\ u(L,t) &= 0, \quad t > 0 \\ u(x,0) &= f(x), \quad 0 \leq x \leq L \end{aligned}$$

Apply the method of separation of variables to this initial boundary value problem and determine the product solutions which satisfy the homogeneous boundary conditions.

Solution (1 of 8)

- Assume the product solution $u(x,t) = X(x)T(t)$.
- Differentiating and substituting into the heat equation yields

$$
\begin{aligned}
u_t &= k\, u_{xx} \\
X(x)T'(t) &= k\, X''(x)T(t) \\
\frac{1}{k}\,\frac{T'(t)}{T(t)} &= \frac{X''(x)}{X(x)} = -c
\end{aligned}
$$

  where c is a constant and the minus sign is introduced for convenience.
- The resulting ODEs for x and t are

$$
\begin{aligned}
X''(x) + c\,X(x) &= 0 \\
T'(t) + c\,k\,T(t) &= 0.
\end{aligned}
$$

Solution (2 of 8)

Consider the ODE for x and the boundary conditions.

$$X''(x) + c\,X(x) = 0$$
$$u(0,t) = X(0)T(t) = 0 \iff X(0) = 0$$
$$u(L,t) = X(L)T(t) = 0 \iff X(L) = 0$$

Find solutions to the ODE which satisfy the boundary conditions.
Consider the three cases:

- $c = 0$,
- $c < 0$,
- $c > 0$.

Solution (3 of 8)

Case: $c = 0$.

$$\begin{aligned} X''(x) + c\,X(x) = X''(x) &= 0 \\ X(x) &= Ax + B \end{aligned}$$

Solution (3 of 8)

Case: $c = 0$.

$$X''(x) + c\,X(x) = X''(x) = 0$$
$$X(x) = Ax + B$$

When $x = 0$ we have $0 = X(0) = B$.

Solution (3 of 8)

Case: $c = 0$.

$$X''(x) + c\,X(x) = X''(x) = 0$$
$$X(x) = Ax + B$$

When $x = 0$ we have $0 = X(0) = B$.

When $x = L$ we have $0 = X(L) = A\,L$ which implies $A = 0$. Thus when $c = 0$ we have only the trivial solution $X(x) = 0$.

Solution (4 of 8)

Case: $c < 0$. Let $c = -\lambda^2$ where $\lambda > 0$.

$$\begin{aligned} X''(x) + c\,X(x) = X''(x) - \lambda^2 X(x) &= 0 \\ X(x) &= A\cosh(\lambda x) + B\sinh(\lambda x) \end{aligned}$$

Solution (4 of 8)

Case: $c < 0$. Let $c = -\lambda^2$ where $\lambda > 0$.

$$X''(x) + c\,X(x) = X''(x) - \lambda^2 X(x) = 0$$
$$X(x) = A\cosh(\lambda x) + B\sinh(\lambda x)$$

When $x = 0$ we have $0 = X(0) = A$.

Solution (4 of 8)

Case: $c < 0$. Let $c = -\lambda^2$ where $\lambda > 0$.

$$X''(x) + c\,X(x) = X''(x) - \lambda^2 X(x) = 0$$
$$X(x) = A\cosh(\lambda x) + B\sinh(\lambda x)$$

When $x = 0$ we have $0 = X(0) = A$.

When $x = L$ we have $0 = X(L) = B\sinh(\lambda L)$ which implies $B = 0$. Thus when $c < 0$ we have only the trivial solution $X(x) = 0$.

Solution (5 of 8)

Case: $c > 0$. Let $c = \lambda^2$ where $\lambda > 0$.

$$\begin{aligned} X''(x) + c\,X(x) = X''(x) + \lambda^2 X(x) &= 0 \\ X(x) &= A\cos(\lambda x) + B\sin(\lambda x) \end{aligned}$$

Solution (5 of 8)

Case: $c > 0$. Let $c = \lambda^2$ where $\lambda > 0$.

$$\begin{aligned} X''(x) + c\,X(x) = X''(x) + \lambda^2 X(x) &= 0 \\ X(x) &= A\cos(\lambda x) + B\sin(\lambda x) \end{aligned}$$

When $x = 0$ we have $0 = X(0) = A$.

Solution (5 of 8)

Case: $c > 0$. Let $c = \lambda^2$ where $\lambda > 0$.

$$X''(x) + c\,X(x) = X''(x) + \lambda^2 X(x) = 0$$
$$X(x) = A\cos(\lambda x) + B\sin(\lambda x)$$

When $x = 0$ we have $0 = X(0) = A$.

When $x = L$ we have $0 = X(L) = B\sin(\lambda L)$ which implies

$$\lambda L = n\pi$$
$$\lambda \equiv \lambda_n = \frac{n\pi}{L}$$

for $n \in \mathbb{N}$. Thus when $c = n^2\pi^2/L^2$ for $n \in \mathbb{N}$ we have the nontrivial solutions

$$X_n(x) = \sin\frac{n\pi x}{L}.$$

Function $X_n(x)$ is called an **eigenfunction** corresponding to **eigenvalue** $n^2\pi^2/L^2$.

Solution (6 of 8)

Using the known eigenvalue in the ODE for t yields

$$T'(t) + \frac{n^2\pi^2 k}{L^2} T(t) = 0$$
$$T(t) \equiv T_n(t) = C_n e^{-n^2\pi^2 k\, t/L^2}$$

for $n \in \mathbb{N}$.

Solution (6 of 8)

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$$T'(t) + \frac{n^2\pi^2 k}{L^2} T(t) = 0$$
$$T(t) \equiv T_n(t) = C_n e^{-n^2\pi^2 k\, t/L^2}$$

for $n \in \mathbb{N}$.

The product solutions which satisfy the boundary conditions have the form

$$u_n(x,t) = X_n(x) T_n(t) = B_n e^{-n^2\pi^2 k\, t/L^2} \sin\left(\frac{n\pi x}{L}\right).$$

These are called **fundamental solutions**.

Solution (7 of 8)

Using the Principle of Superposition, a finite linear combination of fundamental solutions will likewise solve the PDE and satisfy the BCs.

$$u(x,t) = \sum_{n=1}^{N} B_n e^{-n^2\pi^2 k\,t/L^2} \sin\left(\frac{n\pi x}{L}\right)$$

Solution (7 of 8)

Using the Principle of Superposition, a finite linear combination of fundamental solutions will likewise solve the PDE and satisfy the BCs.

$$u(x,t) = \sum_{n=1}^{N} B_n e^{-n^2\pi^2 k\, t/L^2} \sin\left(\frac{n\pi x}{L}\right)$$

Now consider the initial condition:

$$u(x,0) = f(x)$$

$$\sum_{n=1}^{N} B_n \sin\left(\frac{n\pi x}{L}\right) = f(x)$$

As long as $f(x)$ contains a finite sum of sine functions of the appropriate periods we can equate coefficients and solve for the B_n's.

Solution (8 of 8)

Take the case in which

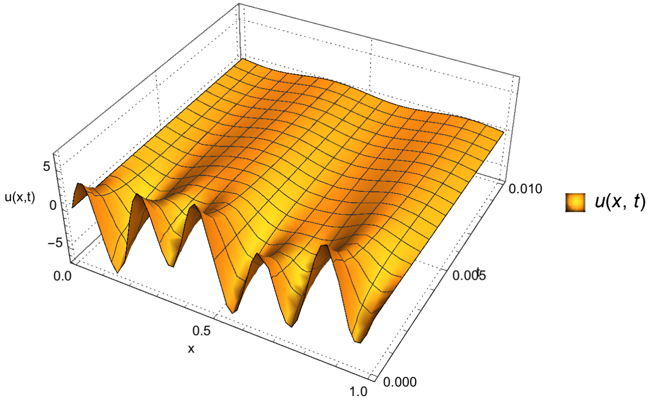
$$f(x) = -2 \sin \frac{4\pi x}{L} + 5 \sin \frac{10\pi x}{L}.$$

Then $B_4 = -2$, $B_{10} = 5$ and all other coefficients are 0.

$$u(x,t) = -2e^{-16\pi^2 k t/L^2} \sin \frac{4\pi x}{L} + 5e^{-100\pi^2 k t/L^2} \sin \frac{10\pi x}{L}.$$

Illustration

$$u(x,t) = -2e^{-16\pi^2 k\,t/L^2} \sin \frac{4\pi\,x}{L} + 5e^{-100\pi^2 k\,t/L^2} \sin \frac{10\pi\,x}{L}.$$



Homework

- Read Section 1.6
- Exercises: 14, 18, 20