Computing the Flow Lines of a Vector Field

Math 311

To find the flow lines of a given vector field $\mathbf{F}(x,y) = \langle f_1(x,y), f_2(x,y)\rangle$:

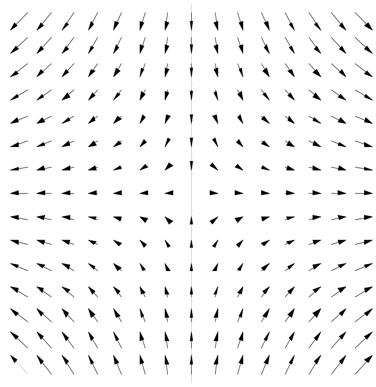
1. Write
$$\frac{dy}{dx} = \frac{f_2(x,y)}{f_1(x,y)}.$$
2. Separate variables, i.e., move things involving x to one side and things involving y to the other.
3. Integrate both sides and simplify to obtain an equation in x and y.

Example 1: Find the flow lines of the vector field $\mathbf{F}(x,y) = \langle x, y\rangle$.

1. Assume $x \neq 0$ and write:
$$\frac{dy}{dx} = \frac{y}{x}$$
2. Assume $y \neq 0$ and separate variables:
$$\frac{dy}{y} = \frac{dx}{x}$$
3. Integrate and simplify:
$$\begin{aligned}
\int \frac{dy}{y} &= \int \frac{dx}{x} \\
\ln|y| + C_1 &= \ln|x| + C_2 \\
\ln|y| &= \ln|x| + C \\
e^{\ln|y|} &= e^{\ln|x|+C} = e^C e^{\ln|x|} \\
|y| &= a|x|, \text{ where } a > 0 \text{ and } x \neq 0 \\
y &= \pm ax, \text{ where } a > 0 \text{ and } x \neq 0.
\end{aligned}$$

Note that the vectors $\mathbf{F}(x,0) = \langle x, 0\rangle$ are tangent to the x-axis, so the x-axis is a flow line; likewise, vectors $\mathbf{F}\langle 0, y\rangle$ are tangent to the y-axis, so the y-axis is also a flow line. We conclude that the flow lines are all lines through the origin.

Example 2. Let $\mathbf{F} = \langle x, -y \rangle$.



Compute the equations of the flow lines:

$$\frac{dy}{dx} = -\frac{y}{x}$$

$$\int \frac{dy}{y} = -\int \frac{dx}{x}$$

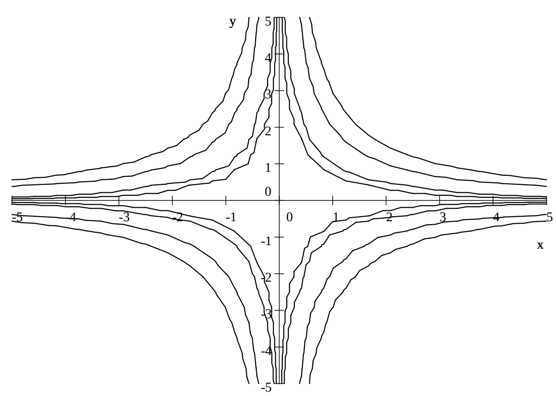
$$\ln|y| = -\ln|x| + A$$

$$\ln|xy| = A$$

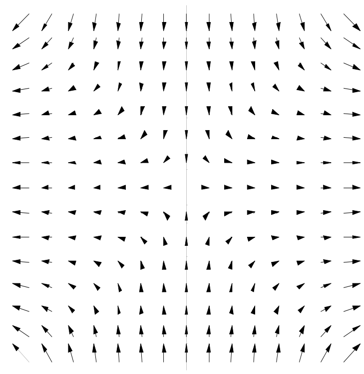
$$|xy| = e^A$$

$$xy = C;\ C \neq 0.$$

Now plot $xy = C$ with $C = \pm.5, \pm 1, \pm 2, \pm 3$:



Example 3. Let $\mathbf{F} = \left\langle x^3, -y^3 \right\rangle$.



Compute the equations of the flow lines:

$$\frac{dy}{dx} = -\frac{y^3}{x^3}$$

$$\int y^{-3} dy = -\int x^{-3} dx$$

$$\left(-2x^2y^2\right)\frac{y^{-2}}{-2} = \left(-2x^2y^2\right)\left(-\frac{x^{-2}}{-2} + A\right)$$

$$x^2 = -y^2 + Cx^2y^2$$

$$x^2 + y^2 = Cx^2y^2.$$

Plot $x^2 + y^2 = Cx^2y^2$ with $C = .25, .5, 1, 2, 3$