Finding Potential Functions in 3-variables

Math 311

Section 14.1, #35. Determine whether or not the vector field $\mathbf{F} = \langle 4x - z, 3y + z, y - x \rangle$ is conservative; if it is, find a potential function f.

1. Compute:

$$\begin{aligned} \mathbf{Curl}\,(\mathbf{F}) &= \left\langle \frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z} \right\rangle \times \langle 4x - z, 3y + z, y - x \rangle \\ &= \left\langle \begin{vmatrix} \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ 3y + z & y - x \end{vmatrix}, - \begin{vmatrix} \frac{\partial}{\partial x} & \frac{\partial}{\partial z} \\ 4x - z & y - x \end{vmatrix}, \begin{vmatrix} \frac{\partial}{\partial x} & \frac{\partial}{\partial y} \\ 4x - z & 3y + z \end{vmatrix} \right\rangle \\ &= \langle 1 - 1, (-1) - (-1), 0 - 0 \rangle = \langle 0, 0, 0 \rangle . \end{aligned}$$

Since $\mathbf{Curl}\,(\mathbf{F}) = \langle 0, 0, 0 \rangle$, $\mathbf{F}$ is conservative and there exists a potential function f, i.e., $\mathbf{F} = \left\langle \frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}, \frac{\partial f}{\partial z} \right\rangle$.

2. To find f :

(a) Integrate the first component with respect to x :

$$f(x, y, z) = \int 4x - z \; dx = 2x^2 - xz + g(y, z) .$$

(b) Differentiate with respect to y and z and equate with the respective component of $\mathbf{F}$:

$$\frac{\partial f}{\partial y} = \frac{\partial g}{\partial y} = 3y + z$$

$$\frac{\partial f}{\partial z} = -x + \frac{\partial g}{\partial z} = y - x$$

(c) Solve for $\left\langle \frac{\partial g}{\partial y}, \frac{\partial g}{\partial z} \right\rangle$ and apply the two-variable procedure to find $g(y, z)$:

$$\left\langle \frac{\partial g}{\partial y}, \frac{\partial g}{\partial z} \right\rangle = \langle 3y + z, y \rangle$$

1. Integrate the first component with respect to y :

$$g(y, z) = \int 3y + z \; dy = \frac{3}{2} y^2 + yz + h(z)$$

2. Differentiate with respect to z and equate with $\frac{\partial g}{\partial z} = y$:

$$\begin{aligned} \frac{\partial g}{\partial z} &= y + h'(z) = y \\ h(z) &= \int 0 \; dz = C \end{aligned}$$

3. $g(y, z) = \frac{3}{2} y^2 + yz + C$

(d) Conclude:

$$f(x, y, z) = 2x^2 - xz + \frac{3}{2} y^2 + yz + C$$